ON EXTENDABILITY OF CAYLEY GRAPHS

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Abstract

A connected graph $\Gamma$ of even order is $n$-extendable, if it contains a matching of size $n$ and if every such matching is contained in a perfect matching of $\Gamma$. Furthermore, a connected graph $\Gamma$ of odd order is $n_{\frac{1}{2}}$-extendable, if for every vertex $v$ of $\Gamma$ the graph $\Gamma - v$ is $n$-extendable.

It is proved that every connected Cayley graph of an abelian group of odd order which is not a cycle is $1_{\frac{1}{2}}$-extendable. This result is then used to classify 2-extendable connected Cayley graphs of generalized dihedral groups.

1 Introductory remarks

Throughout this paper graphs are assumed to be finite and simple. A connected graph $\Gamma$ of even order is $n$-extendable, if it contains a matching of size $n$ and if every such matching is contained in a perfect matching of $\Gamma$. The concept of $n$-extendable graphs was introduced by Plummer [8] in 1980. Since then a number of papers on this topic have appeared (see [2, 10, 11, 12] and the references therein). In 1993 Yu [11] introduced an analogous concept for graphs of odd order. A connected graph $\Gamma$ of odd order is $n_{\frac{1}{2}}$-extendable, if for every vertex $v$ of $\Gamma$ the graph $\Gamma - v$ is $n$-extendable.

The problem of $n$-extendability of Cayley graphs was first considered in [3] where a classification of 2-extendable Cayley graphs of dihedral groups was obtained. (For a definition of a Cayley graph see Section 2.) A few years later a classification of 2-extendable Cayley graphs of abelian groups was obtained in [2]. In this paper we generalize these results in two different ways. First, we consider $n_{\frac{1}{2}}$-extendability for Cayley graphs of abelian groups of odd order. In particular, we prove the following theorem.

Theorem 1 Let $\Gamma$ be a connected Cayley graph on an abelian group of odd order $n \geq 3$. Then either $\Gamma$ is a cycle, or $\Gamma$ is $1_{\frac{1}{2}}$-extendable.
Second, using Theorem 1 we generalize the result of [3] to generalized dihedral groups as follows.

**Theorem 2** Let $\Gamma$ be a connected Cayley graph on a generalized dihedral group which is not a cycle. Then $\Gamma$ is 2-extendable unless it is isomorphic to one of the following Cayley graphs on cyclic groups, also called circulants: $\text{Circ}(2n; \{\pm 1, \pm 2\}) (n \geq 3)$, $\text{Circ}(4n; \{\pm 1, 2n\}) (n \geq 2)$, $\text{Circ}(4n + 2; \{\pm 2, 2n + 1\})$ and $\text{Circ}(4n + 2; \{\pm 1, \pm 2n\})$.

### 2 Preliminaries

In this section we introduce the notation and some results needed in the rest of the paper.

A *Cayley graph* $\text{Cay}(G; S)$ of a group $G$ with respect to the connection set $S \subseteq G \setminus \{1\}$, where $S^{-1} = S$, is a graph with vertex-set $G$ in which $g \sim gs$ for all $g \in G$, $s \in S$. In the case that $G = \mathbb{Z}_n$ the graph $\text{Cay}(G; S)$ is called a circulant and is denoted by $\text{Circ}(n; S)$. Let $M$ be a subset of edges of $\text{Cay}(G; S)$ and let $g \in G$. Then $Mg$ denotes the set of all edges of the form $\{ug, vg\}$, where $\{u, v\} \in M$.

A *Hamilton path* of a graph is a path visiting all of its vertices. The question of existence of Hamilton paths in vertex-transitive graphs and in particular Cayley graphs has been extensively studied over the last forty years (see for instance [1, 5, 6, 7] and the references therein). The following result on this topic is of particular interest to us.

**Proposition 3** [4] Let $\Gamma$ be a connected Cayley graph of an abelian group and of valency at least three. If $\Gamma$ is not bipartite then for any pair of its vertices $u$ and $v$ there exists a Hamilton path of $\Gamma$ from $u$ to $v$. If $\Gamma$ is bipartite then for any pair of vertices $u$ and $v$ from different parts of bipartition of $\Gamma$ there exists a Hamilton path of $\Gamma$ from $u$ to $v$.

