GA$_2$ INDEX OF SOME GRAPH OPERATIONS

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Abstract

Let $G = (V, E)$ be a graph. For $e = uv \in E(G)$, $n_u(e)$ is the number of vertices of $G$ lying closer to $u$ than to $v$ and $n_v(e)$ is the number of vertices of $G$ lying closer to $v$ than $u$. The GA$_2$ index of $G$ is defined as $\sum_{uv \in E(G)} \frac{2\sqrt{n_u(e)n_v(e)}}{n_u(e)+n_v(e)}$. We explore here some mathematical properties and present explicit formulas for this new index under several graph operations.

1 Introduction

In this paper, we only consider simple connected graphs. As usual, the distance between the vertices $u$ and $v$ of $G$ is denoted by $d_G(u, v)$ ($d(u, v)$ for short). It is defined as the length of a minimum path connecting them and $d_G(u)(d(u)$ for short) denotes the degree of $u$ in $G$. The Wiener index of a graph $G$ is defined as $W(G) = \sum_{\{u, v\}} d(u, v)$[7, 17, 20, 23]. GA$_2$ index of the graph of $G$ is defined by $GA_2(G) = \sum_{uv \in E(G)} \frac{2\sqrt{n_u(e) n_v(e)}}{n_u(e) + n_v(e)}$[4] that $n_u(e|G)(n_u(e)$ for short) is the number of vertices of $G$ lying closer to $u$ and $n_v(e|G)$ is the number of vertices of $G$ lying closer to $v$. Notice that vertices equidistance from $u$ and $v$ are not taken into account.

The Cartesian product $G \times H$ of graphs $G$ and $H$ is a graph such that $V(G \times H) = V(G) \times V(H)$, and any two vertices $(a, b)$ and $(u, v)$ are adjacent in $G \times H$ if and only if either $a = u$ and $b$ is adjacent with $v$, or $b = v$ and $a$ is adjacent with $u$, see [10] for details. The join $G = G_1 + G_2$ of graphs $G_1$ and $G_2$ with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ is the graph union $G_1 \cup G_2$ together with all the edges joining $V_1$ and $V_2$. The composition $G = G_1[G_2]$ of graphs $G_1$ and $G_2$ with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever $(u_1$ is adjacent with $u_2)$ or $(u_1 = u_2$ and $v_1$ is adjacent with $v_2)$[10, p. 185]. For given graphs $G_1$ and $G_2$ we define their corona product $G_1 \circ G_2$ as the

2010 Mathematics Subject Classifications. 05C12, 05A15, 05A20, 05C05.

Key words and Phrases. GA$_2$ index, Szeged index, $C_4$-nanotorus, Cartesian product, join, composition, corona.

Received: October 20, 2009
Communicated by Dragan Stevanović
Some properties of GA graph. Our other notations are standard and taken mainly from [3, 8, 21].

Proposition 2. GA graphs are contribute to u has diameter 2. (10,3,0,1), respectively. It is easy to see that a non-trivial strongly regular graph of vertex u is equal to 2. Hence they cannot contribute to u(e). By the same reasoning, n(e) = k − a. Therefore, by definition GA2(G) = |E(G)| = 1/2k|V(G)|. This if the end of the proof.

Proposition 2. [4, Theorem 3] For any connected graph G with m edges,

GA2(G) ≤ \sqrt{mSz(G)},

with equality if and only if G \cong Kn.

\[ \text{GA}_2(G) = \frac{1}{2} k |V(G)| \]

A k-regular graph G on n vertices is called strongly regular with parameters (n,k,a,c) if and only if each pair of adjacent vertices have a common neighbors and any two distinct non-adjacent vertices have c common neighbors ([6], p.177). We also say that G is (n,k,a,c)-strongly regular. A strongly regular graph is primitive if both G and its complement \( \overline{G} \) is primitive; otherwise it is imprimitive or trivial. We restrict our attention to primitive strongly regular graphs, since an imprimitive strongly regular graph is either a complete multipartite graph or its complement, i.e., the disjoint union of some copies of \( K_m \), for some m. This restriction allows us to assume c = 0 and c = k. The simplest non-trivial examples of strongly regular graphs are c and the Petersen graph, with 5 the parameter vectors (5,2,0,1) and (10,3,0,1), respectively. It is easy to see that a non-trivial strongly regular graph has diameter 2.

