BI-LIPSCHICITY OF QUASICONFORMAL HARMONIC MAPPINGS IN THE PLANE

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Abstract

We show that quasiconformal harmonic mappings on the proper domains in \( \mathbb{R}^2 \) are bi-Lipschitz with respect to the quasihyperbolic metric.

1 Introduction

Continuity properties of quasiconformal mappings \( f : D \rightarrow D' \), where \( D \) and \( D' \) are domains in plane, with respect to various natural metrics have been studied extensively in [AKM], [KM], [KP] and [P].

Since the inverse of a \( K \)-quasiconformal mapping is also \( K \)-quasiconformal mapping, such results apply at the same time to \( f \) and \( f^{-1} \).

In this paper we deal with harmonic quasiconformal mappings \( f : D \rightarrow D' \), note that \( f^{-1} \) is not, in general, harmonic.

Our main result is that harmonic \( K \)-quasiconformal mapping \( f : D \rightarrow D' \) in plane is bi-Lipschitz with respect to quasihyperbolic metric.

We note that in [M] this result is proved in \( n \)-dimensional setting, but only in the case where \( D \) and \( D' \) are the upper half space in \( \mathbb{R}^n \).

In the case \( n = 2 \), in [M] this result is proved for \( D = D' = \mathbb{D} = \{ z : |z| < 1 \} \), with explicit bounds in terms of \( K \).

2 Result

Theorem 1. Suppose \( D \) and \( D' \) are proper domains in \( \mathbb{R}^2 \). If \( f : D \rightarrow D' \) is \( K \)-qc and harmonic, then it is bi-Lipschitz with respect to quasihyperbolic metrics on \( D \) and \( D' \).
We recall definition from [AG, Definition 1.5]

\[ \alpha_f(z) = \exp \left( \frac{1}{n}(\log J_f)_{B_z} \right), \]

where

\[ (\log J_f)_{B_z} = \frac{1}{m(B_z)} \int_{B_z} \log J_f \, dm, \quad B_z = B(z, d(z, \partial D)). \]

In the case \( n = 2 \) we have

\[ \frac{1}{\alpha_f(z)} = \exp \left( \frac{1}{2} \frac{1}{m(B_z)} \int_{B_z} \frac{1}{J_f(w)} \, dm(w) \right). \quad (1) \]

We are going to use the following result:

**Theorem 2.** [AG, Theorem 1.8] Suppose that \( D \) and \( D' \) are domains in \( \mathbb{R}^n \) if \( f : D \rightarrow D' \) is \( K \)-qc, then

\[ \frac{1}{c} \frac{d(f(z), \partial D')}{d(z, \partial D)} \leq \alpha_f(z) \leq c \frac{d(f(z), \partial D')}{d(z, \partial D)} \]

for \( z \in D \), where \( c \) is a constant which depends only on \( K \) and \( n \).

### 3 Proof of Theorem 1

Our proof is based on the theorem of Astala and Gehring.

**Proof.** Since \( f \) is harmonic we have a local representation

\[ f(z) = g(z) + \bar{h}(z), \]

where \( g \) and \( h \) are analytic functions. Then Jacobian \( J_f(z) = |g'(z)|^2 - |h'(z)|^2 > 0 \) (note that \( g'(z) \neq 0 \)).

Further,

\[ J_f(z) = |g'(z)|^2 \left( 1 - \frac{|h'(z)|^2}{|g'(z)|^2} \right) = |g'(z)|^2 \left( 1 - |\omega(z)|^2 \right), \]

where \( \omega(z) = \frac{h'(z)}{g'(z)} \) is analytic and \( |\omega| < 1 \). Now we have

\[ \log \frac{1}{J_f(z)} = -2 \log |g'(z)| - \log(1 - |\omega(z)|^2). \]

The first term is harmonic function (it is well known that logarithm of moduli of analytic function is harmonic everywhere except where that analytic function vanishes, but \( g'(z) \neq 0 \) everywhere).
The second term can be expanded in series
\[ \sum_{k=1}^{\infty} \frac{|\omega(z)|^{2k}}{k}, \]
and each term is subharmonic (note that \( \omega \) is analytic).

So, \( -\log(1 - |\omega(z)|^2) \) is a continuous function represented as a locally uniform sum of subharmonic functions. Thus it is also subharmonic.

Hence
\[ \log \frac{1}{J_f(z)} \]

is a subharmonic function. \hfill (2)

Note that representation \( f(z) = g(z) + \overline{h(z)} \) is local, but that suffices for our conclusion (2).

From (2) we have
\[ \frac{1}{m(B_z)} \int_{B_z} \log \frac{1}{J_f(w)} \, dm(w) \geq \log \frac{1}{J_f(z)}. \]
Combining this with (1) we have
\[ \frac{1}{\alpha_f(z)} \geq \exp \left( \frac{1}{2} \log \frac{1}{J_f(z)} \right) = \frac{1}{\sqrt{J_f(z)}} \]
and therefore
\[ \sqrt{J_f(z)} \geq \alpha_f(z). \]

Applying the first inequality from Theorem 2 we have
\[ \sqrt{J_f(z)} \geq \frac{1}{c} \frac{d(f(z), \partial D')}{d(z, \partial D)}. \]

Note that
\[ J_f(z) = |g'(z)|^2 - |h'(z)|^2 \leq |g'(z)|^2 \]
and by \( K \)-quasiconformality of \( f \), \( |h'| \leq k|g'| \), \( 0 \leq k < 1 \), where \( K = \frac{1+k}{1-k} \).

This gives \( J_f \geq (1 - k^2)|g'|^2 \). Hence,
\[ \sqrt{J_f} \asymp |g'| \asymp |g'| + |h'| = L(f, z), \]
where
\[ L(f, z) = \max_{|h|=1} |f'(z)h|. \]

Finally (3) and the above asymptotic relation give
\[ L(f, z) \geq \frac{1}{c} \frac{d(f(z), \partial D')}{d(z, \partial D)}, \quad c = c(k). \]
For the reverse inequality we again use $J_f(z) \geq (1 - k^2)|g'(z)|^2$, i.e.

$$\sqrt{J_f(z)} \geq \sqrt{1 - k^2}|g'(z)|$$

Further, we know that for $n = 2$

$$\alpha_f(z) = \exp \left( \frac{1}{m(B_z)} \int_{B_z} \log \sqrt{J_f(x)} \, dm(w) \right).$$

Using (4)

$$\frac{1}{m(B_z)} \int_{B_z} \log \sqrt{J_f(x)} \, dm(w) \geq \frac{1}{m(B_z)} \int_{B_z} \log \sqrt{1 - k^2} + \log |g'(w)| \, dm(w)$$

$$= \log \sqrt{1 - k^2} + \frac{1}{m(B_z)} \int_{B_z} \log |g'(w)| \, dm(w)$$

$$= \log \sqrt{1 - k^2} + \log |g'(z)|.$$
This pointwise result, combined with integration along curves, easily gives
\[ k_D'(f(z_1), f(z_2)) \asymp k_D(z_1, z_2). \]

\[ \square \]

**Problem 1.** Is Theorem 1 true in dimensions \( n \geq 3? \)

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**References**


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