INCLUSION AND NEIGHBORHOOD PROPERTIES
OF A CERTAIN SUBCLASSES OF P-VALENT
FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

By means of Ruscheweyh derivative operator, we introduced and investigated two new subclasses of p-valent analytic functions. The various results obtained here for each of these function classes include coefficient bounds and distortion inequalities, associated inclusion relations for the \((n, \theta)\)-neighborhoods of subclasses of analytic and multivalent functions with negative coefficients, which are defined by means of non-homogenous differential equation.

1 Introduction

Let \(T_p(n)\) denote the class of functions of the form:

\[
f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0; p, n \in \mathbb{N} = \{1, 2, \ldots\}),
\]

which are analytic and p-valent in the open unit disc \(U = \{z : |z| < 1\}\). The modified Hadamard product (or convolution) of the function \(f(z)\) given by (1.1) and the function \(g(z) \in T_p(n)\) given by

\[
g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad (b_k \geq 0; p, n \in \mathbb{N})
\]

is defined by

\[
(f \ast g)(z) = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k = (g \ast f)(z).
\]
We introduce here an extended linear derivative operator of Ruscheweyh type (see [14]):

\[ D^{\mu,p} : T_p \to T_p \quad (T_p = T_p(1)), \]

which is defined by the following convolution:

\[ D^{\mu,p}f(z) = \frac{z^p}{(1-z)^{\mu+p}} * f(z) \quad (\mu > -p; f(z) \in T_p), \tag{1.4} \]

which in view of (1.1) (with \( n = 1 \)) becomes

\[ D^{\mu,p}f(z) = z^p - \sum_{k=p+1}^{\infty} \left( \frac{k + \mu - 1}{k - p} \right) a_k z^k \]

\[ = z^p - \sum_{k=p+1}^{\infty} \frac{\Gamma(k + \mu)}{(k - p)! \Gamma(p + \mu)} a_k z^k \quad (\mu > -p; f(z) \in T_p). \tag{1.5} \]

In particular, when \( \mu = n \ (n \in N_0 = N \cup \{0\}) \), it is easy observed from (1.4) and (1.5) that

\[ D^{n,p}f(z) = \frac{z^p(z^n-pf(z))^{(n)}}{n!} \quad (p \in N; n \in N_0), \tag{1.6} \]

so that

\[ D^{1-p,p}f(z) = f(z) \quad \text{and} \quad D^{1,p}f(z) = (1-p)f(z) + zf'(z). \tag{1.7} \]

For a function \( f(z) \in T_p(n) \), we have (see [9])

\[ (D^{\mu,p}f(z))^{(q)} = \delta(p,q)z^{p-q} - \sum_{k=n+p}^{\infty} \left( \frac{k + \mu - 1}{k - p} \right) \delta(k,q) a_k z^{k-q}, \]

\[ = \delta(p,q)z^{p-q} - \sum_{k=n+p}^{\infty} \frac{\Gamma(k + \mu)}{(k - p)! \Gamma(p + \mu)} \delta(k,q) a_k z^{k-q} \]

\[ (p \in N; q \in N_0; p > q), \tag{1.8} \]

where

\[ \delta(p,q) = \begin{cases} 1 & (q = 0) \\ p(p-1) \ldots \ldots(p-q+1) & (q \neq 0). \end{cases} \tag{1.9} \]

Now, making use of the operator \( D^{\mu,p}f(z)(\mu > -p, p \in N) \) given by (1.5), we now introduce a new subclass \( T^\beta_p(n, p, \lambda, \beta) \) of the \( p \)-valent analytic function class \( T_p(n) \) which consist of functions \( f(z) \in T_p(n) \) satisfying the inequality:

\[ \left| \left\{ \lambda z(D^{\mu,p}f(z))^{(q+1)} + (1-\lambda) z(D^{1+p,p}f(z))^{(q+1)} \right/ \lambda(D^{\mu,p}f(z))^{(q)} + (1-\lambda)(D^{1+p,p}f(z))^{(q)} - (p-q) \right\} \right| < \beta \]
Inclusion and neighborhood properties of a certain subclasses of p-valent...  3

\[ (p \in \mathbb{N}; q \in \mathbb{N}_0; 0 \leq \lambda \leq 1; p > \max(q, \mu); 0 < \beta \leq 1). \]  