Proposition 1. If G is a strongly k-regular graph then GA2(G) = \( \frac{1}{2} k |V(G)| \).

Proof. We assume c \neq 0 and c \neq k. Let us consider an edge e = uv of G. Its endvertices have a common neighbors and all of them are equidistant to u and v. The vertex u has another k − 1 − a neighbors and all of them are closer to u than to v. Together with u itself, this gives us n(e) = k − a. We need not bother to consider other u vertices: those at the distance 2 from u are either adjacent to v, or are at the distance 2 from v, since the diameter of G is equal to 2. Hence they cannot contribute to u(e). By the same reasoning, n(e) = k − a. Therefore, by definition GA2(G) = |E(G)| = 1/2k|V(G)|. This if the end of the proof.

The Geometric-Arithmetic inequality \( \sqrt{n_u(e)n_v(e)} \leq \frac{n_u(e) + n_v(e)}{2} \), implies that GA2(G) \leq |E(G)|, with equality if and only if for all e ∈ E(G), n_u(e) = n_v(e).

The Szeged index was originally defined as \( Sz(G) = \sum_{e=uv \in E(G)} [n_u(e)n_v(e)] \) [5, 13, 16, 17] where \( n_u(e) \) and \( n_v(e) \) are the same as the definition of GA2. Now, we define \( GA_1(G) = GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u) + d(v)} \) [22] where d(u) is the degree of vertex u. Throughout this paper, \( C_n \), \( P_n \), \( K_n \) and \( W_n \) denote the cycle, path, complete graphs and wheel on n vertices. Also, \( K_{m,n} \) denotes the complete bipartite graph. Our other notations are standard and taken mainly from [3, 8, 21].

2 Some properties of GA2 index
Proposition 3. [4, Theorem 4] For any connected graph $G$ with $m$ edges,
\[
GA_2(G) \leq \sqrt{Sz(G) + m(m - 1)},
\]
with equality if and only if $G \cong K_n$.

Proposition 4. [4, Theorem 6] Let $G$ be a connected graph with $n$ vertices and $m \geq 1$ edges. Then
\[
GA_2(G) \geq \frac{2}{n} \sqrt{Sz(G) + m(m - 1)}.
\]
The equality is attained if and only if $G \cong K_2$.

Proposition 5. [17] If $T$ is a tree then $Sz(T) = W(T)$.

Corollary 6. If $T$ is a $n$-vertex tree then
\[
GA_2(T) \leq \sqrt{n - 1} W(T) + \frac{|E(T)| - 1}{2} + Sz(T),
\]
$GA_2(T) \leq \sqrt{\frac{|E(T)| - 1}{2} + Sz(T)}$, and
\[
GA_2(T) \geq 2n \sqrt{W(T) + (n - 1)(n - 2)}.
\]
The equality is attained if and only if $G \cong K_2$.

Proposition 7. Suppose $G$ is a connected graph. Then
\[
GA_2(G) \leq \left\lceil \sqrt{|E(G)| - 1} \right\rceil + \sqrt{\frac{|E(G)| - 1}{2} + Sz(G)},
\]
and equality holds if and only if $G$ is a union of the odd number of $K_2$.

Proof. By definition,
\[
[GA_2(G)]^2 = \sum_{u \in E(G)} \frac{4n_u(e)n_v(e)}{n_u(e) + n_v(e)} + 2 \sum_{u \neq v \in E(G)} \frac{2n_u(e)n_v(e)}{n_u(e) + n_v(e)} \frac{2\sqrt{n_u(e)n_v(e)}}{n_u(e) + n_v(e)}
\]
\[
\leq \sum_{u \in E(G)} n_u(e)n_v(e) + 2\left[ \frac{|E(G)| - 1}{2} \right] \cdot GA_2(G)
\]
\[
= Sz(G) + 2\left[ \frac{|E(G)| - 1}{2} \right] \cdot GA_2(G)
\]
\[
\Rightarrow [GA_2(G) - \left[ \frac{|E(G)| - 1}{2} \right]]^2 \leq \left[ \frac{|E(G)| - 1}{2} \right]^2 + Sz(G).
\]
Therefore,
\[
GA_2(G) \leq \left[ \frac{|E(G)| - 1}{2} \right] + \sqrt{\frac{|E(G)| - 1}{2}} + Sz(G)
\]
and equality holds if and only if $G$ is a union of the odd number of $K_2$.

