SOME RELATIONS IN THE GENERALIZED KÄHLERIAN SPACES OF THE SECOND KIND

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Abstract

Starting from the definition of generalized Riemannian space \((GR_N)\) [1], in which a non-symmetric basic tensor \(g_{ij}\) is introduced, in the present paper a generalized Kählerian space \(GK^2_N\) of the second kind is defined, as a \(GR_N\) with almost complex structure \(F_{hi}\), that is covariantly constant with respect to the second kind of covariant derivative (equation (2.3)).

Several theorems are proved. These theorems are generalizations of the corresponding theorems relating to \(K_N\). The relations between \(F_{hi}\) and four curvature tensors from \(GR_N\) are obtained.

1 Introduction

A generalized Riemannian space \(GR_N\) in the sense of Eisenhart’s definition [1] is a differentiable \(N\)-dimensional manifold, equipped with a non-symmetric basic tensor \(g_{ij}\). Connection coefficients of this space are generalized Cristoffel’s symbols of the second kind. Generally, \(\Gamma_{ij}^k \neq \Gamma_{kj}^i\).

In a generalized Riemannian space one can define four kinds of covariant derivatives [3], [4]. For example, for a tensor \(a^i_{j}^{\cdot m}\) in \(GR_N\) we have

\[
\begin{align*}
    a_{j}^{i}_{\cdot 1} &= a_{j,m}^{i} + \Gamma_{pm}^{i}a_{j}^{p} - \Gamma_{jm}^{p}a_{j}^{p}, \quad a_{j}^{i}_{\cdot 2} = a_{j,m}^{i} + \Gamma_{mp}^{i}a_{j}^{p} - \Gamma_{jm}^{p}a_{j}^{p}, \\
    a_{j}^{i}_{\cdot 3} &= a_{j,m}^{i} + \Gamma_{pm}^{i}a_{j}^{p} - \Gamma_{jm}^{p}a_{j}^{p}, \quad a_{j}^{i}_{\cdot 4} = a_{j,m}^{i} + \Gamma_{mp}^{i}a_{j}^{p} - \Gamma_{jm}^{p}a_{j}^{p}.
\end{align*}
\]

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In the case of the space $GR_N$ we have five independent curvature tensors [5]:

$$R^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm} \Gamma^i_{pn} - \Gamma^p_{jn} \Gamma^i_{mp},$$

$$R^i_{jmn} = \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj} \Gamma^i_{np} - \Gamma^p_{nj} \Gamma^i_{mp},$$

$$R^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm} \Gamma^i_{np} - \Gamma^p_{nj} \Gamma^i_{mp} + \Gamma^p_{nm} (\Gamma^i_{pj} - \Gamma^i_{jp}),$$

$$R^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm} \Gamma^i_{np} - \Gamma^p_{nj} \Gamma^i_{mp} + \Gamma^p_{nm} (\Gamma^i_{pj} - \Gamma^i_{jp}),$$

$$R^i_{jmn} = \frac{1}{2} (\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm} \Gamma^i_{pn} + \Gamma^p_{nj} \Gamma^i_{pm} - \Gamma^p_{jm} \Gamma^i_{np} - \Gamma^p_{nj} \Gamma^i_{mp}).$$

Kählerian spaces and their mappings were investigated by many authors, for example K. Yano [14], [15], M. Prvanović [12], T. Otsuki [11], N. S. Sinyukov [13], J. Mikeš [2] and many others.

In [6], [7], [8] we defined a generalized Kählerian space $GK_N$ as a generalized $N$-dimensional Riemannian space with a (non-symmetric) metric tensor

$$g_{ij} = g_{ij} + g^\perp_{ij},$$

where $g_{ij}$ is symmetric part, and $g^\perp_{ij}$ anty-symmetric one of the metric tensor.

The lowering and the raising of indices one defines by the tensors $g_{ij}$ and $g^{\perp}_{ij}$ respectively, where $g^{\perp}_{ij}$ is defined by the equation

$$g_{ij} g^{\perp}_{jk} = \delta^k_i, \; i, k = 1, \ldots, m,$$

and $\delta^k_i$ is Kronecker symbol. Therefore, since the matrix $(g^{\perp}_{ij})$ is inverse for $(g_{ij})$ it is necessary to be

$$g = det(g_{ij}) \neq 0.$$

There exist an almost complex structure $F_i^j$ such that

$$(1.1) \quad F^h_p(x) F^p_i(x) = -\delta^h_i,$$

$$(1.2) \quad g_{pq} F^p_i F^q_j = g_{ij}, \quad g^{\perp}_{pq} F^p_i F^q_j,$$

$$(1.3) \quad F^h_p(x) F^p_i(x) = 0, \quad (\theta = 1, 2),$$

where $|$ denotes the covariant derivative of the kind $\theta$ with respect to the metric tensor $g_{ij}$.