(1.10)

We note that:

(i) \( T_\lambda^\lambda(n, 1, \lambda, \beta) = T_\mu(n, \lambda, \beta) \) (Irmak et al. [10];

(ii) \( \mathcal{T}_n^b(n, p, \lambda, \beta, b) = \mathcal{S}_n(b, \lambda, \beta) (b \in C \backslash \{0\}) \) (Altintas et al. [5]);

(iii) \( T_\mu(n, p, 1, \beta | b) = S(b, \mu, \beta) (b \in C \backslash \{0\}) \) (Murugusundaramoorthy and Srivastava [13]).

Also in this paper we shall derive several results for functions in the subclass \( H_\mu^p(n, p, \lambda, \beta; \gamma) \) of the function class \( T_p(n) \), which is defined as follows:

A function \( f(z) \in T_p(n) \) is said to belong to the class \( H_\mu^p(n, p, \lambda, \beta; \gamma) \) if \( w = f(z) \) satisfies the following non-homogenous Cauchy-Euler differential equation:

\[
z^2 \frac{d^{k+q}w}{dz^{k+q}} + 2(1 + \gamma)z \frac{d^{k+q}w}{dz^{k+q}} + \gamma(1 + \gamma) \frac{d^{q+1}w}{dz^{q+1}} = (p-q+\gamma)(p-q+\gamma+1) \frac{d^2g(z)}{dz^2},
\]

(1.11)

where \( g(z) \in T_\mu^\mu(n, p, \lambda, \beta) \) and \( \gamma > q-p, \gamma \in \mathbb{R} \).

Several other interesting subclasses of the class \( T_p(n) \) were investigated recently, for example, by Chen et al. [8], Chen [7], Srivastava and Aouf [16], Murugusundaramoorthy et al. [12], Altintas [1], and Altintas et al. (3 and 4), (see also Srivastava and Owa [17]).

In this paper we investigate the geometric characteristics of the classes \( T_\mu^\mu(n, p, \lambda, \beta) \) and \( H_\mu^p(n, p, \lambda, \beta; \gamma) \) also we investigate some \((n, \theta)\)-neighborhood properties.

2 Basic properties of the class \( T_\mu^\mu(n, p, \lambda, \beta) \)

We begin by proving a necessary and sufficient condition for a function belonging to the class \( T_p(n) \) to be in the class \( T_\mu^\mu(n, p, \lambda, \beta) \).

**Theorem 1.** Let the function \( f(z) \) be defined by (1.1). Then \( f(z) \) is in the class \( T_\mu^\mu(n, p, \lambda, \beta) \) if and only if

\[
\sum_{k=n+p}^{\infty} \frac{(k + \beta - p) ([k + \mu] - \lambda(k - p)] \Gamma(k + \mu) \delta(k, q)}{(k - p)!} \alpha_k \leq \beta \Gamma(p + 1 + \mu) \delta(p, q). 
\]

(2.1)

**Proof.** If the condition (2.1) holds true, we find from (1.1) and (2.1) that

\[
\begin{align*}
& \left| \lambda (D^{\mu, p}f(z))^{(q+1)} + (1 - \lambda)z(D^{1+\mu, p}f(z))^{(q+1)} - (p-q) \left[ \lambda (D^{\mu, p}f(z))^{(q)} - \right. \\
& \left. (1 - \lambda)(D^{1+\mu, p}f(z))^{(q)} \right] \right| - \beta \left| \lambda (D^{\mu}f(z))^{(q)} + (1 - \lambda)(D^{1+\mu, p}f(z))^{(q)} \right| \\
& = \sum_{k=n+p}^{\infty} \frac{(k - p) ([k + \mu] - \lambda(k - p)] \Gamma(k + \mu) \delta(k, q)}{(k - p)! \Gamma(p + 1 + \mu)} \alpha_k z^{k-p} \\
& \quad - \beta \left| \delta(p, q) - \sum_{k=n+p}^{\infty} \frac{([k + \mu] - \lambda(k - p)] \Gamma(k + \mu) \delta(k, q)}{(k - p)! \Gamma(p + 1 + \mu)} \alpha_k z^{k-p} \right|
\end{align*}
\]
\[ \sum_{k=n+p}^{\infty} \frac{(k + \beta - p) \left( (k + \mu) - \lambda(k - p) \right) \Gamma(k + \mu) \delta(k, q)}{(k - p)! \Gamma(p + 1 + \mu)} a_k - \beta \delta(p, q) \]
\[ \leq 0 \quad (z \in \partial U = \{ z : z \in C \text{ and } |z| = 1 \}). \]

Hence, by the maximum modulus theorem, \( f(z) \in T^n_p(n, p, \lambda, \beta) \).

