UNIVALENCE OF TWO GENERAL INTEGRAL OPERATORS

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Abstract

In this paper, we give some sufficient conditions for general two integral operators to be univalent in the open unit disk.

1 Introduction and definitions

Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ defined in the open unit disk $\mathcal{U} = \{ z : |z| < 1 \}$ and normalized by the condition $f(0) = 0 = f'(0) - 1$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Recently, Breaz and Breaz [6] and Breaz et al. [10] introduced and studied the integral operators

$$F_n(z) = \int_0^z \left( \frac{f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{f_n(t)}{t} \right)^{\alpha_n} dt$$

and

$$F_{\alpha_1, \ldots, \alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} \cdots (f_n'(t))^{\alpha_n} dt$$

where $f_i \in \mathcal{A}$ and for $\alpha_i > 0$, for all $i = 1, \ldots, n$ (see also [3, 4, 5, 7, 9]).

Breaz and Güney [8] considered the above integral operators and they obtained their properties on the classes $\mathcal{S}_\alpha^*(b)$, $\mathcal{C}_\alpha(b)$ of starlike and convex functions of complex order $b$ and type $\alpha$ introduced and studied by Frasin [11].

Very recently, Frasin [12] obtained some sufficient conditions for the above integral operators to be in the classes $\mathcal{S}^*$, $\mathcal{C}(\alpha)$ and $\mathcal{UCV}$, where $\mathcal{C}(\alpha)$ and $\mathcal{UCV}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, close-to-convex of order $\alpha(0 \leq \alpha < 1)$ in $\mathcal{U}$ and uniformly convex functions.

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In the present paper, we obtain some sufficient conditions for the above integral operators \( F_n(z) \) and \( F_{\alpha_1,\ldots,\alpha_n}(z) \) to be univalent in \( U \).

In order to derive our main results, we have to recall here the following lemma:

**Lemma 1.1.** ([1]) Let \( f \in A, \beta \in \mathbb{C}, \text{Re}(\beta) > 0 \). If for some \( \theta \in [0,2\pi] \) the inequality

\[
\text{Re}\left\{ e^{i\theta} \frac{zf''(z)}{f'(z)} \right\} \leq \begin{cases} \frac{1}{2} \text{Re}(\beta) & \text{for } 0 < \text{Re}(\beta) < 1 \\ \frac{1}{4} & \text{for } \text{Re}(\beta) \geq 1 \end{cases} \quad (z \in U)
\]

is valid, then the function

\[
G_\beta(z) = \left\{ \beta \int_0^z u^{\beta-1} f'(u) du \right\}^{1/\beta}
\]

is in \( S \), for all \( \theta \in [0,2\pi] \).

## 2 Main results.

**Theorem 2.1.** Let \( \alpha_j > 0 \) be real numbers for all \( j = 1,2,\ldots,n \), \( \beta \in \mathbb{C}, \text{Re}(\beta) > 0 \). If \( f_j \in A \) for all \( j = 1,2,\ldots,n \) satisfies

\[
\text{Re}\left\{ e^{i\theta} \frac{zf_j''(z)}{f_j'(z)} \right\} \leq \begin{cases} \frac{\text{Re}(\beta)}{2} \sum_{j=1}^{n} \alpha_j + \cos \theta & \text{for } 0 < \text{Re}(\beta) < 1 \\ \frac{1}{4} \sum_{j=1}^{n} \alpha_j + \cos \theta & \text{for } \text{Re}(\beta) \geq 1 \end{cases} \quad (3)
\]

for all \( z \in U \) and for some \( \theta \in [0,2\pi] \), then the function

\[
\left\{ \beta \int_0^z u^{\beta-1} \prod_{j=1}^{n} \left( \frac{f_j(u)}{u} \right)^{\alpha_j} du \right\}^{1/\beta} \in S
\]

for all \( \theta \in [0,2\pi] \).