In [9] we defined a generalized Kählerian space of the first kind $GK_1^N$ if there exists an almost complex structure $F_i^j(x)$, such that

$$(1.4) \quad F^h_p(x) F^p_i(x) = -\delta^h_i,$$

$$(1.5) \quad g_{pq} F^p_i F^q_j = g_{ij}, \quad g^{\perp}_{pq} F^p_i F^q_j,$$

$$(1.6) \quad F^h_p(x) F^p_i(x) = 0,$$
where \( \ddagger \) denotes the covariant derivative of the first kind with respect to the metric tensor \( g_{ij} \).

### 2 Generalized Kählerian spaces of the second kind

A generalized \( N \)-dimensional Riemannian space with (non-symmetric) metric tensor \( g_{ij} \) is a generalized Kählerian space of the second kind \( GK^2_N \) if there exists an almost complex structure \( F^i_j(x) \) such that

\[
\begin{align*}
(2.1) \quad & F^h_p(x) F^p_i(x) = -\delta^h_i, \\
(2.2) \quad & g_{pq} F^p_i F^q_j = g_{ij}, \quad g_{ij} = g_{pq} F^p_i F^q_j, \\
(2.3) \quad & F^h_{ij} = 0,
\end{align*}
\]

where \( \ddagger \) denotes the covariant derivative of the first kind with respect to the metric tensor \( g_{ij} \). From (2.2), using (2.1), we get

\[
\begin{align*}
(2.4) \quad & F_{ij} = -F_{ji}, \quad F^{ij} = -F^{ji},
\end{align*}
\]

where we denote

\[
\begin{align*}
(2.5) \quad & F^{ij} = F^p_j g_{pi}, \quad F_{ji} = F^p_j g^p_i.
\end{align*}
\]

From here we prove the following theorems.

**Theorem 2.1.** For the almost complex structure \( F^i_j \) of a generalized Kählerian space of the second kind the relations

\[
\begin{align*}
F^h_{ij} &= 2(F^p_i \Gamma^h_{pj} + F^h_p \Gamma^p_{ij}), \\
F^h_{ij} &= 2F^p_i \Gamma^h_{pj}, \\
F^h_{ij} &= 2F^h_p \Gamma^p_{ij}
\end{align*}
\]

are valid, where \( \Gamma^h_{ij} \) is the torsion tensor.

**Proof.** We get the relations (2.6) by using the condition (2.3) \( \square \)

Let us denote \( F^h_{ij} = F^p_i \Gamma^h_{pj} \) and \( \widetilde{F}^h_{ij} = F^h_p \Gamma^p_{ij} \). Then we have
Theorem 2.2. For the curvature tensors $R^\theta_h$, $(\theta = 1, \ldots, 4)$ of a generalized Kählerian space of the second kind the relations

$$
F^h_i \frac{\partial}{\partial x^i} R^p_j \frac{\partial}{\partial x^j} \equiv F^h_j \frac{\partial}{\partial x^j} R^p_i \frac{\partial}{\partial x^i},
$$

$$
F^h_i \frac{\partial}{\partial x^i} R^p_l \frac{\partial}{\partial x^l} - \frac{\partial}{\partial x^l} F^p_l \frac{\partial}{\partial x^l} R^h_i \frac{\partial}{\partial x^i} - 4F^p_l \frac{\partial}{\partial x^l} (F^q_m \frac{\partial}{\partial x^m} + F^q_l \frac{\partial}{\partial x^l})
$$

$$
= 2(F^h_i \frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i} F^h_i) + 2(F^h_i \frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i} F^h_i),
$$

$$
F^h_i \frac{\partial}{\partial x^i} R^p_l \frac{\partial}{\partial x^l} - \frac{\partial}{\partial x^l} F^p_l \frac{\partial}{\partial x^l} R^h_i \frac{\partial}{\partial x^i} = -2(F^h_i \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^i} F^h_i),
$$

$$
F^h_i \frac{\partial}{\partial x^i} R^p_l \frac{\partial}{\partial x^l} + \frac{\partial}{\partial x^l} R^h_i \frac{\partial}{\partial x^i} = 2(F^h_i \frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i} F^h_i)
$$

are valid.