Note that it follows from this proposition that every connected Cayley graph of an abelian group is 1-extendable if the order of the group is even and is 02-extendable otherwise. However, as the following proposition (which will be used in the proofs of our main results) shows, not all Cayley graphs on abelian groups of even order are 2-extendable.

**Proposition 4** [2] Let $\Gamma$ be a connected Cayley graph of an abelian group of even order and valency at least three. Then $\Gamma$ is 2-extendable if and only if it is not isomorphic to any of $\text{Circ}(2n; \{\pm 1, \pm 2\}) (n \geq 3)$, $\text{Circ}(4n; \{\pm 1, 2n\}) (n \geq 2)$, $\text{Circ}(4n + 2; \{\pm 2, 2n + 1\})$ and $\text{Circ}(4n + 2; \{\pm 1, \pm 2n\})$.

We remark that none of the exceptional graphs from Proposition 4 is bipartite. This fact will be used in the proof of Theorem 2.
3 Cayley graphs of abelian groups

In this section we prove Theorem 1. To do this, we first need the following result.

Proposition 5 Let $G$ be an abelian group of odd order with identity $1$, and let $S \subseteq G \setminus \{1\}$ be a nonempty set such that $S = S^{-1}$. Then $\Gamma = \text{Cay}(\langle S \cup \{g, g^{-1}\}; S \cup \{g, g^{-1}\} \rangle)$ is $1_{\frac{1}{2}}$-extendable for every $g \in G \setminus \langle S \rangle$.

Proof. Recall first that $\text{Cay}(\langle S \rangle; S)$ is $0_{\frac{1}{2}}$-extendable by the comment following Proposition 3. Let $m$ be the smallest positive integer such that $g^m \in \langle S \rangle$. Note that the subgraphs of $\Gamma$ induced on cosets $g^i \langle S \rangle$, $i \in \{0, 1, \ldots, m-1\}$ are all isomorphic to $\text{Cay}(\langle S \rangle; S)$. Furthermore, for every $i \in \{0, 1, \ldots, m-1\}$ and $s \in \langle S \rangle$, the vertex $g^i s$ of $\Gamma$ is adjacent to the vertex $g^{i+1} s$. Finally, since the order of $G$ is odd, both $|\langle S \rangle|$ and $m$ are also odd and $m \geq 3$.

Pick an edge $e = \{g^i s_1, g^j s_2\}$, $i, j \in \{0, 1, \ldots, m-1\}$, $s_1, s_2 \in \langle S \rangle$, and a vertex $x$ of $\Gamma$. We show that there exists a perfect matching of $\Gamma - x$ containing $e$. Since $\Gamma$ is vertex-transitive, we can assume that $x = 1$. The proof is split into four cases depending on the numbers $i$ and $j$. Note that we can assume $i \leq j$. Observe also that if $i \neq j$, then $s_1 = s_2$ and either $j - i = 1$ or $i = 0$, $j = m - 1$.

Case 1: $i = j = 0$. Since the subgraph of $\Gamma$ induced on the coset $g^2 \langle S \rangle$ is isomorphic to $\text{Cay}(\langle S \rangle; S)$, which is $0_{\frac{1}{2}}$-extendable, there exists an almost perfect matching $M$ of this subgraph missing the vertex $g^2$. But then

$$\{\{s_1, s_2\}, \{gs_1, gs_2\}, \{g, g^2\}\} \cup \{\{s, gs\} : s \in \langle S \rangle \setminus \{1, s_1, s_2\}\} \cup M \cup \{\{g^k s, g^{k+1} s\} : k \in \{3, 5, \ldots, m-2\}, s \in \langle S \rangle\}$$

is an almost perfect matching of $\Gamma$ missing 1 and containing $e$.

Case 2: $i = j \neq 0$. Since $\text{Cay}(\langle S \rangle; S)$ is $0_{\frac{1}{2}}$-extendable, there exists an almost perfect matching $M$ of $\text{Cay}(\langle S \rangle; S)$ missing 1. If $i$ is odd, then

$$M \cup \{\{g^k s, g^{k+i} s\} : k \in \{1, 3, \ldots, i-2, i+2, \ldots, m-2\}, s \in \langle S \rangle\} \cup \{\{g^i s_1, g^j s_2\} \cup \{g^{i+1} s_1, g^{i+1} s_2\}\}$$

is an almost perfect matching of $\Gamma$ missing 1 and containing $e$. If $i$ is even, then

$$M \cup \{\{g^k s, g^{k+i} s\} : k \in \{1, 3, \ldots, i-3, i+1, \ldots, m-2\}, s \in \langle S \rangle\} \cup \{\{g^{i-1} s_1, g^i s_2\} \cup \{g^{i-1} s_1, g^{i-1} s_2\}\}$$

is an almost perfect matching of $\Gamma$ missing 1 and containing $e$.