Conversely, let \( f(z) \in T^n_p(n, p, \lambda, \beta) \) be given by \( (1.1) \). Then, from \( (1.8) \) and \( (1.10) \), we have

\[
\left| \frac{\lambda z (D^{\mu}\varphi f(z))^{(q+1)} + (1 - \lambda)z(D^{1+\mu}\varphi f(z))^{(q+1)}}{\lambda (D^{\mu}\varphi f(z))^{(q)} + (1 - \lambda)(D^{1+\mu}\varphi f(z))^{(q)}} - (p - q) \right|
= \left| - \sum_{k=n+p}^{\infty} \frac{(k - p) [(k + \mu) - \lambda(k - p)] \Gamma(k + \mu) \delta(k, q)}{(k - p)! \Gamma(p + 1 + \mu)} a_k \right| < \beta. \tag{2.2}
\]

Putting \( z = r(0 \leq r < 1) \) on the right-hand side of \( (2.2) \), and noting the fact that for \( r = 0 \), the resulting expression in the denominator is positive, and remains so for all \( r \in (0, 1) \), the desired inequality \( (2.1) \) follows upon letting \( r \to 1^- \).

**Corollary 1.** Let the function \( f(z) \in T_p(n) \) be given by \( (1.1) \). If \( f(z) \in T^n_p(n, p, \lambda, \beta) \), then

\[
a_k \leq \frac{(k - p)! \Gamma(p + 1 + \mu) \delta(p, q)}{(k + \beta - p)((k + \mu) - \lambda(k - p)) \Gamma(k + \mu) \delta(k, q)} (k \geq n + p; p, n \in N). \tag{2.3}
\]

The result is sharp for the function \( f(z) \) given by

\[
f(z) = z^p - \frac{(k - p)! \Gamma(p + 1 + \mu) \delta(p, q)}{(k + \beta - p)((k + \mu) - \lambda(k - p)) \Gamma(k + \mu) \delta(k, q)} z^k
\]

\[
(k \geq n + p; p, n \in N). \tag{2.4}
\]

We next prove the following growth and distortion property for the functions of the form \( (1.1) \) belonging to the class \( T^n_p(n, p, \lambda, \beta) \).

**Theorem 2.** If a function \( f(z) \) defined by \( (1.1) \) is in the class \( T^n_p(n, p, \lambda, \beta) \). Then

\[
||f(z)|| - |z|^p \leq \frac{n! \beta \Gamma(p + 1 + \mu) \delta(p, q)}{(n + \beta)((p + \mu) + n(1 - \lambda)) \Gamma(n + p + \mu) \delta(n + p, q)} |z|^{n+p}, \tag{2.5}
\]

and (in general),

\[
\left| |f^{(m)}(z)| - \delta(p, m) |z|^{p-m} \right| \leq \frac{n! \beta \Gamma(p + 1 + \mu) \delta(p, q)(n + p - q)!}{(n + \beta)((p + \mu) + n(1 - \lambda)) \Gamma(n + p + \mu)(n + p - m)!} |z|^{n+p-m}, \tag{2.6}
\]

\((z \in U; p, n \in N; m, q \in N_0; m \leq q < p; p > \max(m, q, -\mu)).\)
Inclusion and neighborhood properties of a certain subclasses of p-valent...  

