SUBORDINATION AND SUPERORDINATION RESULTS
FOR THE FAMILY OF JUNG-KIM-SRIVASTAVA
INTEGRAL OPERATORS

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Abstract
In this paper, we derive some subordination and superordination results associated with the family of Jung-Kim-Srivastava integral operators defined on the space of meromorphic functions. Several sandwich-type results are also obtained.

1 Introduction
Let $\Sigma$ denote the class of functions of the form
\[ f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1} \]
which are analytic in the punctured open unit disk
\[ U^* := \{ z : z \in \mathbb{C} \text{ and } 0 < |z| < 1 \} =: U \setminus \{0\}. \]

Let $\mathcal{H}$ be the linear space of all analytic functions in $U$. For a positive integer number $n$ and $a \in \mathbb{C}$, we let
\[ \mathcal{H}[a, n] := \{ f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}. \]

Denote by $Q$ the set of all functions $f$ that are analytic and injective on $\overline{U \setminus E(f)}$, where
\[ E(f) = \left\{ \varepsilon \in \partial U : \lim_{z \to \varepsilon} f(z) = \infty \right\}, \]
and are such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial U \setminus E(f)$.

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Let $f, g \in \Sigma$, where $f$ is given by (1.1) and $g$ is defined by
\[ g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k. \]

Then the Hadamard product (or convolution) $f \ast g$ of the functions $f$ and $g$ is defined by
\[ (f \ast g)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k =: (g \ast f)(z). \]

For two functions $f$ and $g$, analytic in $U$, we say that the function $f$ is subordinate to $g$ in $U$, and write
\[ f(z) \prec g(z), \]
if there exists a Schwarz function $\omega$, which is analytic in $U$ with
\[ \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in U) \]
such that
\[ f(z) = g(\omega(z)) \quad (z \in U). \]

Indeed, it is known that
\[ f(z) \prec g(z) \implies f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U). \]

Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence:
\[ f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U). \]

Let $h, \kappa \in \mathcal{H}$ and let
\[ \phi(r, s, t; z) : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}. \]

If $h$ and $\phi(h(z), zh'(z), z^2 h''(z); z)$ are univalent and $h$ satisfies the second-order superordination
\[ \kappa(z) \prec \phi(h(z), zh'(z), z^2 h''(z); z), \quad (1.2) \]
then $h$ is a solution of the differential superordination (1.2). Note that if $f$ is subordinate to $g$, then $g$ is superordinate to $f$. An analytic function $q$ is called a subordinant if $q \prec h$ for all $h$ satisfying (1.2). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants $q$ of (1.2) is said to be the best subordinant.

Analogous to the integral operator defined by Jung et al. [10], Lashin [11] recently introduced and investigated the integral operator
\[ Q_{\alpha, \beta} : \Sigma \longrightarrow \Sigma \]
defined, in terms of the familiar Gamma function, by

\[
\mathcal{Q}_{\alpha,\beta} f(z) = \frac{\Gamma(\beta + \alpha)}{\Gamma(\alpha) \Gamma(\beta + 1)} \frac{1}{z^{\beta + 1}} \int_0^z t^{\beta} \left( 1 - \frac{t}{z} \right)^{\alpha - 1} f(t) \, dt
\]

\[
= \frac{1}{z} + \frac{\Gamma(\beta + \alpha)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k + \beta + 1)}{\Gamma(k + \beta + \alpha + 1)} a_k z^k \quad (\alpha > 0; \ \beta > 0; \ z \in U^*). 
\]  

(1.3)

By setting

\[
f_{\alpha,\beta}(z) := \frac{1}{z} + \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} \sum_{k=1}^{\infty} \frac{\Gamma(k + \beta + 1)}{\Gamma(k + \beta + 1)} \frac{\Gamma(\beta + \alpha + 1)}{\Gamma(\beta + \alpha + 1)} a_k z^k \quad (\alpha > 0; \ \beta > 0; \ z \in U^*), 
\]  