**Proof.** From (2.6) by using the covariant derivative of the first kind we have

$$
F^h_i \frac{\partial}{\partial x^i} = -2\frac{\partial}{\partial x^i} F^h_i - 2\frac{\partial}{\partial x^i} F^h_i,
$$

and also

$$
F^h_i \frac{\partial}{\partial x^i} = -2\frac{\partial}{\partial x^i} F^h_i - 2\frac{\partial}{\partial x^i} F^h_i.
$$

Now, from (2.8) and (2.9) we obtain

$$
F^h_i \frac{\partial}{\partial x^i} - F^h_i \frac{\partial}{\partial x^i} = 2(F^h_i \frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i} F^h_i) + \frac{\partial}{\partial x^i} F^h_i - \frac{\partial}{\partial x^i} F^h_i.
$$

Using the Ricci identity [5], we get from (2.10)

$$
F^h_i \frac{\partial}{\partial x^i} R^p_j \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} F^p_j \frac{\partial}{\partial x^j} R^h_i \frac{\partial}{\partial x^i} - 2\frac{\partial}{\partial x^j} F^h_i = 2(F^h_i \frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i} F^h_i),
$$

and from here the second equality (2.7) is valid.

The first equality we get directly from the Ricci identity obtained by virtue of the second kind of covariant derivative and using (2.3).

By the same procedure like in the previous two cases, it is easy to prove the third and the fourth equation.

If we denote with $(;)$ covariant derivative wrt the symmetric connection, then the next theorem follows.
Theorem 2.3. For the Ricci tensor $R_{ij}$, given by $g_{ij}$ the relation

$$R_{hk} = F^{p}_{h} F^{q}_{k} R_{pq} - g^{pq} F^{p}_{k} \left( \mathcal{D}_{s.pqk} + \mathcal{D}_{h.pqs} \right)$$

is valid, where

$$\mathcal{D}_{ijh} = F^{p}_{i} \Gamma_{hjp}^{h} - F^{p}_{i} \Gamma_{kjp}^{h} + F^{p}_{h} \left( \Gamma_{ijp}^{h} - \Gamma_{kjp}^{h} \right)$$

(2.13)

and $R_{hk} = R^{p}_{hkp}$, $\mathcal{D}_{h.ijk} = g_{ph} \mathcal{D}^{p}_{ijk}$.

Proof. From (2.3) and (2.6) we get

$$F^{h}_{i} = -(F^{p}_{i} \Gamma_{jph} + F^{h}_{p} \Gamma_{ijp}).$$

(2.14)

The integrability conditions of the equation (2.14) are given by

$$F^{h}_{i} = F^{h}_{i},$$

(2.15)

Using the Ricci identity, from (2.15) we obtain

$$F^{h}_{i} R^{p}_{ijk} - F^{p}_{i} R^{h}_{phjk} = -\mathcal{D}^{h}_{ijk}.$$  

(2.16)

Here $R^{h}_{ijk}$ is the curvature tensor with respect to the symmetric affine connection $\Gamma_{hij}$. Composition with $F^{i}_{p}$ in (2.16) gives

$$F^{h}_{p} F^{q}_{i} R^{p}_{qjk} + \mathcal{D}^{h}_{ijk} = -F^{p}_{i} \mathcal{D}^{h}_{pjk}.$$  

(2.17)

Now, from (2.17) by composition with $g_{hk}$ we get

$$F^{h}_{i} F^{p}_{i} R^{p}_{qjk} + R^{h}_{hij} = -F^{p}_{i} \mathcal{D}^{h}_{pjk}.$$  

(2.18)

From here we obtain

$$-F^{h}_{p} F^{q}_{i} R^{p}_{qjk} + R^{h}_{hij} = -F^{p}_{i} \mathcal{D}^{h}_{pjk}.$$  

(2.19)

From (2.19) by composition with $F^{i}_{p}$ we have

$$F^{p}_{i} R_{pqk} - F^{p}_{i} R_{phjk} = \mathcal{D}^{h}_{h.ijk}.$$  

(2.20)

Using composition with $g_{pq}$ in (2.20) we obtain

$$F^{p}_{i} R_{pk} - F^{p}_{i} R_{phq} = g^{pq} \mathcal{D}^{h}_{h.pqk}.$$  

(2.21)

The symmetrization in (2.21) with respect to $h$, $k$ gives the relation (2.12).
Theorem 2.4. The Ricci tensors $R^{\theta}_{\ j\ m}$ ($\theta = 1, \cdots, 5$) of the space $GK_{\ 2}^{\ N}$ satisfy the relations

\begin{align}
R^{\alpha}_{\ \alpha} F^p_m &= R^{\alpha}_{\ j\ m} - 2\Gamma^p_{\ rg} \Gamma^g_{\ \alpha} F^r_m \\
&\quad + 2\Gamma^p_{\ jq} \Gamma^q_{\ pm} + 2g^{pq} F^s_h (D_{s.pqk} + D_{k.pqs}), \ \alpha = 1, 2, 3,
\end{align}