3 Main Results

In this section, some exact formulas for the $GA_2$ index of the Cartesian product, composition, join and corona of graphs are presented.

The Wiener index of the Cartesian product of graphs was studied in [7, 20]. In [17], Klavžar, Rajapakse and Gutman computed the Szeged index of the Cartesian
Proof. We notice that if $GA$ is an arbitrary edge of $P_n$ or $C_n$ then $n_u(e) = n_v(e)$. Thus $2\sqrt{n_u(e)n_v(e)} = 1$ for each edge of $P_n$ or $C_n$. Therefore, $GA_2(P_n) = \frac{1}{n}\sum_{i=1}^{n-1} \sqrt{i(n-i)}$ and $GA_2(C_n) = n$. Now, Proposition 8 completes the proof. □

Proposition 12. Let $G = G_1 + G_2$, where $G_i$ are $r_i$-regular, $i = 1, 2$. Then

$GA_2(G) = GA_2(G_1) + GA_2(G_2) + 2|V(G_1)||V(G_2)|\frac{\sqrt{|V(G_1)|-r_1||V(G_2)|-r_2}}{|V(G_1)|+|V(G_2)|-r_1+r_2}$.

Corollary 9. Suppose $G_1, G_2, ..., G_n$ are graphs. Then

$GA_2(\prod_{i=1}^{k} G_i) = (\prod_{i=1}^{k} |V(G_i)|) \sum_{i=1}^{k} GA_2(G_i) \frac{|V(G_i)|}{|V(G_i)|}$.

Corollary 10. Suppose $G$ is a graph. Then $GA_2(G^n) = nGA_2(G)|V(G)|^{n-1}$. In particular, $GA_2(Q_n) = n2^{n-1}$.

Corollary 11. If $G_1 = P_m \times P_n$, $G_2 = P_m \times C_n$ and $G_3 = C_m \times C_n$ are $C_4$-net, $C_4$-nanotube and $C_4$-nanotorus, respectively. Then

$GA_2(G_1) = \frac{|E(G_1)|}{|V(G_1)|} \sum_{i=1}^{|V(G_1)|-1} \sqrt{|V(G_1)| - i + |E(G_1)| - i}$,

$GA_2(G_2) = \frac{|E(G_2)|}{|V(G_2)|} \sum_{i=1}^{|E(G_2)|-1} \sqrt{|E(G_2)| - i + |E(G_2)||V(G_2)|}$,

$GA_2(G_3) = 2|E(G_3)||V(G_3)|$.

Proof. We notice that if $e = uv$ is an arbitrary edge of $P_n$ or $C_n$ then $n_u(e) = n_v(e)$. Thus $2\sqrt{n_u(e)n_v(e)} = 1$ for each edge of $P_n$ or $C_n$. Therefore, $GA_2(P_n) = \frac{1}{n}\sum_{i=1}^{n-1} \sqrt{i(n-i)}$ and $GA_2(C_n) = n$. Now, Proposition 8 completes the proof. □

GA Corollary 14.