(1.4)

we define a new function \( f_{\alpha,\beta}^\lambda(z) \) in terms of the Hadamard product (or convolution)

\[
f_{\alpha,\beta}(z) * f_{\alpha,\beta}^\lambda(z) = \frac{1}{z(1 - z)^\lambda} \quad (\alpha > 0; \ \beta > 0; \ \lambda > 0; \ z \in U^*). 
\]

(1.5)

Then, motivated essentially by the operator \( \mathcal{Q}_{\alpha,\beta} \), Wang et al. [21] introduced the operator

\[
\mathcal{Q}_{\alpha,\beta}^\lambda : \Sigma \rightarrow \Sigma,
\]

which is defined as

\[
\mathcal{Q}_{\alpha,\beta}^\lambda f(z) := f_{\alpha,\beta}^\lambda(z) * f(z)
\]

\[
= \frac{1}{z} + \frac{\Gamma(\beta + \alpha)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{(\lambda)_{k+1}}{(k+1)!} \frac{\Gamma(k + \beta + 1)}{\Gamma(k + \beta + \alpha + 1)} a_k z^k \quad (z \in U^*; \ f \in \Sigma),
\]

(1.6)

where (and throughout this paper unless otherwise mentioned) the parameters \( \alpha, \beta \) and \( \lambda \) are constrained as follows:

\[
\alpha > 0; \ \beta > 0 \quad \text{and} \quad \lambda > 0,
\]

and \( (\lambda)_k \) is the Pochhammer symbol defined by

\[
(\lambda)_k := \begin{cases} 
1 & (k = 0), \\
\lambda(\lambda + 1) \cdots (\lambda + k - 1) & (k \in \mathbb{N} := \{1, 2, \cdots \}). 
\end{cases}
\]

Clearly, we know that \( \mathcal{Q}_{\alpha,\beta}^1 = \mathcal{Q}_{\alpha,\beta} \).

It is readily verified from (1.6) that

\[
z(\mathcal{Q}_{\alpha,\beta}^\lambda f)'(z) = \lambda \mathcal{Q}_{\alpha,\beta}^{\lambda+1} f(z) - (\lambda + 1) \mathcal{Q}_{\alpha,\beta}^\lambda f(z),
\]

(1.7)
and
\[ z(Q_{\alpha,1}^\lambda f)'(z) = (\beta + \alpha)Q_{\alpha,1}^\lambda f(z) - (\beta + \alpha + 1)Q_{\alpha+1,1}^\lambda f(z). \quad (1.8) \]

In [21], Wang et al. obtained several inclusion relationships and integral-preserving properties associated with some subclasses involving the operator \( Q_{\alpha,\beta}^\lambda \). Several subordination and superordination results involving this family of integral operators are also derived. For some other recent sandwich-type results in analytic function theory, one can find in [1, 2, 3, 5, 6, 7, 8, 9, 16, 17, 18, 19, 20, 22] and the references cited therein.

The main purpose of the present paper is to derive some other new subordination and superordination results involving the operator \( Q_{\alpha,\beta}^\lambda \).

2 Preliminary Results

In order to establish our main results, we need the following lemmas.

**Lemma 1.** (See [15]) Let \( q \) be convex univalent in \( U \) and \( \psi, \gamma \in \mathbb{C} \) with
\[ \Re \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, - \Re \left( \frac{\psi}{\gamma} \right) \right\}. \]
If \( p \) is analytic in \( U \) and
\[ \psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z), \]
then \( p \prec q \), and \( q \) is the best dominant.

**Lemma 2.** (See [12]) Let \( q \) be univalent in \( U \), and let \( \theta \) and \( \phi \) be analytic in the domain \( D \) containing \( q(U) \) with \( \phi(\omega) \neq 0 \) when \( \omega \in q(U) \). Setting
\[ Q(z) = z q'(z) \phi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z). \]
Suppose also that
1. \( Q \) is starlike univalent in \( U \);
2. \( \Re \left( \frac{zh'(z)}{Q(z)} \right) > \Re \left( \frac{\phi'(q(z))}{\phi(q(z))} + \frac{Q'(z)}{Q(z)} \right) > 0 \quad (z \in U). \]
If \( p \) is analytic in \( U \) with \( p(0) = q(0) \), \( p(U) \in D \), and
\[ \theta(p(z)) + z p'(z) \phi(p(z)) \prec \theta(q(z)) + z q'(z) \phi(q(z)), \]
then \( p \prec q \), and \( q \) is the best dominant.