Case 3: $i = j - 1 \neq 0$. Recall that in this case $s_1 = s_2$. Since $\text{Cay}(\langle S \rangle; S)$ is $0_{\frac{1}{2}}$-extendable, there exists an almost perfect matching $M$ of $\text{Cay}(\langle S \rangle; S)$ missing 1. If $i$ is odd, then

$$M \cup \{\{g^k s, g^{k+i} s\} : k \in \{1, 3, \ldots, m-2\}, s \in \langle S \rangle\}$$

is an almost perfect matching of $\Gamma$ missing 1 and containing $e$.

Case 4: $i = j - 2 \neq 0$. Recall that in this case $s_1 = s_2$. Since $\text{Cay}(\langle S \rangle; S)$ is $0_{\frac{1}{2}}$-extendable, there exists an almost perfect matching $M$ of $\text{Cay}(\langle S \rangle; S)$ missing 1. If $i$ is odd, then

$$M \cup \{\{g^k s, g^{k+i} s\} : k \in \{1, 3, \ldots, m-2\}, s \in \langle S \rangle\}$$

is an almost perfect matching of $\Gamma$ missing 1 and containing $e$. Observe also that if $i \neq j$, then $s_1 = s_2$ and either $j - i = 1$ or $i = 0$, $j = m - 1$.

Finally, since the order of $G$ is odd, both $|\langle S \rangle|$ and $m$ are also odd and $m \geq 3$.
is an almost perfect matching of $\Gamma$ missing 1 and containing $e$.

Assume now that $i$ is even. Pick an edge $\{s, s'\}$ of $M$. Since subgraphs of $\Gamma$ induced on the cosets $g(S)$ and $g^{m-1}(S)$ are $0\frac{1}{2}$-extendable, there exist almost perfect matchings $M_1$ and $M_{m-1}$ of these subgraphs, which miss vertices $gs$ and $g^{-1}s'$, respectively. But now

$$\left( M \setminus \{s, s'\} \right) \cup \{s, gs\} \cup \{s', g^{-1}s'\} \cup M_1 \cup M_{m-1} \cup \{g^k s, g^{k+1} s\} : k \in \{2, 4, \ldots, m-3\}, s \in \langle S \rangle$$

is an almost perfect matching of $\Gamma$ missing 1 and containing $e$.

**Case 4:** $j = 0, j \in \{1, m-1\}$. Without loss of generality we can assume $j = 1$ (otherwise replace $g$ by $g^{-1}$). Since a subgraph of $\Gamma$ induced on the coset $g^2(S)$ is isomorphic to $Cay(G; S)$, there exist an almost perfect matching $M$ of this subgraph missing the vertex $g^2$. But then

$$\{s, gs : s \in \langle S \rangle \setminus \{0\} \cup \{g, g^2\} \cup M \cup \{g^k s, g^{k+1} s\} : k \in \{3, 5, \ldots, m-2\}, s \in \langle S \rangle \}$$

is an almost perfect matching of $\Gamma$ missing 1 and containing $e$.

**Proof.** [Of Theorem 1] Assume that $\Gamma = Cay(G; S)$ is not a cycle and note that this implies $|S| \geq 4$. We show that $\Gamma$ is $1\frac{1}{2}$-extendable using induction on $|S|$.

Suppose first that $|S| = 4$. If for some $s \in S$ we have that $\langle s \rangle \neq G$, then $Cay(G; S)$ is $1\frac{1}{2}$-extendable by Proposition 5. We are left with the possibility that $S = \{s, s^{-1}, t, t^{-1}\}$ where $\langle s \rangle = \langle t \rangle = G$. Pick a vertex $x$ and an edge $e$ of $Cay(G; S)$. Let $n$ denote the order of $G$. Without loss of generality we can assume that $x = 1$, that $s = t^\ell$ for some $\ell \in \{2, 3, \ldots, n-2\}$, and that $e = \{t^i, t^j s\}$ for some $i \in \{1, 2, \ldots, n - \ell - 1, n - \ell + 1, \ldots, n-1\}$. We now construct an almost perfect matching $M$ of $\Gamma$ containing $e$ and missing $x$ depending on the parity of $i$ and $\ell$.