The results are sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{n!\beta \Gamma(p+1+\mu)\delta(p,q)}{(n + \beta)((p+\mu) + n(1-\lambda))\Gamma(n + p + \mu)\delta(n + p, q)} z^{n+p}. \quad (2.7)$$

**Proof.** In view of Theorem 1, we have

$$\frac{(n + \beta)((p+\mu) + n(1-\lambda))\Gamma(n + p + \mu)\delta(n + p, q)}{n!} \sum_{k=n+p}^{\infty} a_k$$

$$\leq \sum_{k=n+p}^{\infty} \frac{(k + \beta - p)((k + \mu) - \lambda(k - p))\Gamma(k + \mu)\delta(k, q)}{(k-p)!} a_k$$

$$\leq \beta \Gamma(p+1+\mu)\delta(p,q),$$

which readily yields

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{n!\beta \Gamma(p+1+\mu)\delta(p,q)}{(n + \beta)((p+\mu) + n(1-\lambda))\Gamma(n + p + \mu)\delta(n + p, q)} \quad (2.8)$$

Also, (2.1) yields

$$\sum_{k=n+p}^{\infty} k!a_k \leq \frac{n!(n + p - q)!\beta \Gamma(p+1+\mu)\delta(p,q)}{(n + \beta)((p+\mu) + n(1-\lambda))\Gamma(n + p + \mu)} \quad (2.9)$$

Now, by differentiating both sides of (1.1) m-times, we have

$$f^{(m)}(z) = \delta(p,m) z^{p-m} - \sum_{k=n+p}^{\infty} \delta(k,m) a_k z^{k-m} \quad (p, n \in \mathbb{N}; m \in \mathbb{N}_0; p > m). \quad (2.10)$$

Theorem 2 follows from (2.8), (2.9) and (2.10).

Finally, it is easy to see that the bounds in Theorem 2 are attained for the function $f(z)$ given by (2.7).

**Theorem 3.** Let the function $f(z)$ defined by (1.1) be in the class $T^q_{p,\mu}(n, p, \lambda, \beta)$, then

(i) $f(z)$ is p-valently close-to-convex of order $\alpha$ ($0 \leq \alpha < p$) in $|z| < r_1$, where

$$r_1 = \inf_k \left\{ \left( \frac{p - \alpha}{k} \right)^{\frac{p}{k-p}} \theta(p, q, \lambda, \beta; k) \right\}^{\frac{1}{k-p}}, \quad (2.11)$$

(ii) $f(z)$ is p-valently starlike of order $\alpha$ ($0 \leq \alpha < p$) in $|z| < r_2$, where

$$r_2 = \inf_k \left\{ \left( \frac{p - \alpha}{k - \alpha} \right)^{\frac{1}{k-p}} \theta(p, q, \lambda, \beta; k) \right\}^{\frac{1}{k-p}}, \quad (2.12)$$
(iii) $f(z)$ is $p$-valently convex of order $\alpha(0 \leq \alpha < p)$ in $|z| < r_3$, where

$$ r_3 = \inf_k \left\{ \frac{p(p-\alpha)}{k(k-\alpha)} \theta(p,q,\lambda,\mu,\beta;k) \right\} \frac{1}{k-p}, $$

(2.13)

where

$$ \theta(p,q,\lambda,\mu,\beta;k) = \frac{(k+\beta-p)(k+\mu-\lambda(k-p))\Gamma(k+\mu)\delta(k,q)}{\beta(k-p)\Gamma(p+1+\mu)\delta(p,q)} $$

(2.14)

$(k \geq n + p; p, n \in N; q \in N_0; p > q; \mu > -p; 0 \leq \lambda \leq 1; 0 \leq \alpha < p; 0 < \beta \leq 1)$.

Each of these results is sharp for the function $f(z)$ given by (2.4).

**Proof.** It is sufficient to show that

$$ \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \alpha \quad (|z| < r_1; 0 \leq \alpha < p; p \in N), $$

(2.15)

$$ \left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \alpha \quad (|z| < r_2; 0 \leq \alpha < p; p \in N), $$

(2.16)

and that

$$ \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p - \alpha \quad (|z| < r_3; 0 \leq \alpha < p; p \in N), $$

(2.17)

for a function $f(z) \in T_q^n(p, \lambda, \beta)$, where $r_1, r_2$ and $r_3$ are defined by (2.11), (2.12) and (2.13), respectively. The details involved are fairly straightforward and may be omitted.