**Lemma 3.** (See [13]) Let \( q \) be convex univalent in \( U \) and \( \zeta \in \mathbb{C} \). Further assume that \( \Re(\zeta) > 0 \). If \( p \in \mathcal{H} \cap Q \) and \( p(z) + \zeta z p'(z) \) is univalent in \( U \), then
\[ q(z) + \zeta z q'(z) \prec p(z) + \zeta z p'(z), \]
which implies that \( q \prec p \) and \( q \) is the best subordinant.
Lemma 4. (See [4]) Let $q$ be convex univalent in $U$, and let $\vartheta$ and $\varphi$ be analytic in the domain $\mathbb{D}$ containing $q(U)$. Suppose that

1. $\Re \left( \frac{\vartheta'(q(z))}{\varphi(q(z))} \right) > 0$ for $z \in U$;
2. $zq'(z)\varphi(q(z))$ is starlike univalent in $U$.

If $p \in H_q(0) \cap Q$ with $p(U) \subseteq \mathbb{D}$, and $\varphi(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $U$, and

$$\varphi(q(z)) + q'(z)\varphi(q(z)) < \varphi(p(z)) + zp'(z)\varphi(p(z)),$$

then $q \prec p$, and $q$ is the best subordinant.

Lemma 5. (See [14]) The function

$$(1 - z)^{\nu} \equiv e^{\nu \log(1 - z)} \quad (\nu \neq 0)$$

is univalent in $U$ if and only if $\nu$ is either in the closed disk $|\nu - 1| \leq 1$ or in the closed disk $|\nu + 1| \leq 1$.

3 Main Results

Firstly, we derive some subordination results involving the integral operator $Q^\lambda_{\alpha,\beta}$.

Throughout this section, without otherwise mentioned, we assume that the parameters $\gamma$, $\mu$, $\sigma$, $\delta$, $a$ and $b$ satisfy the conditions:

$$\gamma \neq 0; \mu \neq 0; \sigma, \delta, a, b \in \mathbb{C} \quad \text{with} \quad a + b \neq 0.$$

Theorem 1. Let $q$ be convex univalent in $U$ with

$$\Re \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\Re \left( \frac{\lambda}{\eta} \right) \right\} \quad (\eta \neq 0). \quad (3.1)$$

If $f \in \Sigma$ satisfies the subordination

$$\eta zQ^\lambda_{\alpha,\beta}f(z) + (1 - \eta)zQ^\lambda_{\alpha,\beta}f(z) \prec q(z) + \frac{\eta zq'(z)}{\lambda}, \quad (3.2)$$

then

$$zQ^\lambda_{\alpha,\beta}f(z) \prec q(z), \quad (3.3)$$

and $q$ is the best dominant.

Proof. Define the function $h$ by

$$h(z) := zQ^\lambda_{\alpha,\beta}f(z). \quad (3.4)$$

Differentiating both sides of (3.4) with respect to $z$ logarithmically, we have

$$\frac{zh'(z)}{h(z)} = 1 + \frac{z(Q^\lambda_{\alpha,\beta}f)'(z)}{Q^\lambda_{\alpha,\beta}f(z)}. \quad (3.5)$$
It now follows from (1.7), (3.2) and (3.5) that
\[ h(z) + \frac{\eta z h'(z)}{\lambda} < q(z) + \frac{\eta q'(z)}{\lambda}. \]

An application of Lemma 1, with \( \gamma = \frac{\lambda}{\eta} \) and \( \psi = 1 \), leads to (3.3).

Taking \( q(z) = \frac{1+A}{1+Az} \) in Theorem 1, we get the following result.