If $i$ and $\ell$ are both odd, then

$$M = \{e\} \cup \{\{t^j, t^{j+1}\} : j \in J\},$$

where $J = \{1, 3, \ldots, i - 2, i + 1, i + 3, \ldots, i + \ell - 2, i + \ell + 1, i + \ell + 3, \ldots, n - 2\}$. If $i$ is odd and $\ell$ is even, then

$$M = \{e, \{t^{i+1}, t^{i+\ell+1}\}\} \cup \{\{t^j, t^{j+1}\} : j \in J\},$$

where $J = \{1, 3, \ldots, i - 2, i + 2, i + 4, \ldots, i + \ell - 2, i + \ell + 2, i + \ell + 4, \ldots, n - 2\}$. If $i$ and $\ell$ are both even, then

$$M = \{e, \{t^{i-1}, t^{i+\ell-1}\}\} \cup \{\{t^j, t^{j+1}\} : j \in J\},$$

where $J = \{1, 3, \ldots, i - 3, i + 1, i + 3, \ldots, i + \ell - 3, i + \ell + 1, i + \ell + 3, \ldots, n - 2\}$. Finally, if $i$ is even and $\ell$ is odd, then

$$M = \{e, \{t^{i-1}, t^{i+\ell-1}\}, \{t^{i+1}, t^{i+\ell+1}\}\} \cup \{\{t^j, t^{j+1}\} : j \in J\},$$

where $J = \{1, 3, \ldots, i - 3, i + 2, i + 4, \ldots, i + \ell - 3, i + \ell + 2, i + \ell + 4, \ldots, n - 2\}$.
Now suppose $|S| \geq 6$ and pick a vertex $x$ and an edge $e = \{u, us\}$, $s \in S$, of $Cay(G; S)$. We will show that there exists an almost perfect matching of $Cay(G; S)$ which contains $e$ and misses $x$. Let $t \in S \setminus \{s, s^{-1}\}$, let $S' = S \setminus \{t, t^{-1}\}$ and consider the subgraph $\Gamma' = Cay(G'; S')$, which, by induction, is $1\frac{1}{2}$-extendable. If $\langle S' \rangle = G$, then an almost perfect matching of $\Gamma'$, containing $e$ and missing $x$, is also an almost perfect matching of $\Gamma$ containing $e$ and missing $x$. If however $\langle S' \rangle \neq G$, then $\Gamma$ is $1\frac{1}{2}$-extendable by Proposition 5.

4 Cayley graphs of generalized dihedral groups

A group $G$ containing an abelian subgroup $H$ of index 2 and an involution $t \notin H$ such that $tht = h^{-1}$ for each $h \in H$ is called a generalized dihedral group. In this case we denote $G$ by $D_H$. Observe that if $\Gamma = Cay(D_H; S)$ is a Cayley graph of a generalized dihedral group $D_H$ and $h, t \in H$ then for any vertex $x$ of $\Gamma$, $(x, xh, xh^{-1}t', xh')$ is a 4-cycle of $\Gamma$. Note also that for each $ta \in S$ and for each subgroup $H' \leq H$ the edges corresponding to $ta$ introduce perfect matchings between components of the subgraph $Cay(D_H; S \cap H')$.

Proof. [Of Theorem 2] Let $\Gamma = Cay(D_H; S)$ and let $S_1 = H \cap S$ and $S_2 = S \setminus S_1$. Let $\Gamma_1$ be the subgraph of $\Gamma$ induced by $H$ on $S_2$ and let $\Gamma_2$ be the subgraph of $\Gamma$ induced on $tH$. Furthermore pick any two disjoint edges $e_1$ and $e_2$ of $\Gamma$. We distinguish four cases depending on whether the edges $e_i$ belong to $\Gamma_1$ or $\Gamma_2$ or neither of them.

Case 1: $e_1 \in \Gamma_1$ and $e_2 \notin \Gamma_1 \cup \Gamma_2$. (The case $e_1 \in \Gamma_2$, $e_2 \notin \Gamma_1 \cup \Gamma_2$ is done analogously.)