(2.22a, b, c)

\begin{align}
R^{\alpha}_{\ \alpha} F^p_m &= R^{\alpha}_{\ j\ m} + 6\Gamma^p_{\ jq} \Gamma^q_{\ pm} F^p_m \\
&\quad - 6\Gamma^p_{\ jn} \Gamma^q_{\ pm} + 2g^{pq} F^s_h (D_{s.pqk} + D_{k.pqs}),
\end{align}

(2.22d)

\begin{align}
R^{\alpha}_{\ \alpha} F^p_m &= R^{\alpha}_{\ j\ m} + 2\Gamma^p_{\ jq} \Gamma^q_{\ pm} F^p_m \\
&\quad - 2\Gamma^p_{\ jn} \Gamma^q_{\ pm} + 2g^{pq} F^s_h (D_{s.pqk} + D_{k.pqs}),
\end{align}

(2.22c)

where $(j\ m)$ denotes the symmetrization without division with respect to the indices $j, m$.

Proof. (a) We can express the tensor $R^{i}_{\ j\ mn}$ in the form [5]:

$$R^{i}_{\ j\ mn} = R^{i}_{\ j\ mn} + \Gamma^i_{\ jm,n} - \Gamma^i_{\ jn,m} + \Gamma^p_{\ jn} \Gamma^i_{\ pn} - \Gamma^p_{\ jm} \Gamma^i_{\ pm}.$$  

By contraction with respect to the indices $i, n$, and by symmetrization with respect to $j, m$, we get

$$R^{\alpha}_{\ \alpha} = R^{\alpha}_{\ j\ m} - 2\Gamma^p_{\ jq} \Gamma^q_{\ pm}.  \quad \text{(2.23)}$$

From (2.12) and (2.23) we have (2.22a).

(b) The tensor $R^{i}_{\ j\ mn}$ can be expressed in the form [5]:

$$R^{i}_{\ j\ mn} = R^{i}_{\ j\ mn} - \Gamma^i_{\ jm,n} + \Gamma^i_{\ jm,n} - \Gamma^p_{\ jm} \Gamma^i_{\ pm} + \Gamma^p_{\ jm} \Gamma^i_{\ pm}.$$  

By contraction with respect to $i, n$, and then by symmetrization with respect to $j, m$, we get

$$R^{\alpha}_{\ \alpha} = R^{\alpha}_{\ j\ m} - 2\Gamma^p_{\ jq} \Gamma^q_{\ pm},$$

from where, using (2.12), we get the relation (2.22b).

(c) For the tensor $R^{i}_{\ j\ mn}$ we have [5]:

$$R^{i}_{\ j\ mn} = R^{i}_{\ j\ mn} + \Gamma^i_{\ jm,n} + \Gamma^i_{\ jm,n} - \Gamma^p_{\ jm} \Gamma^i_{\ pm} + \Gamma^p_{\ jm} \Gamma^i_{\ pm} - 2\Gamma^p_{\ mn} \Gamma^i_{\ pj}.$$
Contracting with respect to \(i, n\), and then symmetrizing in relation to \(j, m\), we get

\[
R_{(jm)} = R_{(jm)} - 2\Gamma_{jq}^{p} \Gamma_{pm}^{q},
\]

from where, using (2.12), we can see that the relation (2.22c) is valid.

\((d)\) The tensor \(R_{4}^{i} jmn\) can be expressed in the form [5]:

\[
R_{4}^{i} jmn = R_{4}^{i} jmn + \Gamma_{jm,n}^{i} + \Gamma_{jn,m}^{i} - \Gamma_{jm}^{p} \Gamma_{pn}^{i} + \Gamma_{jn}^{p} \Gamma_{pm}^{i} + 2\Gamma_{qp}^{i} \Gamma_{pj}^{i}.
\]

Contracting with respect to \(i, n\), and symmetrizing with respect to \(j, m\), we get

\[
R_{4}^{i} (jm) = R_{(jm)} + 6\Gamma_{jq}^{p} \Gamma_{pm}^{q}.
\]

Using (2.12) we get the relation (2.22d).

\((e)\) The tensor \(R_{5}^{i} jmn\) satisfies the relation [5]:

\[
R_{5}^{i} jmn = R_{5}^{i} jmn + \Gamma_{jm}^{p} \Gamma_{pn}^{i} + \Gamma_{jn}^{p} \Gamma_{pm}^{i}.
\]

Contracting with respect the indices \(i, n\), and than symmetrizing with respect to \(j, m\), we get

\[
R_{5}^{i} (jm) = R_{(jm)} + 2\Gamma_{jq}^{p} \Gamma_{pm}^{q},
\]

from where, using (2.12), we get (2.22e). \(\square\)

References

Some relations in the generalized Kählerian spaces of the second kind


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