**Corollary 1.** Let \( -1 \leq B < A \leq 1 \) and
\[ \Re \left( \frac{1-Bz}{1+Bz} \right) > \max \left\{ 0, -\Re \left( \frac{\lambda}{\eta} \right) \right\} \quad (\eta \neq 0). \]
If \( f \in \Sigma \) satisfies the subordination
\[ \eta z Q_{\alpha,\beta} f(z) + (1-\eta) z Q_{\alpha+1,\beta} f(z) < \frac{1+Az}{1+Bz} + \frac{\eta (A-B)z}{\lambda (1+Bz)^2}, \]
then
\[ z Q_{\alpha,\beta} f(z) < \frac{1+Az}{1+Bz}, \]
and \( \frac{1+Az}{1+Bz} \) is the best dominant.

In view of (1.8) and Lemma 1, and by similarly applying the method of proof of Theorem 1, we easily get the following results.

**Corollary 2.** Let \( q \) be convex univalent in \( U \) with
\[ \Re \left( \frac{1+q'(z)}{q(z)} \right) > \max \left\{ 0, -\Re \left( \frac{\beta+\alpha}{\eta} \right) \right\} \quad (\eta \neq 0). \] (3.6)
If \( f \in \Sigma \) satisfies the subordination
\[ \eta z Q_{\alpha,\beta} f(z) + (1-\eta) z Q_{\alpha+1,\beta} f(z) \prec \frac{1+Az}{1+Bz} + \frac{\eta q(z)}{\beta+\alpha}, \]
then
\[ z Q_{\alpha+1,\beta} f(z) \prec q(z) \]
and \( q(z) \) is the best dominant.

**Corollary 3.** Let \( -1 \leq B < A \leq 1 \) and
\[ \Re \left( \frac{1-Bz}{1+Bz} \right) > \max \left\{ 0, -\Re \left( \frac{\beta+\alpha}{\eta} \right) \right\} \quad (\eta \neq 0). \]
If \( f \in \Sigma \) satisfies the subordination
\[ \eta z Q_{\alpha,\beta} f(z) + (1-\eta) z Q_{\alpha+1,\beta} f(z) \prec \frac{1+Az}{1+Bz} + \frac{\eta (A-B)z}{\beta+\alpha (1+Bz)^2}, \]
then
\[ z Q_{\alpha+1,\beta} f(z) \prec \frac{1+Az}{1+Bz}, \]
and \( \frac{1+Az}{1+Bz} \) is the best dominant.
Theorem 2. Let \( q \) be univalent in \( \mathbb{U} \). Suppose that \( q \) satisfies

\[
\Re \left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right) > 0.
\]  

(3.7)

Let

\[
\varrho(z) = 1 + \gamma \mu \left( 1 + \frac{az(Q_{\alpha,\beta}^{\lambda+1} f)'(z) + bz(Q_{\alpha,\beta}^{\lambda} f)'(z)}{aQ_{\alpha,\beta}^{\lambda+1} f(z) + bQ_{\alpha,\beta}^{\lambda} f(z)} \right).
\]  

(3.8)

If

\[
\varrho(z) \prec \left( 1 + \gamma \frac{zq'(z)}{q(z)} \right),
\]  

then

\[
\left( \frac{azQ_{\alpha,\beta}^{\lambda+1} f(z) + bzQ_{\alpha,\beta}^{\lambda} f(z)}{a + b} \right)^{\mu} < q(z),
\]  

(3.9)

and \( q \) is the best dominant.

Proof. Let us consider a function \( p \) defined by

\[
p(z) := \left( \frac{azQ_{\alpha,\beta}^{\lambda+1} f(z) + bzQ_{\alpha,\beta}^{\lambda} f(z)}{a + b} \right)^{\mu} \quad (\mu \neq 0; \ a + b \neq 0).
\]  

(3.10)

Now, Differentiating (3.10) logarithmically, we get

\[
\frac{z p'(z)}{p(z)} = \mu \left( 1 + \frac{az(Q_{\alpha,\beta}^{\lambda+1} f)'(z) + bz(Q_{\alpha,\beta}^{\lambda} f)'(z)}{aQ_{\alpha,\beta}^{\lambda+1} f(z) + bQ_{\alpha,\beta}^{\lambda} f(z)} \right).
\]

Setting

\[
\theta(\omega) = 1 \quad \text{and} \quad \phi(\omega) = \frac{\gamma}{\omega},
\]

by observing that \( \theta(\omega) \) is analytic in \( \mathbb{C} \) and that \( \phi(\omega) \neq 0 \) is analytic in \( \mathbb{C} \ \setminus \{0\} \).