Let $ta \in S_2$ be the unique element such that $e_2 = \{x, tx^{-1}a\}$ for some $x \in H$ and let $h \in S_1$ be such that $e_1 = \{y, yh\}$ for some $y \in H$. Then a perfect matching of $\Gamma$ containing $e_1$ and $e_2$ is

$\{e_1, e_2\} \cup \{\{z, tz^{-1}a\} : z \in H \setminus \{y, yh, x\} \} \cup \{\{ty^{-1}a, ty^{-1}h^{-1}a\}\}$.

Case 2: $e_1, e_2 \in \Gamma_1$. (The case $e_1, e_2 \in \Gamma_2$ is done analogously.)

Since $\Gamma$ is connected, $S_2$ is nonempty. With no loss of generality we can assume $t \in S_2$. Letting $h, h' \in S_1$ be such that $e_1 = \{x, xh\}$ and $e_2 = \{y, yh'\}$ for some $x, y \in H$ a perfect matching of $\Gamma$ containing $e_1$ and $e_2$ is

$\{e_1, e_2\} \cup \{\{z, tz^{-1}\} : z \in H \setminus \{x, xh, yh'\} \} \cup \{\{tx^{-1}, xh^{-1}t\}, \{ty^{-1}, ty^{-1}h^{-1}t\}\}$.

Case 3: $e_1 \in \Gamma_1$, $e_2 \in \Gamma_2$.

If $H' = \langle S_1 \rangle$ is of even order, then each of the $[H : H']$ components of $\Gamma_1$ (and $\Gamma_2$), and thus $\Gamma_1$ (and $\Gamma_2$) itself, is 1-extendable by the remark following Proposition 3. Thus in this case $\Gamma$ clearly contains a desired perfect matching. We can therefore assume that $H'$ is of odd order. Moreover, we can also assume that $e_1 = \{1, h\}$ for some $h \in S_1$. Let $x, xh' \in H$ be such that $e_2 = \{tx, txh'\}$. If there exists an element
that each contradict a subset $X$

With no loss of generality we can assume that $x^{-1}a, x^{-1}h^{-1}a \in \Gamma$ and so a perfect matching of $\Gamma$ containing $e_1$ and $e_2$ is

$$\{e_1, e_2\} \cup \{\{z, tz^{-1}a\} : z \in H \setminus \{1, h, x^{-1}a, x^{-1}h^{-1}a\}\} \cup \{\{ta, th^{-1}a\}, \{x^{-1}a, x^{-1}h^{-1}a\}\}.$$  

Similarly, if for some $ta \in S_2$ we have that $e_2 = \{ta, th^{-1}a\}$, then a desired perfect matching of $\Gamma$ is

$$\{e_1, e_2\} \cup \{\{z, tz^{-1}a\} : z \in H \setminus \{1, h\}\}.$$  

We are left with the possibility that for each $ta \in S_2$ we have $\{ta, th^{-1}a\} \cap e_2 = \emptyset$.

In view of the connectedness of $\Gamma$ this implies $H' = H$. Suppose first that $|S_1| > 2$. Then $|H| > 4$, and so there exists an edge $e = \{y, ty^{-1}a\}$ such that $e \cap (e_1 \cup e_2) = \emptyset$.

By Theorem 1 both $\Gamma_1$ and $\Gamma_2$ are 12-extendable, and so a desired perfect matching of $\Gamma$ clearly exists. Suppose now that $S_1 = \{h, h^{-1}\}$. In this case each of $\Gamma_1$ and $\Gamma_2$ is isomorphic to a cycle of odd length, say $2n + 1$. Using the remarks from the beginning of this section it is easy to see that the above assumptions imply $|S_2| \leq 2$ and $\Gamma \cong \text{Cay}(\mathbb{Z}_{2n+1} \times \mathbb{Z}_2; \{(\pm 1, 0), (0, 1)\}) \cong \text{Circ}(4n + 2; \{(\pm 2, 2n + 1)\})$ in the case of $|S_2| = 1$, and $\Gamma \cong \text{Cay}(\mathbb{Z}_{2n+1} \times \mathbb{Z}_2; \{(\pm 1, 0), (\pm 1, 1)\}) \cong \text{Circ}(4n + 2; \{(\pm 1, \pm 2n)\})$ in the case of $|S_2| = 2$. Hence, in either case $\Gamma$ is a Cayley graph of an abelian group, so that Proposition 4 applies.