**Proof.** From (1) we observe that \( F_n \in A \), i.e. \( F_n(0) = F_n'(0) - 1 = 0 \). On the other hand, it is easy to see that

\[
F_n'(z) = \prod_{j=1}^{n} \left( \frac{f_j(z)}{z} \right)^{\alpha_j}
\]
Univalence of two general integral operators

and

\[
\left( \frac{zF''(z)}{F'_n(z)} \right) = \sum_{j=1}^{n} \alpha_j \left( \frac{zf'_j(z)}{f_j(z)} \right) - \sum_{j=1}^{n} \alpha_j
\]

thus we have

\[
\left( \frac{e^{i\theta} zF''(z)}{F'_n(z)} \right) = \sum_{j=1}^{n} \alpha_j \left( \frac{e^{i\theta} zf'_j(z)}{f_j(z)} \right) - e^{i\theta} \sum_{j=1}^{n} \alpha_j
\]

It follows from (4) and the hypothesis (3) that

\[
\text{Re} \left( \frac{e^{i\theta} zF''(z)}{F'_n(z)} \right) = \sum_{j=1}^{n} \alpha_j \text{Re} \left( \frac{e^{i\theta} zf'_j(z)}{f_j(z)} \right) - (\cos \theta) \sum_{j=1}^{n} \alpha_j
\]

for all \( z \in \mathcal{U} \) and for some \( \theta \in [0,2\pi] \). Applying Lemma 1.1, we have

\[
\left\{ \beta \int_0^z u^{\beta-1} F'_n(u) du \right\}^{1/\beta} \in \mathcal{S}
\]

or, equivalently

\[
\left\{ \beta \int_0^z u^{\beta-1} \prod_{j=1}^{n} \left( \frac{f_j(u)}{u} \right)^{\alpha_j} du \right\}^{1/\beta} \in \mathcal{S}
\]

for all \( \theta \in [0,2\pi] \).

This completes the proof.

Letting \( n = 1, \alpha_1 = \alpha \) and \( f_1 = f \) in Theorem 2.1, we have

**Corollary 2.2.** Let \( \alpha > 0 \) be real number, \( \beta \in \mathbb{C}, \text{Re}(\beta) > 0 \). If \( f \in \mathcal{A} \) satisfies

\[
\text{Re} \left( \frac{e^{i\theta} zf'(z)}{f(z)} \right) \leq \begin{cases} 
\frac{\text{Re}(\beta)}{2\alpha} + \cos \theta & \text{for } 0 < \text{Re}(\beta) < 1 \\
\frac{1}{4\alpha} + \cos \theta & \text{for } \text{Re}(\beta) \geq 1
\end{cases}
\]

for all \( z \in \mathcal{U} \) and for some \( \theta \in [0,2\pi] \), then the function

\[
\left\{ \beta \int_0^z u^{\beta-1} \left( \frac{f(u)}{u} \right)^{\alpha} du \right\}^{1/\beta} \in \mathcal{S}
\]

for all \( \theta \in [0,2\pi] \).

Letting \( \alpha = 1 \) in Corollary 2.2, we have
Corollary 2.3. Let $\beta \in \mathbb{C}$, $\text{Re}(\beta) > 0$. If $f \in \mathcal{A}$ satisfies

$$
\text{Re}\left(\frac{e^{i\theta}zf'(z)}{f(z)}\right) \leq \begin{cases} 
\frac{\text{Re}(\beta)}{2} + \cos \theta & \text{for } 0 < \text{Re}(\beta) < 1 \\
\frac{1}{4} + \cos \theta & \text{for } \text{Re}(\beta) \geq 1
\end{cases}
$$

for all $z \in U$ and for some $\theta \in [0, 2\pi]$, then the function

$$
\left\{ \beta \int_0^z u^{\beta-2} f(u) du \right\}^{1/\beta} \in \mathcal{S}
$$

for all $\theta \in [0, 2\pi]$.

Letting $\beta = 1$ in Corollary 2.3, we have

Corollary 2.4. If $f \in \mathcal{A}$ satisfies

$$
\text{Re}\left(\frac{e^{i\theta}zf'(z)}{f(z)}\right) \leq \frac{1}{4} + \cos \theta
$$

for all $z \in U$ and for some $\theta \in [0, 2\pi]$, then the function

$$
\int_0^z f(u) \frac{u}{u} du \in \mathcal{S}
$$

for all $\theta \in [0, 2\pi]$.

Next, we have

Theorem 2.5. Let $\alpha_j > 0$ be real numbers for all $j = 1, 2, \ldots, n$, $\beta \in \mathbb{C}$, $\text{Re}(\beta) > 0$. If $f_j \in \mathcal{A}$ for all $j = 1, 2, \ldots, n$ satisfies

$$
\text{Re}\left(\frac{e^{i\theta}zf''(z)}{f'(z)}\right) \leq \begin{cases} 
\frac{\text{Re}(\beta)}{2} \sum_{j=1}^n \alpha_j & \text{for } 0 < \text{Re}(\beta) < 1 \\
\frac{1}{4} \sum_{j=1}^n \alpha_j & \text{for } \text{Re}(\beta) \geq 1
\end{cases}
$$