Furthermore, we let

\[
Q(z) := zq'(z)\phi(q(z)) = \gamma \frac{zq'(z)}{q(z)},
\]

and

\[
h(z) := \theta(q(z)) + Q(z) = 1 + \gamma \frac{zq'(z)}{q(z)}.
\]

From (3.7), we see that \( Q(z) \) is starlike univalent in \( \mathbb{U} \), and

\[
\Re \left( \frac{zh'(z)}{Q(z)} \right) = \Re \left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right) > 0.
\]

Thus, an application of Lemma 2 to (3.8) yields the desired result. \( \square \)

Putting \( a = 0, \ b = 1, \ \gamma = 1 \) and \( q(z) = \frac{1+Az}{1+Bz} \) in Theorem 2, we obtain the following corollary.
Corollary 4. Let $-1 \leq B < A \leq 1$, $\mu \neq 0$. If $f \in \Sigma$, and

$$1 + \mu \left( 1 + \frac{z(Q_{\alpha,\beta}^\lambda f)'(z)}{Q_{\alpha,\beta}^\lambda f(z)} \right) \prec 1 + \frac{(A - B)z}{(1 + Az)(1 + Bz)},$$

then

$$Q_{\alpha,\beta}^\lambda f(z)^\mu \prec \frac{1 + Az}{1 + Bz}$$

and $\frac{1 + Az}{1 + Bz}$ is the best dominant.

By similarly applying the method of proof of Theorem 2, we easily get the following result.

Corollary 5. Let $q$ be univalent in $U$. Suppose that $q$ satisfies (3.7). Let

$$\chi(z) = 1 + \gamma \mu \left( 1 + \frac{az(Q_{\alpha,\beta}^\lambda f)'(z) + bz(Q_{\alpha+1,\beta}^\lambda f)'(z)}{aQ_{\alpha,\beta}^\lambda f(z) + bQ_{\alpha+1,\beta}^\lambda f(z)} \right). \quad (3.11)$$

If

$$\chi(z) \prec 1 + \frac{zq'(z)}{q(z)},$$

then

$$\left( \frac{azQ_{\alpha,\beta}^\lambda f(z) + bzQ_{\alpha+1,\beta}^\lambda f(z)}{a + b} \right)^\mu \prec q(z)$$

and $q$ is the best dominant.

Theorem 3. Let $q$ be univalent in $U$. Suppose that $q$ satisfies

$$\Re \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\Re \left( \frac{\sigma}{\gamma} \right) \right\}. \quad (3.12)$$

Let

$$\psi(z) = \left( \frac{azQ_{\alpha,\beta}^{\lambda+1} f(z) + bzQ_{\alpha,\beta}^\lambda f(z)}{a + b} \right)^\mu \cdot \left( \sigma + \gamma \mu \left( 1 + \frac{az(Q_{\alpha,\beta}^{\lambda+1} f)'(z) + bz(Q_{\alpha,\beta}^\lambda f)'(z)}{aQ_{\alpha,\beta}^{\lambda+1} f(z) + bQ_{\alpha,\beta}^\lambda f(z)} \right) \right) + \delta \quad (3.13)$$

If

$$\psi(z) \prec \sigma q(z) + \delta + \gamma zq'(z),$$

then

$$\left( \frac{azQ_{\alpha,\beta}^{\lambda+1} f(z) + bzQ_{\alpha,\beta}^\lambda f(z)}{a + b} \right)^\mu \prec q(z),$$

and $q$ is the best dominant.
Proof. Define the function $m$ by
\[
m(z) := \left( \frac{a z Q_{\alpha, \beta}^{\lambda+1}f(z) + b z Q_{\alpha, \beta}^\lambda f(z)}{a + b} \right)^\mu \quad (\mu \neq 0; \ a + b \neq 0).
\]  
(3.15)