**Case 4:** $e_1, e_2 \notin \Gamma_1 \cup \Gamma_2$.

With no loss of generality we can assume that $e_1 = \{1, t\}$ and $e_2 = \{x, tx^{-1}a\}$ for some $x, a \in H$. If $a = 1$ then a perfect matching of $\Gamma$ containing $e_1$ and $e_2$ is $$\{\{z, tz^{-1}\} : z \in H\}.$$ We can thus assume $a \neq 1$ (implying that $|S_2| \geq 2$). We distinguish two subcases depending on whether $|S_2| = 2$ or not.

**Subcase 4.1:** $|S_2| \geq 3$.

We show that in this case a desired perfect matching of $\Gamma$ can be constructed using just some of the edges corresponding to elements of $S_2$. Now, if $x \notin \langle a \rangle$ then a desired perfect matching of $\Gamma$ is given by

$$\{\{z, tz^{-1}\} : z \in H \setminus \langle a \rangle\} \cup \{\{z, tz^{-1}a\} : z \in \langle a \rangle\}.$$  

so that we can assume $x \in \langle a \rangle$. Let $tb \in S_2 \setminus \{t, ta\}$, let $H' = \langle a, b \rangle \leq H$ and consider the subgraph $\Gamma'$ of $\Gamma$ induced on $H' \cup H't$ by the edges corresponding to $t, ta$ and $tb$.

Note that it suffices to prove that $\Gamma'$ is 2-extendable. To prove this we use a result of [9] that a bipartite graph with bipartition $A \cup B$, where $|A| = |B|$, is 2-extendable if and only if for each subset $X \subset A$ with $|X| \leq |A| - 2$ we have that $|N(X)| \geq |X| + 2$ (here $N(X)$ denotes the set of neighbours of vertices from $X$). Suppose there exists a subset $X$ of $H'$ of cardinality at most $|H'| - 2$ for which $|N(X)| \leq |X| + 1$. Since $tx^{-1} \in N(X)$ for each $x \in X$, there cannot exist distinct $x_1, x_2 \in X$ with $x_1a^{-1}, x_2a^{-1} \notin X$ (in this case $\{tx^{-1} : x \in X\} \cup \{tx_1^{-1}a, tx_2^{-1}a\} \subseteq N(X)$ would contradict $|N(X)| \leq |X| + 1$). Hence, except possibly with one exception, for each $x \in X$ we have that $xa^{-1} \in X$. Similarly, except possibly with one exception, for each $x \in X$ we have that $xb^{-1} \in X$. It is easy to see that these two conditions imply that $|X| \geq |H'|-1$, a contradiction, showing that $\Gamma'$ and thus $\Gamma$ is 2-extendable.
Suppose first that \( x \notin \langle S_1 \rangle \) and \( tx^{-1}a \notin t\langle S_1 \rangle \). Then a desired perfect matching of \( \Gamma \) is obtained by taking \( \{z, tz^{-1} \} : z \in \langle S_1 \rangle \} \cup \{xz, txz^{-1}a : z \in \langle S_1 \rangle \} \) together with perfect matchings of the remaining \( 2[(H : \langle S_1 \rangle) - 2] \) components of \( \Gamma_1 \cup \Gamma_2 \) (which exist as they are Cayley graphs of an abelian group of even order).

Next, suppose \( x \in \langle S_1 \rangle \) but \( tx^{-1}a \notin t\langle S_1 \rangle \) (the case \( x \notin \langle S_1 \rangle \), \( tx^{-1}a \in t\langle S_1 \rangle \) is dealt with analogously). If \( |S_1| = 1 \) (that is, \( S_1 \) consists of a single involution), then either \( \langle a \rangle = H \) or \( |H : \langle a \rangle| = 2 \). Hence either \( \Gamma \cong \text{Circ}(4n; \{\pm 1, 2n\}) \) (the cycle of length \( 4n \) corresponding to \( \pm 1 \) is given by the edges corresponding to \( t \) and \( ta \)) or \( \Gamma \cong \text{Cay}(\mathbb{Z}_{2n} \times \mathbb{Z}_2; \{(\pm 1, 0), (0, 1)\}) \), depending on whether \( \langle a \rangle = H \) or not, respectively. This shows that \( \Gamma \) is a Cayley graph of an abelian group of even order and valency three, so Proposition 4 applies. We can thus assume that \( |S_1| \geq 2 \) implying that there exists some \( h \in S_1 \setminus \{x\} \). Since \( \Gamma' = \text{Cay}(\langle S_1 \rangle; S_1) \) is 1-extendable, there exists a perfect matching \( M \) of \( \Gamma' \) containing \( \{1, h\} \). Let \( h' \in H \) be such that \( \{x, xh'\} \in M \). Taking \( M_1 = Mt \setminus \{t, th^{-1}\} \) and \( M_2 = Mta \setminus \{tx^{-1}a, tx^{-1}h^{-1}a\} \) a desired perfect matching of \( \Gamma \) is obtained by taking