### 3 Properties of the class $H_q^n(p, \lambda, \beta; \gamma)$

Applying the results of Section 2, which were obtained for the function $f(z)$ of the form (1.1) belonging to the class $T_q^n(p, \lambda, \beta)$, we now derive the corresponding results for the function $f(z)$ belonging to the class $H_q^n(p, \lambda, \beta; \gamma)$.

**Theorem 4.** If a function $f(z)$ defined by (1.1) is in the class $H_q^n(p, \lambda, \beta; \gamma)$, then

$$ ||f(z)| - |z||^p \leq $$

$$ \frac{n!\beta\Gamma(p+1+\mu)\delta(p,q)(p-q+\gamma)(p-q+\gamma+1)}{(n+\beta)(p+\mu) + n(1-\lambda)\Gamma(n+p+\mu)\delta(n+p,q)(n+p-q+\gamma)} |z|^{n+p} $$

(3.1)

and (in general),
Inclusion and neighborhood properties of a certain subclasses of p-valent... 7

\[\left|f^{(m)}(z) - \delta(p,m) |z|^{p-m}\right| \leq \frac{n! \beta \Gamma(p+1+\mu) \delta(p,q)(p-q+\gamma)(p-q+\gamma+1)}{(n+\beta)[(p+\mu)+n(1-\lambda)] \Gamma(n+p+\mu) \delta(n+p,q)(n+p+\mu)(n+p-q+\gamma)} |z|^{n+p-m} \]

(3.2)

The results in (3.1) and (3.2) are sharp for the function \( f(z) \) given by

\[ f(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \] (b_k \geq 0; p, n \in N). \hspace{1cm} (3.4)

Then, we readily find from (1.11) that

\[ a_k = \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} b_k \] (k \geq n+p; p, n \in N), \hspace{1cm} (3.5)

so that

\[ f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k = z^p - \sum_{k=n+p}^{\infty} \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} b_k z^k, \] (3.6)

and

\[ ||f(z)| - |z|^p|| \leq |z|^{n+p} \sum_{k=n+p}^{\infty} \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} b_k \] (z \in U). \hspace{1cm} (3.7)

Next, since \( g(z) \in T_p^n(n,p,\lambda,\beta) \), therefore, on using the assertion (2.8) of Theorem 2, we get the following coefficient inequality:

\[ b_k \leq \frac{n! \beta \Gamma(p+1+\mu) \delta(p,q)}{(n+\beta)[(p+\mu)+n(1-\lambda)] \Gamma(n+p+\mu) \delta(n+p,q)(n+p+\mu)} \] (k \geq n+p; p, n \in N), \hspace{1cm} (3.8)

which in conjunction with (3.6) and (3.7) yield

\[ ||f(z)| - |z|^p|| \leq \frac{n! \beta \Gamma(p+1+\mu) \delta(p,q)}{(n+\beta)[(p+\mu)+n(1-\lambda)] \Gamma(n+p+\mu) \delta(n+p,q)} |z|^{n+p}. \]
\[
\sum_{k=n+p}^{\infty} \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} (z \in U). \tag{3.9}
\]

By noting the following summation result:
\[
\sum_{k=n+p}^{\infty} \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} = \frac{(p-q+\gamma)(p-q+\gamma+1)}{(n+p-q+\gamma)}, \tag{3.10}
\]

where \( \gamma \in R^* = R\{ -n-p, -n-p-1, ... \} \). The assertion (3.1) of Theorem 4 follows from (3.9) and (3.10). The assertion (3.2) of Theorem 4 can be established by similarly applying (2.9), (3.5) and (3.10).

**Theorem 5.** Let the function \( f(z) \) defined by (1.1) be in the class \( H_\mu^p(n, p, \lambda; \beta; \gamma) \), then \( f(z) \) is \( p \)-valently close-to-convex of order \( \delta (0 \leq \delta < p) \) in \( |z| < r_4 \), where
\[
r_4 = \inf_k \left\{ \theta(p, q, \lambda, \mu, \beta; k) \frac{(p-\delta)(k-q+\gamma)(k-q+\gamma+1)}{k(p-q+\gamma)(p-q+\gamma+1)} \right\} \frac{1}{k-p}, \tag{3.11}
\]

\( (k \geq n+p; p, n \in N; q \in N_0; p > q; \mu > -p; 0 \leq \lambda \leq 1; 0 \leq \delta < p; 0 < \beta \leq 1; \gamma \in R^* \),

where \( \theta(p, q, \lambda, \mu, \beta; k) \) is given by (3.3). The result is sharp for the function \( f(z) \) given by (3.3).