Taking the logarithmic differentiation on both sides of (3.15), we get
\[
\frac{zm'(z)}{m(z)} = \mu \left( 1 + \frac{a z (Q_{\alpha, \beta}^{\lambda+1}f(z) + b z (Q_{\alpha, \beta}^\lambda f(z))'}{a Q_{\alpha, \beta}^{\lambda+1}f(z) + b Q_{\alpha, \beta}^\lambda f(z)} \right),
\]
and hence
\[
zm'(z) = \mu m(z) \left( 1 + \frac{a z (Q_{\alpha, \beta}^{\lambda+1}f(z) + b z (Q_{\alpha, \beta}^\lambda f(z))'}{a Q_{\alpha, \beta}^{\lambda+1}f(z) + b Q_{\alpha, \beta}^\lambda f(z)} \right).
\]

Suppose that
\[
\theta(\omega) = \sigma \omega + \delta \quad \text{and} \quad \phi(\omega) = \gamma.
\]
Also let
\[
Q(z) = z q'(z) \phi(q(z)) = \gamma z q'(z),
\]
and
\[
h(z) = \theta(q(z)) + Q(z) = \sigma q(z) + \delta + \gamma z q'(z).
\]

From (3.12), we see that $Q(z)$ is starlike in $U$, and
\[
\Re \left( \frac{zh'(z)}{Q(z)} \right) = \Re \left( \frac{\sigma}{\gamma} + 1 + \frac{z q''(z)}{q'(z)} \right) > 0.
\]

Thus, by Lemma 2, we get the assertion of Theorem 3.

Taking $a = 0$, $b = \gamma = 1$ and $q(z) = \frac{1+A}{1+Bz}$ in Theorem 3, we obtain the following corollary.

Corollary 6. Let
\[
\Re \left( \frac{1 + Az}{1 + Bz} \right) > \max \{0, -\Re(\sigma)\}.
\]

If
\[
(z Q_{\alpha, \beta}^\lambda f(z))^\mu \left[ \sigma + \mu \left( 1 + \frac{z (Q_{\alpha, \beta}^{\lambda+1}f(z) + b z (Q_{\alpha, \beta}^\lambda f(z))'}{Q_{\alpha, \beta}^\lambda f(z)} \right) \right] + \delta < \sigma \frac{1 + Az}{1 + Bz} + \delta + \frac{(A - B)z}{(1 + Bz)^2},
\]
then
\[
(z Q_{\alpha, \beta}^\lambda f(z))^\mu < \frac{1 + Az}{1 + Bz}
\]
and $\frac{1+A}{1+Bz}$ is the best dominant.
By similarly applying the method of proof of Theorem 3, we easily get the following result.

**Corollary 7.** Let \( q \) be univalent in \( U \). Suppose that \( q \) satisfies (3.12) and

\[
\varphi(z) = \left( \frac{a z \mathcal{Q}_{\alpha,\beta}^\lambda f(z) + b z \mathcal{Q}_{\alpha+1,\beta}^\lambda f(z)}{a+b} \right)^\mu. \tag{3.16}
\]

\[
\left[ \sigma + \gamma \mu \left( 1 + \frac{a z (\mathcal{Q}_{\alpha,\beta}^\lambda f)'(z) + b z (\mathcal{Q}_{\alpha+1,\beta}^\lambda f)'(z)}{a \mathcal{Q}_{\alpha,\beta}^\lambda f(z) + b \mathcal{Q}_{\alpha+1,\beta}^\lambda f(z)} \right) \right] + \delta \tag{3.17}
\]

If

\[
\varphi(z) < \sigma q(z) + \delta + \gamma z q'(z),
\]

then

\[
\left( \frac{a z \mathcal{Q}_{\alpha,\beta}^\lambda f(z) + b z \mathcal{Q}_{\alpha+1,\beta}^\lambda f(z)}{a+b} \right)^\mu < q(z),
\]

and \( q \) is the best dominant.

