SCATTERED DATA POINTS BEST INTERPOLATION
AS A PROBLEM OF THE BEST RECOVERY
IN THE SENSE OF SARD

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Abstract

The problem of the best recovery in the sense of Sard of a linear functional $L f$ on the basis of information $T(f) = \{L_j f, j = 1, 2, \ldots, N\}$ is studied. It is shown that in the class of bivariate functions with restricted $(n, m)$-derivative, known on the $(n, m)$-grid lines, the problem of the best recovery of a linear functional leads to the best approximation of

$$L(K_n K_m)$$

in the space $S = \text{span}\{L_j(K_n K_m), j = 1, 2, \ldots, N\}$, where $K_n(x, t) = K(x, t) - L_n^*(K(., t); x)$ is the difference between the truncated power kernel $K(x, t) = (x-t)^{n-1}/(n-1)!$ and its Lagrange interpolation formula. In particular, the best recovery of a bivariate function is considered, if scattered data points and blending grid are given. An algorithm is designed and realized using the software product MATLAB.

1 Introduction

Let $H$ be a Hilbert space. Suppose that $L$ and $L_1, \ldots, L_N$ are linear functionals defined on $H$. The problem is to find the best method $S f$ of recovery of the functional $L$ on the basis of the information $T f = (L_1 f, L_2 f, \ldots, L_N f)$.

Given a method $S f$, the error is

$$E(S) = \sup_{f \in B} |L f - S f|,$$

where $B$ usually is the unit ball in $H$. The method $S_*$ is the best method of recovery of $L f$ if its error is minimal, i.e.

$$E(T) = \inf_S \sup_{f \in B} |L f - S f| = \sup_{f \in B} |L f - S_* f|.$$  \hspace{1cm} (1)

In accordance with Smolyak’s Lemma [6], a linear best method of recovery of a linear functional exists under certain restrictions:

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Lemma Smolyak: If the linear functionals $L, L_j, j = 1, 2, \ldots, N$ are defined on a convex and centrally symmetric body $B$ in a linear space and

$$
\sup \{ Lf : f : L_j f = 0, j = 1, 2, \ldots, N \} < \infty,
$$

then there exists a best linear method for recovery of $L$, i.e. there exist numbers $A_1, A_2, \ldots, A_N$ such that

$$
E(T) = \inf_{c_j, f \in B} |Lf - \sum C_j L_j f| = \sup_{f \in B} |Lf - \sum A_j L_j f|.
$$

(2)

The linear methods which are exact for a class of functions, are usually studied [5], [7]. This gives an opportunity for Peano’s theorem application. An explicit representation and an algorithm for the best recovery of univariate function can be found in [11].

Results for bivariate functions in some special cases can be found in [1], [2], [3] and [4]. We will not set here any restrictions on the linear approximation methods, but some additional information about the function we will need.

2 Best recovery of a linear functional in the sense of Sard

The Sobolev’s class of functions is defined as usually:

$$
W^n_p[a, A] = \{ f \in C^{n-1}[a, A] : f^{(n-1)} \text{abs.cont.}, \| f^{(n)} \|_p < \infty \}.
$$

We denote the truncated power kernels resp:

$$
K(x, t) := \frac{(x - t)^{n-1}}{(n - 1)!}, \quad K(y, \tau) := \frac{(y - \tau)^{m-1}}{(m - 1)!}.
$$

Set the Lagrange interplation formula $L_n^x(f; x) = \sum_{i=1}^n f(x_i) l_{ni}(x)$, $l_{ni}(x) = \prod_{r=1, r \neq i}^{n} \frac{x - x_r}{x_i - x_r}$, $l_{ni}(x_i) = \delta_{ij}$. We shall denote the difference between the truncated power kernel and its Lagrange formula by $K_n$, resp. $K_m$:

$$
K_n(x, t) := K(x, t) - L_n^x(K(\cdot, t); x), \quad K_m(y, \tau) := K(y, \tau) - L_m^y(K(\cdot, \tau); y).
$$

We shall omit the bar in the symbols $\bar{K}_m$ and $\bar{l}_{jm}$, since there is no chance of confusion.