\[
M \setminus \{1, h\} \cup \{x, xh'\} \} \cup \{e_1, e_2\} \cup \{(h, th^{-1}), \{xh', tx^{-1}h^{-1}a\}\} \cup M_1 \cup M_2
\]

together with perfect matchings of the remaining \( 2[H : \langle S_1 \rangle) - 3 \) components of \( \Gamma_1 \cup \Gamma_2 \), each of which is isomorphic to \( \Gamma' \). Finally, suppose \( x \in \langle S_1 \rangle \) and \( tx^{-1}a \in t\langle S_1 \rangle \) (note that this implies \( \langle S_1 \rangle = H \)). We can clearly assume \( |S_1| > 1 \) (otherwise \( \Gamma = K_4 \)). Now, if \( |S_1| = 2 \), then each of \( \Gamma_1 \) and \( \Gamma_2 \) is isomorphic to a cycle of length \( 2n \) for some \( n \). We can thus identify the vertex set of \( \Gamma \) with the set \( V = \mathbb{Z}_{2n} \times \mathbb{Z}_2 \) in such a way that \( (i, j) \sim (i + 1, j) \) for each \( i \in \mathbb{Z}_{2n} \) and \( j \in \{0, 1\} \), \( (i, 0) \sim (i, 1) \) for each \( i \in \mathbb{Z}_{2n} \) and \( (i, 0) \sim (i + k, 1) \) for each \( i \in \mathbb{Z}_{2n} \) and some fixed nonzero \( k \in \mathbb{Z}_{2n} \). If \( k = 2k_1 \) for some \( k_1 \), then (relabeling the vertices \( (i, 1) \) by \( (i - k_1, 1) \) for \( i \in \mathbb{Z}_{2n} \)) we can thus identify \( \Gamma \cong \text{Cay}(\mathbb{Z}_{2n} \times \mathbb{Z}_2; \{(\pm 1, 0), (\pm k_1, 1)\}) \). If on the other hand \( k = 2k_1 - 1 \) for some \( k_1 \), then the permutation \( \rho \) of the vertex set \( V \) defined by \( \rho(i, 0) = (i + k_1, 1) \) and \( \rho(i, 1) = (i - k_1 + 1, 0) \) for every \( i \in \mathbb{Z}_{2n} \) easily seen to be an automorphism of \( \Gamma \) of order \( 4n \), so that \( \Gamma \) is a circulant in this case (in particular, \( \Gamma \cong \text{Circ}(4n; \{\pm 2, \pm k\}) \)). In either case \( \Gamma \) is a Cayley graph of an abelian group, so that Proposition 4 applies. We can thus assume \( |S_1| \geq 3 \), and so Proposition 3 applies to \( \Gamma_1 \) and \( \Gamma_2 \). If \( \Gamma_1 \) is not bipartite, there is a Hamilton path of \( \Gamma_1 \) between \( 1 \) and \( x \) and there is a Hamilton path of \( \Gamma_2 \) between \( t \) and \( tx^{-1}a \). Together with \( e_1 \) and \( e_2 \) this gives a Hamilton cycle of \( \Gamma \), and so a desired perfect matching of \( \Gamma \) can be obtained by taking every other edge of this cycle, starting with \( e_1 \) (recall that \( \Gamma_1 \) is of even order). If \( \Gamma_1 \) is bipartite then it is 2-extendable by Proposition 4. It is easy to see that, since \( \Gamma_1 \) contains no triangles, there exist disjoint edges \( e = \{1, h\} \) and \( e' = \{x, xh'\} \) such that \( et \) and \( e'ta \) are also disjoint. As \( \Gamma_1 \) is 2-extendable we can now find a perfect matching of \( \Gamma_1 \) containing \( e \) and \( e' \) as well as a perfect matching of \( \Gamma_2 \) containing \( et \) and \( e'ta \). It is now clear how to construct a desired perfect matching of \( \Gamma \).