**Proof.** Assume that \( f(z) \in T_\mu^p(n) \) is given by (1.1). Also, let the function \( g(z) \in T_\mu^p(n, p, \lambda, \beta) \), occurring in the non-homogenous differential equation (1.11), be given by (3.4). Then, it sufficient to show that
\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| < p - \delta \quad (|z| < r_4; 0 \leq \delta < p; p \in N).
\]

Indeed, we have
\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=n+p}^{\infty} k a_k |z|^{k-p},
\]

and by using the coefficient relation (3.5) between the functions \( f(z) \) and \( g(z) \), we get
\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=n+p}^{\infty} \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} k b_k |z|^{k-p} \leq p - \delta. \tag{3.12}
\]

Since \( g(z) \in T_\mu^p(n, p, \lambda, \beta) \), and we know from the assertion (2.1) of Theorem 1 that
\[
\sum_{k=n+p}^{\infty} \frac{(k+\beta-p)((k+\mu)-\lambda(k-p))\Gamma(k+\mu)\delta(k,q)}{(k-p)!} b_k \leq \beta \Gamma(p+1+\mu)\delta(p,q),
\]
hence, (3.11) is true if
\[
\left(\frac{k}{p-\delta}\right) \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} |z|^{k-p} \leq \theta(p, q, \lambda, \mu, \beta; k) \quad (k \geq n+p; p, n \in N),
\]
(3.13)
where \( \theta(p, q, \lambda, \mu, \beta; k) \) is given by (2.14). Solving (3.12) for \(|z|\), we obtain
\[
|z| \leq \left\{ \theta(p, q, \lambda, \mu, \beta; k) \frac{(p-\delta)(k-q+\gamma)(k-q+\gamma+1)}{(k-\delta)(p-q+\gamma)(p-q+\gamma+1)} \right\}^{\frac{1}{p-\delta}} \quad (k \geq n+p; p, n \in N)
\]
which obviously proves Theorem 5.

**Remark 1.** We note that the result obtained by Irmak et al. [10, Theorem 2.3] is not correct. The correct result is given by (3.11) with \( p = 1 \) and \( q = 0 \).

**Theorem 6.** Let the function \( f(z) \) defined by (1.1) be in the class \( H^q_\mu(n, p, \lambda, \beta; \gamma) \), then \( f(z) \) is \( p \)-valently starlike of order \( \delta(0 \leq \delta < p) \) in \(|z| < r_5\), where
\[
r_5 = \inf_k \left\{ \theta(p, q, \lambda, \mu, \beta; k) \frac{(p-\delta)(k-q+\gamma)(k-q+\gamma+1)}{(k-\delta)(p-q+\gamma)(p-q+\gamma+1)} \right\}^{\frac{1}{p-\delta}}
\]
(3.14)
where \( \theta(p, q, \lambda, \mu, \beta; k) \) is given by (2.14). The result is sharp for the function \( f(z) \) given by (3.3).

**Theorem 7.** Let the function \( f(z) \) defined by (1.1) be in the class \( H^q_\mu(n, p, \lambda, \beta; \gamma) \), then \( f(z) \) is \( p \)-valently convex of order \( \delta(0 \leq \delta < p) \) in \(|z| < r_6\), where
\[
r_6 = \inf_k \left\{ \theta(p, q, \lambda, \mu, \beta; k) \frac{p(p-\delta)(k-q+\gamma)(k-q+\gamma+1)}{k(k-\delta)(p-q+\gamma)(p-q+\gamma+1)} \right\}^{\frac{1}{p-\delta}}
\]
(3.15)
where \( \theta(p, q, \lambda, \mu, \beta; k) \) is given by (2.14). The result is sharp for the function \( f(z) \) given by (3.3).

**Remark 2.** We note that the results obtained by Irmak et al. [10, Theorems 3.3 and 3.4] are not correct. The correct results are given by (3.14) and (3.15), respectively, with \( p = 1 \) and \( q = 0 \).