If $G = G_1 + G_2$. We can partition the edges of $G = G_1 + G_2$ into three subsets $E_1, E_2$ and $E_3$, as follows:

$$E_i = \{ e \in E(G_1 + G_2) \mid e \in E(G_i) \}, i = 1, 2$$

$$E_3 = \{ e \in E(G_1 + G_2) \mid e = uv, u \in V(G_1) \text{ and } v \in V(G_2) \}.$$ By [13, Theorem 2], if $e = u_1v_1 \in E_1$ then $n_{u_1}(e|G) = n_{u_1}(e|G_i)$ and $n_{v_1}(e|G) = n_{v_1}(e|G_i)$. If $e = uv \in E_3$ then $n_u(e|G) = |V(G_2)| - d_{G_2}(v)$ and $n_v(e|G) = |V(G_1)| - d_{G_1}(u)$. Therefore,

$$GA_2(G) = \sum_{u \in V(G_1)} \frac{2\sqrt{n_u(e|G_1)n_v(e|G_1)}}{n_u(e|G_1) + n_v(e|G_1)} + \sum_{u \in E(G_2)} \frac{2\sqrt{n_u(e|G_2)n_v(e|G_2)}}{n_v(e|G_2) + n_v(e|G_2)} + \sum_{u \in V(G_1) \cap V(G_2)} \frac{2\sqrt{(|V(G_1)| - d_{G_1}(u))(|V(G_2)| - d_{G_2}(v))}}{|V(G_1)| - |V(G_2)|}.

This is the end of our proof. □

Corollary 13. If $G$ is $r$-regular graph then

$$GA_2(nG) = nGA_2(G) + 2 \sum_{i=2}^{n} |V(G)|^i \sqrt{(|V(G)| - r)(|V(G)|^{r-1} - r)}\sqrt{|V(G)|^{i - 2}}.$$}

Corollary 14. $GA_2(K_{m,n}) = 2(\frac{mn}{m+n})^{\frac{3}{2}}$, $GA_2(K_{m,n,\ldots,n}) = 2 \sum_{i=2}^{t} \sqrt{n_i}$ and $GA_2(W_n) = n - 1 + 2(n - 1)\sqrt{\frac{n-2}{m}}$.

We present formula for $GA_2$ index of open fence, $P_n[K_2]$.

Example 15. $GA_2(P_n[K_2]) = n + \frac{4}{n-1} \sum_{i=1}^{n-1} \sqrt{(2i-1)(2n - 2i - 1)}$.

Proposition 16. If $G_2$ is triangle-free and $r$-regular graph then

$$GA_2(G_1[G_2]) < |V(G_2)|^2 |E(G_1)| \frac{|V(G_2)|(|V(G_1)| - 1)}{|V(G_2)| - 2r} + |V(G_1)|GA_1(G_2).$$

Proof. Suppose $G = G_1[G_2]$ and $t_G(e)$ denotes the number of triangles containing $e$ of the graph $G$. Let

$$A_u = \{ {u, v} \mid v \in V(G_2) \},$$

$$B_u = \{ {u, v_1, v_2} \mid v_1, v_2 \in E(G_2) \},$$

$$T(u_1, u_2) = \{ (x, y) \mid (a, b) \in A_{u_1} \times A_{u_2} \},$$

$$E(G) = (\cup_{u_1, u_2 \in E(G_1)} T(u_1, u_2)) \cup (\cup_{v \in V(G_1)} B_v).$$
By [13, Theorem 3], if \( e = (u_1, v_1)(u_2, v_2) \in T(u_1, u_2) \) then
\[
n_{(u_1, v_1)}(e\mid G) = |V(G_2)|n_{u_1}(u_1u_2G_1) - d_{G_2}(v_2)
\]
and \( n_{(u_2, v_2)}(e\mid G) = |V(G_2)|n_{u_1}(u_1u_2G_1) - d_{G_2}(v_1) \) and if \( e = (u, v_1)(u, v_2) \in B_u \) then \( n_{(u, v_1)}(e\mid G) = d_{G_2}(v_1) \) and \( n_{(u, v_2)}(e\mid G) = d_{G_2}(v_2) \). Therefore
\[
GA_2(G) = \sum_{e=uv \in E \subseteq E(G_1)} 2\sqrt{n_{(u_1, v_1)}(e\mid G)n_{(u_2, v_2)}(e\mid G)} n_{(u_1, v_1)}(e\mid G) + n_{(u_2, v_2)}(e\mid G)
+ \sum_{e=(u, v_1)(u, v_2) \in B_u} 2\sqrt{d_{G_2}(v_1)d_{G_2}(v_2)} d_{G_2}(v_1) + d_{G_2}(v_2)
= \sum_{e=uv \in E \subseteq E(G_1)} |V(G_2)|n_{u_1}(u_1u_2G_1) - r |V(G_2)||V(G_1)| - 2r |V(G_2)|GA_1(G_2)
< |V(G_2)|^2|E(G_1)|/|V(G_2)|^2(1) + |V(G_1)||GA_1(G_2)|,
\]
which completes our proof.