Theorem 1: If $f(x, y) \in W_2^{n,m}[a, A] \times [b, B]$ and $x_1, \ldots, x_n$ are in $(a, A)$, $y_1, \ldots, y_m$ are in $(b, B)$, then for every $(x, y) \in [a, A] \times [b, B]$ the following equality holds:

$$
f(x, y) = \sum_{k=1}^n f(x_k, y) l_{nk}(x) + \sum_{j=1}^m f(x, y_j) l_{mj}(y) - \sum_{i=1}^n \sum_{j=1}^m f(x_i, y_j) l_{ni}(x) l_{mj}(y)
$$

$$
+ \int_a^A \int_b^B K_n(x, t) K_m(y, \tau) f^{(n,m)}(t, \tau) dt d\tau.
$$

(3)
We consider the Hilbert space $H$ with the scalar product 
\[ (f, g) := \int_a^B f(x, y)g(x, y)dx dy \]
and $L_2$-norm 
\[ \|f\| = \left( \int_a^A \int_b^B f^2(x, y)dx dy \right)^{\frac{1}{2}}. \]

The class of functions $B$ is defined as follows:
\[ B = \{ f(x, y) \in W_2^{n,m}[a, A] \times [b, B] : \|f^{(n,m)}\| \leq 1 \}. \quad (4) \]

It is clear that $B$ is a convex and centrally symmetric body and the Smolyak’s lemma holds.

We suppose, that the function $f(x, y) \in B$ is known on the grid lines
\[ x = x_k, k = 1, 2, \ldots, n, \quad y = y_j, j = 1, 2, \ldots, m. \quad (5) \]

The proofs of the following Theorem 2. and Theorem 3. can be found in [12].

**Theorem 2:** The problem of finding the best linear method $Lf \sim \sum A_j L_j$ is equivalent to the problem of the best approximation of $L(K_n(x)K_m(y))$ in the space $\text{span}\{L_j(K_nK_m), j = 1, 2, \ldots, N\}$ in the sense that the coefficients $A_j, j = 1, 2, \ldots, N$ of the best linear method can be obtained after the best approximation of $L(K_n(x)K_m(y))$ in the space $\text{span}\{L_j(K_nK_m), j = 1, 2, \ldots, N\}$.

Further we use Gramm-Schmidt orthogonalization:
\[ Q_1 K := L_1 K, \ldots, Q_i K = \sum_{j=1}^i f_{ij}L_j K. \quad (6) \]

We shall consider here linear methods for approximation without any restrictions, but some additional information we need-the function’s traces must be known:

**Theorem 3:** Let $L$ be a linear functional such that $L(K_nK_m)$ is integrable on $[a, A] \times [b, B]$. If $f \in B$ is known on the grid lines (5), then the best linear method for recovery of $Lf$ on the basis of information $T(f) = (L_1 f, L_2 f, \ldots, L_N f)$ is
\[ Lf \sim Lb_f + \sum_{j=1}^N A_j L_j(f - b_f) \]
\[ = Lb_f + \sum_{j=1}^N \sum_{i=j}^N \left( L(K_nK_m), Q_i(K_nK_m) \right) f_{ij} L_j(f - b_f), \quad (7) \]

where $\{Q_i(K_nK_m), i = 1, 2, \ldots, N\}$ is an orthogonal system in the span $\{L_i(K_nK_m)\}$, $\{f_{ij}\}$ are coefficients and $K_n, K_m$ are the modified kernels.
3 Best recovery of a bivariate function

Let a point \((x, y) \in [a, A] \times [b, B]\) be fixed. We consider the special case \(L f := f(x, y)\), \(L_j f := f(a_j, b_j), j = 1, 2, \ldots, N\). As we assume above, the function \(f\) is known on the grid lines (5). The formula (7) holds.

We take points \(x_1, \ldots, x_n\) in \((a, A)\) and points \(y_1, \ldots, y_m\) in \((b, B)\) and we calculate the terms in formula (7):

\[
K_n(x, t) = \frac{(x - t)_+^{n-1}}{(n-1)!} - \sum_{i=1}^{n} \frac{(x_i - t)_+^{n-1}}{(n-1)!} l_{ni}(x).
\]

\[
K_m(y, \tau) = \frac{(y - \tau)_+^{m-1}}{(m-1)!} - \sum_{j=1}^{m} \frac{(y_j - \tau)_+^{m-1}}{(m-1)!} l_{mj}(y).
\]

\[
L_i(K_nK_m) = K_n(a_i, t)K_m(b_i, \tau)
\]