Subcase 4.2.2: \( \langle S_1 \rangle \) is of odd order.
Consider first the case that \( x \not\in \langle S_1 \rangle \) and \( tx^{-1}a \not\in \langle S_1 \rangle \). Note that this implies \([H : \langle S_1 \rangle] \geq 2\), and therefore the connectivity of \( \Gamma \) forces that \( a \not\in \langle S_1 \rangle \). Let \( y \in \langle S_1 \rangle \), \( y \neq x \), and observe that then \( ya^{-1} \not\in \langle S_1 \rangle \) (otherwise \( tx^{-1}a \in \langle S_1 \rangle \)).

Letting \( e = \{y, ty^{-1}\} \) and \( e' = \{ya^{-1}, ty^{-1}a\} \), a desired perfect matching of \( \Gamma \) is

\[
\{e, e'\} \cup \{\{z, tz^{-1}\} : z \in \langle S_1 \rangle \backslash \{y\}\} \cup \{\{z, tz^{-1}\} : z \in H \setminus (\langle S_1 \rangle \cup \langle S_1 \rangle x a^{-1})\} \cup M_1 \cup M_2,
\]

where \( M_1 \) is an almost perfect matching of the component of \( \Gamma_1 \) containing \( xa^{-1} \) which misses \( ya^{-1} \) and \( M_2 \) is an almost perfect matching of the component of \( \Gamma_2 \) containing \( tx^{-1} \) which misses \( ty^{-1} \). In the case that \( x \in \langle S_1 \rangle \) and \( tx^{-1}a \in \langle S_1 \rangle \) we clearly have \( \langle S_1 \rangle = H \). The existence of a desired perfect matching of \( \Gamma \) then depends on \( |S_1| \). If \( |S_1| = 2 \), then each of \( \Gamma_1 \) and \( \Gamma_2 \) is just a cycle. Similar argument as in Subsubcase 4.2.1 shows that then \( \Gamma \) is a Cayley graph of an abelian group, so that Proposition 4 applies. If \( |S_1| > 2 \), then \( \Gamma_1 \) and \( \Gamma_2 \) are \( \frac{1}{4} \)-extendable by Theorem 1. Taking \( h \in S_1 \setminus \{x, xa^{-1}\} \) (which exists since \( |S_1| > 2 \)) there thus exists an almost perfect matching of \( \Gamma_1 \) which contains \( \{1, h\} \) but misses \( x \) and there exists an almost perfect matching of \( \Gamma_2 \) which contains \( \{t, th^{-1}\} \) but misses \( tx^{-1}a \).

It is now clear how to obtain a desired perfect matching of \( \Gamma \). We are left with the possibility that \( x \in \langle S_1 \rangle \) but \( tx^{-1}a \not\in \langle S_1 \rangle \) (the case \( x \not\in \langle S_1 \rangle \), \( tx^{-1}a \in \langle S_1 \rangle \) is dealt with analogously). Let \( M_1 \) be an almost perfect matching of the component of \( \Gamma_2 \) containing \( t \) which misses \( t \). By Proposition 3 each component of \( \Gamma_1 \cup \Gamma_2 \) contains a Hamilton cycle (if \( |S_1| = 2 \), then each component of \( \Gamma_1 \cup \Gamma_2 \) consists of a single cycle). Take a Hamilton cycle \( C \) of the component containing \( 1 \) and let \( y \) be the neighbor of \( x \) on this cycle, such that the length of the subpath of \( C \) from \( 1 \) to \( x \) not passing through \( y \) consists of an even number of vertices. Let \( M_2 \) be the unique matching in \( \text{Cay}(\langle S_1 \rangle; S_1) \) consisting of edges of \( C \) which misses \( 1, x \), and \( y \). Furthermore, let \( M_3 = M_2ta \) and let \( M_4 \) be an almost perfect matching of the component containing \( a^{-1} \) which misses \( a^{-1} \). Then a desired perfect matching of \( \Gamma \) is

\[
\{\{z, tz^{-1}\} : z \in H \setminus (\langle S_1 \rangle \cup \langle S_1 \rangle a^{-1})\} \cup \{e_1, e_2, \{y, ty^{-1}a\}, \{a^{-1}, ta\}\} \cup M_1 \cup M_2 \cup M_3 \cup M_4.
\]

\[\Box\]

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**References**


On extendability of Cayley graphs  


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