Proposition 17. If \( H \) is triangle-free and \( r \)-regular graph then
\[
GA_2(G \circ H) = GA_1(H) + GA_2(G) + |V(G)||V(H)| \frac{2\sqrt{|V(G)| + |V(H)|r - 1}}{|V(G)| + |V(H)| - r - 1}.
\]

Proof. The edges of \( G \circ H \) are partitioned into three subsets \( E_1, E_2 \) and \( E_3 \) as follows:
\[
E_1 = \{ e \in E(G \circ H) \mid e \in E(H_i) \text{ for some } i \}
E_2 = \{ e \in E(G \circ H) \mid e \in E(G) \}
E_3 = \{ e \in E(G \circ H) \mid e = uv, u \in V(H_i), v \in V(G) \}
\]
Suppose \( e = uv \in E(H) \). If there exists \( w \in V(H) \) such that \( uw \neq E(H) \) and \( uv \neq E(H) \) then \( d_{G \circ H}(u, w) = d_{G \circ H}(v, w) = 2 \). Also, if there is \( w \in V(H) \) such that \( uw \in E(H) \) and \( vw \in E(H) \) then \( d_{G \circ H}(u, w) = d_{G \circ H}(v, w) = 1 \). Moreover, if \( e = uv \in E_1 \) then
\[
n_u(e\mid G \circ H) = d_H(u) - t_H(u),
\]
\[
n_v(e\mid G \circ H) = d_H(v) - t_H(v),
\]
and if \( e = uv \in E_2 \) then \( n_u(e\mid G \circ H) = (|V(H)| + 1)n_u(e\mid G),
\]
\[
n_v(e\mid G \circ H) = (|V(H)| + 1)n_v(e\mid G)\),
\]
\[
n_u(e\mid G \circ H) = n_v(e\mid G \circ H) = |V(G \circ H)| - (d_H(u) + 1) \text{ and }
n_u(e\mid G \circ H) + n_v(e\mid G \circ H) = |V(G \circ H)| - d_H(u),
\]
and if \( e = uv \in E_3 \) then by above calculations,
\[
GA_2(G \circ H) = \sum_{uv \in E(G \circ H)} 2\sqrt{n_u(e\mid G \circ H)n_v(e\mid G \circ H)}
/ n_u(e\mid G \circ H) + n_v(e\mid G \circ H)
\[ \begin{align*}
&= \sum_{uv \in E_1} \frac{2\sqrt{(d_H(u) - t_H(uv))(d_H(v) - t_H(uv))}}{d_H(u) + d_H(v) - 2t_H(uv)} \\
&+ \sum_{uv \in E_2} \frac{2\sqrt{|V(H)| + 1)^2n_u(e|G)n_v(e|G)}}{(|V(H)| + 1)(n_u(e|G) + n_v(e|G))} \\
&+ \sum_{uv \in E_3} \frac{2\sqrt{|V(G \circ H)|} - (d_H(u) + 1)}{|V(G \circ H)| - d_H(u)} \\
&= GA_1(H) + GA_2(G) + |V(G)||V(H)| \frac{2\sqrt{|V(G)| + |V(H)||V(H)| - r - 1}}{|V(G)| + |V(G)||V(H)| - r}.
\end{align*} \]

as desired. □

As an application of this result, we present the formulae for \( GA_2 \) index of thorny cycle \( C_n \circ \overline{K}_m \).

**Corollary 18.** \( GA_2(C_n \circ \overline{K}_m) = n + nm^{2\sqrt{nm+n-1}}_{n(m+1)} \).

**References**


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