### 4 Inclusion relations involving \((n, \theta)-neighborhood for the class \( T^q_\mu(n, p, \lambda, \beta)\))

Following the works of Goodman[11], Ruscheweyh [15] and Altintas [2] (see also [5], [6], [9], and [13]) we define the \((n, \theta)-neighborhood of a function \( f^{(q)}(z) \) when \( f \in T^q_\mu(n) \) by
\[
N^\theta_{n,p}(f^{(q)}, g^{(q)}) =
\]
\[ \left\{ g \in T_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} \delta(k, q) k |a_k - b_k| \leq \theta \right\}. \] (4.1)

It follows from (4.1) that, if
\[ h(z) = z^p \quad (p \in N), \] (4.2)
then
\[ N_{n, p}^\theta(h) = \left\{ g \in T_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} \delta(k, q) k |b_k| \leq \theta \right\}. \] (4.3)

Next, we establish inclusion relationships for the function class \( T_{\mu}^q(n, p, \lambda, \beta) \) involving the \((n, \theta)\)-neighborhood \( N_{n, p}^\theta(h) \) defined by (4.3).

**Theorem 8.** If
\[ \theta = \frac{\beta \Gamma(p + 1 + \mu) \delta(p, q)n!}{[p + \mu + n(1 - \lambda)] \Gamma(n + p + \mu)} \left( \frac{n + p}{n + \beta} \right), \] (4.4)
then
\[ T_{\mu}^q(n, p, \lambda, \beta) \subset N_{n, p}^\theta(h). \] (4.5)

**Proof.** Let \( f \in T_{\mu}^q(n, p, \lambda, \beta) \). Then, in view of the assertion (2.1) of Theorem 1, we have
\[ \frac{(n + \beta)[p + \mu + n(1 - \lambda)] \Gamma(n + p + \mu)}{n!} \sum_{k=n+p}^{\infty} \delta(k, q) a_k \leq \beta \Gamma(p + 1 + \mu) \delta(p, q) \] (4.6)
so that
\[ \sum_{k=n+p}^{\infty} \delta(k, q) a_k \leq \frac{\beta \Gamma(p + 1 + \mu) \delta(p, q)n!}{(n + \beta)[p + \mu + n(1 - \lambda)] \Gamma(n + p + \mu)}. \] (4.7)

On the other hand, we also find from (2.1) and (4.7) that
\[ \frac{[p + \mu + n(1 - \lambda)] \Gamma(n + p + \mu)}{n!} \sum_{k=n+p}^{\infty} \delta(k, q) k a_k \leq \beta \Gamma(p + 1 + \mu) \delta(p, q) + \]
\[ \frac{(p - \beta)(p + \mu + n)] \Gamma(n + p + \mu)}{n!} \sum_{k=n+p}^{\infty} \delta(k, q) a_k \leq \beta \Gamma(p + 1 + \mu) \delta(p, q) + \]
\[ (p - \beta) \frac{[p + \mu + n(1 - \lambda)] \Gamma(n + p + \mu)}{n!} \frac{\beta \Gamma(p + 1 + \mu) \delta(p, q)n!}{(n + \beta)[p + \mu + n(1 - \lambda)] \Gamma(n + p + \mu)} \]
\[ = \beta \Gamma(p + 1 + \mu) \delta(p, q) \left( \frac{n + p}{n + \beta} \right). \]
that is
\[
\sum_{k=n+p}^{\infty} \delta(k,q)k\alpha_k \leq \beta \frac{\Gamma(p+1+\mu)\delta(p,q)n!}{[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)} \left(\frac{n+p}{n+\beta}\right) = \theta. \tag{4.8}
\]

**Remark 3.** Putting \(q = 0\) and \(p = 1\) in Theorem 8, we obtain the following corollary.

**Corollary 2.** If
\[
\theta = \frac{\beta\Gamma(2+\mu)n!}{[1+\mu+n(1-\lambda)]\Gamma(n+1+\mu)} \left(\frac{n+1}{n+\beta}\right), \tag{4.9}
\]
then
\[
T_{\mu}(n, \lambda, \beta) \subset N_n^\theta(h). \tag{5.4}
\]