With the aid of Lemma 2 and Lemma 5, we can obtain the following results.

**Theorem 4.** Let \( 0 \leq \rho < 1 \). Suppose that \( \gamma \in \mathbb{C} \) with \( \gamma \neq 0 \) and satisfy either

\[
|2\lambda \gamma (1 - \rho) + 1| \leq 1 \text{ or } |2\lambda \gamma (1 - \rho) - 1| \leq 1.
\]

If \( f \) satisfies

\[
\Re \left( \frac{\mathcal{Q}_{\alpha,\beta}^{\lambda+1} f(z)}{\mathcal{Q}_{\alpha,\beta}^\lambda f(z)} \right) > \rho, \tag{3.18}
\]

then

\[
(z \mathcal{Q}_{\alpha,\beta}^\lambda f(z))^{-\gamma} < \frac{1}{(1-z)^{2\lambda \gamma(1-\rho)}} = q(z),
\]

and \( q \) is the best dominant.

**Proof.** Let

\[
\mathbb{H}(z) = (z \mathcal{Q}_{\alpha,\beta}^\lambda f(z))^{-\gamma}(z \in \mathbb{U}). \tag{3.19}
\]

Combining (1.7), (3.18) and (3.19), we have

\[
1 + \frac{z \mathbb{H}'(z)}{\lambda \gamma \mathbb{H}(z)} = \frac{1 + (1 - 2\rho)z}{1 - z}(z \in \mathbb{U}). \tag{3.20}
\]

If we take

\[
q(z) = \frac{1}{(1-z)^{2\lambda \gamma(1-\rho)}}, \quad \theta(\omega) = 1 \quad \text{and} \quad \phi(\omega) = \frac{1}{\lambda \gamma \omega},
\]

then \( q \) is univalent by the condition of the theorem and Lemma 5. Further, it is easy to show that \( q, \theta(\omega) \) and \( \phi(\omega) \) satisfy the conditions of Lemma 2. Since

\[
Q(z) = zq'(z)\phi(q(z)) = \frac{2(1-\rho)z}{1-z}
\]
is univalent starlike in $\mathbb{U}$ and
\[ h(z) = \theta(q(z)) + Q(z) = \frac{1 + (1 - 2\rho)z}{1 - z} \]
satisfy the conditions of Lemma 2. Thus the result follows from (3.20) immediately. The proof is complete.

**Corollary 8.** Let $0 \leq \rho < 1$ and $\gamma \geq 1$. If $f \in \Sigma$ satisfies the condition (3.18), then
\[ \Re \left( zQ_{\alpha,\beta}^\lambda f(z) \right)^{2\lambda\gamma(1-\rho)} > 2^{-1/\gamma}, \]
and the bound $2^{-1/\gamma}$ is the best possible.

By similarly applying the method of proof of Theorem 4, we easily get the following results.

**Corollary 9.** Let $0 \leq \rho < 1$. Suppose that $\gamma \in \mathbb{C}$ with $\gamma \neq 0$ and satisfy either $|2\gamma(\alpha + \beta)(1 - \rho) + 1| \leq 1$ or $|2\gamma(\alpha + \beta)(1 - \rho) - 1| \leq 1$. If $f$ satisfies
\[ \Re \left( \frac{Q_{\alpha,\beta}^\lambda f(z)}{Q_{\alpha+1,\beta}^\lambda f(z)} \right) > \rho, \quad (3.21) \]
then
\[ (zQ_{\alpha+1,\beta}^\lambda f(z))^\gamma \prec \frac{1}{(1 - z)^{2\gamma(\alpha + \beta)(1 - \rho)}} = q(z), \]
and $q$ is the best dominant.

**Corollary 10.** Let $0 \leq \rho < 1$ and $\gamma \geq 1$. If $f \in \Sigma$ satisfies the condition (3.21), then
\[ \Re \left( zQ_{\alpha+1,\beta}^\lambda f(z) \right)^{2\gamma(\alpha + \beta)(1 - \rho)} > 2^{-1/\gamma}, \]
and the bound $2^{-1/\gamma}$ is the best possible.