(5)

for all $z \in U$ and for some $\theta \in [0, 2\pi]$, then the function

$$
\left\{ \beta \int_0^z u^{\beta-3} \prod_{j=1}^n (f_j'(u))^{\alpha_j} du \right\}^{1/\beta} \in \mathcal{S}
$$

for all $\theta \in [0, 2\pi]$. 
Proof. It follows from (2) that $F_{\alpha_1,...,\alpha_n}(0) = F'_{\alpha_1,...,\alpha_n}(0) - 1 = 0$. Also a simple computation yields

$$
\left( z F''_{\alpha_1,...,\alpha_n}(z) \right) = \sum_{j=1}^{n} \alpha_j \left( \frac{zf_j''(z)}{f_j(z)} \right).
$$

Thus we have

$$
\text{Re} \left( e^{i\theta} z F''_{\alpha_1,...,\alpha_n}(z) \right) = \sum_{j=1}^{n} \alpha_j \text{Re} \left( e^{i\theta} z f_j''(z) \right). \quad (6)
$$

Since $f_j$ satisfies the condition (5) for every $j = 1, \ldots, n$, then from (7), we obtain

$$
\text{Re} \left( e^{i\theta} z F''_{\alpha_1,...,\alpha_n}(z) \right) \leq \begin{cases}
\frac{1}{2} \text{Re}(\beta) & \text{for } 0 < \text{Re}(\beta) < 1 \\
\frac{1}{4} \alpha & \text{for } \text{Re}(\beta) \geq 1
\end{cases}
$$

for all $z \in \mathcal{U}$ and for some $\theta \in [0,2\pi]$. Lemma 1.1 implies that

$$
\left\{ \beta \int_{0}^{z} u^{\beta-1} F'_{\alpha_1,...,\alpha_n}(u) du \right\}^{1/\beta} \in \mathcal{S}
$$

or, equivalently

$$
\left\{ \beta \int_{0}^{z} u^{\beta-1} \prod_{j=1}^{n} (f_j(u))^{\alpha_j} du \right\}^{1/\beta} \in \mathcal{S}
$$

for all $\theta \in [0,2\pi]$. \hfill \Box

Letting $n = 1$, $\alpha_1 = \alpha$, and $f_1 = f$ in Theorem 2.5, we have

**Corollary 2.6.** Let $\alpha > 0$ be real number, $\beta \in \mathbb{C}$, $\text{Re}(\beta) > 0$. If $f \in \mathcal{A}$ satisfies

$$
\text{Re} \left( e^{i\theta} z f''(z) \right) \leq \begin{cases}
\frac{\text{Re}(\beta)}{2\alpha} & \text{for } 0 < \text{Re}(\beta) < 1 \\
\frac{1}{4\alpha} & \text{for } \text{Re}(\beta) \geq 1
\end{cases}
$$

for all $z \in \mathcal{U}$ and for some $\theta \in [0,2\pi]$, then the function

$$
\left\{ \beta \int_{0}^{z} u^{\beta-1} (f'(u))^\alpha du \right\}^{1/\beta} \in \mathcal{S}
$$

for all $\theta \in [0,2\pi]$. Letting $\alpha = 1$ in Corollary 2.6, we have
Corollary 2.7. Let $\beta \in \mathbb{C}$, $\Re(\beta) > 0$. If $f \in \mathcal{A}$ satisfies
\[
\Re\left(e^{i\theta} \frac{zf''(z)}{f'(z)}\right) \leq \begin{cases} 
\frac{\Re(\beta)}{2} & \text{for } 0 < \Re(\beta) < 1 \\
\frac{1}{4} & \text{for } \Re(\beta) \geq 1 
\end{cases}
\]
for all $z \in U$ and for some $\theta \in [0, 2\pi]$, then the function
\[
\left\{ \beta \int_0^z \frac{u^{\beta-1}f'(u)du}{u} \right\}^{1/\beta} \in \mathcal{S}
\]
for all $\theta \in [0, 2\pi]$.

Letting $\beta = 1$ in Corollary 2.7, we obtain the following result of Blezu and Pascu [2].

Corollary 2.8. ([2]) If $f \in \mathcal{A}$ satisfies
\[
\Re\left(e^{i\theta} \frac{zf''(z)}{f'(z)}\right) \leq \frac{1}{4}
\]
for all $z \in U$ and for some $\theta \in [0, 2\pi]$, then $f \in \mathcal{S}$ for all $\theta \in [0, 2\pi]$.

References


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