### 5 Neighborhood for the class \(T_{\mu}^{q,\alpha}(n, p, \lambda, \beta)\)

In this section we determine the neighborhood for the class \(T_{\mu}^{q,\alpha}(n, p, \lambda, \beta)\) which we define as follows. A function \(f \in T_{\mu}^p(n)\) is said to be in the class \(T_{\mu}^{q,\alpha}(n, p, \lambda, \beta)\) if there exist a function \(g \in T_{\mu}^q(n, p, \lambda, \beta)\) such that
\[
\left|\frac{f(z)}{g(z)} - 1\right| < p - \alpha \quad (z \in U; \ 0 \leq \alpha < p). \tag{5.1}
\]

**Theorem 9.** If \(g \in T_{\mu}^q(n, p, \lambda, \beta)\) and
\[
\alpha = p - \frac{\theta(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)}{(n+p)[[n+\beta]\Gamma(n+\mu)]\Gamma(n+\mu+\delta(n+p+\mu))} - \beta\Gamma(p+1+\mu)\delta(p,q)n!\left[\frac{\delta(n+p, q) - \beta\Gamma(p+1+\mu)\delta(p,q)n!}{(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)}\right]^{-1}, \tag{5.2}
\]
where
\[
\theta \leq p(n+p) \times \left\{\delta(n+p, q) - \beta\Gamma(p+1+\mu)\delta(p,q)n!\left[\frac{\delta(n+p, q) - \beta\Gamma(p+1+\mu)\delta(p,q)n!}{(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)}\right]^{-1}\right\},
\]
then
\[
N_{n,p}^\theta(g) \subset T_{\mu}^{q,\alpha}(n, p, \lambda, \beta). \tag{5.4}
\]

**Proof.** Suppose that \(f \in N_{n,p}^\theta(g)\), then we find from the definition (4.1) that
\[
\sum_{k=n+p}^{\infty} \delta(k,q)k|a_k - b_k| \leq \theta, \tag{5.5}
\]
which implies the coefficient inequality
\[
\sum_{k=n+p}^{\infty} |a_k - b_k| \leq \frac{\theta}{(n+p)\delta(n+p,q)} \quad (p > q, n, p \in N, q \in N_0). \tag{5.6}
\]
Next, since \( g \in T_\mu^\alpha(n, p, \lambda, \beta) \), we have

\[
\sum_{k=n+p}^\infty b_k \leq \frac{\beta \Gamma(p + 1 + \mu) \delta(p, q) n!}{(n + \beta)[p + \mu + n(1 - \lambda)] \Gamma(n + p + \mu) \delta(n + p, q)},
\]

so that

\[
\left| \frac{f(z)}{g(z)} - 1 \right| \leq \frac{\sum_{k=n+p}^\infty |a_k - b_k|}{1 - \sum_{k=n+p}^\infty |b_k|} \leq \frac{\theta(n + \beta)|p + \mu + n(1 - \lambda)| \Gamma(n + p + \mu)}{(n + \beta)[p + \mu + n(1 - \lambda)] \Gamma(n + p + \mu) \delta(n + p, q) - \beta \Gamma(p + 1 + \mu) \delta(p, q) n!} \leq p - \alpha,
\]

where \( \alpha \) given by (5.2). This implies that \( f \in T_\mu^\alpha(n, p, \lambda, \beta) \).

**Remark 4.** Putting \( q = 0 \) and \( p = 1 \) in Theorem 9, we obtain the following corollary.

**Corollary 3.** If \( g \in T_\mu(n, \lambda, \beta) \), and

\[
\alpha = 1 - \frac{\theta(n + \beta)[1 + \mu + n(1 - \lambda)] \Gamma(n + 1 + \mu)}{(n + 1)((n + \beta)[1 + \mu + n(1 - \lambda)] \Gamma(n + 1 + \mu) - \beta \Gamma(2 + \mu) n!)},
\]

where

\[
\theta \leq (n + 1)\{1 - \beta \Gamma(2 + \mu) n![(n + \beta)[1 + \mu + A(1 - \lambda)] \Gamma(n + 1 + \mu)]^{-1}\}
\]

then

\[
N^\alpha_\mu(g) \subset T_\mu^\alpha(n, \lambda, \beta).
\]

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**References**


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