\[
(L(K_nK_m), L_i(K_nK_m)) = (K_n(x, t)K_m(y, \tau), K_n(a_i, t)K_m(b_i, \tau)) =
\]

\[
= \int_a^A [K(x, t) - \sum_{k=1}^{n} K(x_k, t)l_{nk}(x)][K(a_i, t) - \sum_{r=1}^{n} K(x_r, t)l_{nr}(a_i)]dt.
\]

\[
\cdot \int_b^B [K(y, \tau) - \sum_{j=1}^{m} K(y_j, \tau)l_{mj}(y)][K(b_i, \tau) - \sum_{s=1}^{m} K(y_s, \tau)l_{ms}(b_i)]d\tau.
\]

(8)

Set

\[
I_n(x, y) = I_n(x, y) := \int_a^A (x - t)_+^{n-1}(y - t)_+^{m-1}dt.
\]

It is easy to see that the first integral in equation (8) is equal to \(\frac{1}{(n-1)!m!}S(x, a_i)\), where

\[
S(x, a_i) := I_n(x, a_i) - \sum_{r=1}^{n} l_{nr}(a_i)I_n(x, x_r) - \sum_{k=1}^{n} l_{nk}(x)I_n(x_k, a_i)
\]

\[
+ \sum_{k=1}^{n} \sum_{r=1}^{n} l_{nk}(x)l_{nr}(a_i)I_n(x_k, x_r).
\]

(10)

By analogy we obtain the second integral in [8]. The equation (8) is equivalent to

\[
(K_nK_m, L_i(K_nK_m)) = \frac{1}{(n-1)!^2(m-1)!^2}S(x, a_i)S(y, b_i).
\]

The calculation of the coefficients \(f_{ij}\) depends on the terms

\[
(L_i(K_nK_m), L_j(K_nK_m)) = (K_n(a_i, t)K_m(b_i, \tau), K_n(a_j, t)K_m(b_j, \tau))
\]

\[
= \frac{1}{(n-1)!^2(m-1)!^2}S(a_i, a_j)S(b_i, b_j).
\]

(11)
From (6) it follows that 
\[(K_n K_m, Q_i(K_n K_m)) = \sum_{k=1}^i f_{ik}(K_n K_m, L_k(K_n K_m)) \]
and 
\[(Q_i(K_n K_m), Q_i(K_n K_m)) = \frac{1}{(n-1)!^2(m-1)!^2} \sum_{r=1}^i \sum_{s=1}^i f_{ir} f_{is} S(a_r, a_s) S(b_r, b_s). \] 
(12)

Finally, replacing these terms in formula (7), we obtain
\[
f(x, y) \sim b_f(x, y) + \sum_{j=1}^N \sum_{i=j}^N f_{ij} \frac{\sum_{k=1}^i f_{ik} S(x, a_k) S(y, b_k)}{\sum_{r=1}^i \sum_{s=1}^i f_{ir} f_{is} S(a_r, a_s) S(b_r, b_s)} (f(a_j, b_j) - b_f(a_j, b_j)), \] 
(13)
where \(S(x, a_k)\) is given by (10). We find that the integral in (9) is equal to
\[
I_n(x, y) = \sum_{k=0}^{n-1} \frac{2n-k-1}{2n-k-1} (\min(x, y) - a)^{2n-k-1} |x - y|^k. \] 
(14)

From this representation it follows that the integral \(I_n(x, x_r)\) in (10) is a polynomial of degree \(2n-1\) for \(x < x_r\) and of degree \(n-1\) for \(x > x_r\).

The error of the best recovery in an arbitrary point \((x, y)\) is obtained in the proof of Theorem 2 [12]. After finding the coefficients \(A_j\) we replace them in
\[
E(T) = \| K_n K_m - \sum A_j L_j(K_n K_m) \|
\]

The formula (11) is exact for blending functions \(f \equiv b_f\), belonging to the class
\[
B_{n,m} = \{ f(x, y) \in W_2^{n,m}[a, A] \times [b, B] : f^{(n,m)} \equiv 0 \}. \] 
(15)

A program has been realized for interpolation of scattered data, based on formulae (11) and (10), (12), using the software product MATLAB v.4. See on the fig.
the interpolating surface for the so called Mexican hat, if $N = 36$ points, $n = 2$, $m = 1$.

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**References**


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