In the following, we provide some superordination results involving the integral operator $Q_{\alpha,\beta}^\lambda$.

**Theorem 5.** Let $q$ be convex univalent in $\mathbb{U}$ and $\Re(\eta) > 0$. Also let
\[ zQ_{\alpha,\beta}^\lambda f(z) \in \mathcal{H}[q(0), 1] \cap Q \]
and
\[ \eta zQ_{\alpha,\beta}^{\lambda+1} f(z) + (1 - \eta)zQ_{\alpha,\beta}^\lambda f(z) \]
is univalent in $\mathbb{U}$. If
\[ q(z) + \frac{\eta zq'(z)}{\lambda} \prec \eta zQ_{\alpha,\beta}^{\lambda+1} f(z) + (1 - \eta)zQ_{\alpha,\beta}^\lambda f(z), \quad (3.22) \]
then
\[ q(z) \prec zQ_{\alpha,\beta}^\lambda f(z) \quad (3.23) \]
and $q$ is the best subordinant.
Proof. Let $f \in \Sigma$ and suppose that
\[ \varpi(z) = zQ^\lambda_{\alpha, \beta} f(z). \]
We easily find that
\[ \varpi(z) + \eta z \varpi'(z) = \eta zQ^{\lambda+1}_{\alpha, \beta} f(z) + (1 - \eta)zQ^\lambda_{\alpha, \beta} f(z). \]
(3.24)
Next, by means of (3.22), (3.24) and Lemma 3, we readily arrive at the assertion
(3.23) of Theorem 5.

In view of (1.8) and Lemma 3, and by similarly applying the method of proof
of Theorem 5, we can get the following result.

Corollary 11. Let $q$ be convex univalent in $U$ and $\Re(\eta) > 0$. Also let
\[ zQ^\lambda_{\alpha+1, \beta} f(z) \in \mathcal{H}[q(0), 1] \cap \mathcal{Q} \]
and
\[ \eta zQ^\lambda_{\alpha, \beta} f(z) + (1 - \eta)zQ^\lambda_{\alpha+1, \beta} f(z) \]
is univalent in $U$. If
\[ q(z) + \frac{\eta z q'(z)}{\beta + \alpha} \prec \eta zQ^\lambda_{\alpha, \beta} f(z) + (1 - \eta)zQ^\lambda_{\alpha+1, \beta} f(z), \]
then
\[ q(z) \prec zQ^\lambda_{\alpha+1, \beta} f(z) \]
and $q$ is the best subordinant.

In view of Lemma 4, and by similarly applying the method of proof of Theorem 5, we get the following results.

Corollary 12. Let $q$ be convex univalent in $U$. Also let
\[ \left( \frac{azQ^{\lambda+1}_{\alpha, \beta} f(z) + bzQ^\lambda_{\alpha, \beta} f(z)}{a + b} \right)^\mu \in \mathcal{H}[q(0), 1] \cap \mathcal{Q} \]
and $\varphi$ be defined by (3.8) is univalent in $U$. If
\[ 1 + \gamma \frac{z q'(z)}{q(z)} \prec g(z), \]
then
\[ q(z) \prec \left( \frac{azQ^{\lambda+1}_{\alpha, \beta} f(z) + bzQ^\lambda_{\alpha, \beta} f(z)}{a + b} \right)^\mu, \]
and $q$ is the best subordinant.
Corollary 13. Let $q$ be convex univalent in $U$. Also let

$$\left( \frac{azQ_{\alpha,\beta}^{\lambda}f(z) + bq_{n+1,\beta}f(z)}{a+b} \right)^\mu \in \mathcal{H}[q(0),1] \cap Q$$

and $\chi$ be defined by (3.11) is univalent in $U$. If

$$1 + \gamma \frac{q'(z)}{q(z)} < \chi(z),$$

then

$$q(z) \prec \left( \frac{azQ_{\alpha,\beta}^{\lambda}f(z) + bq_{n+1,\beta}f(z)}{a+b} \right)^\mu,$$

and $q$ is the best subordinant.

Corollary 14. Let $q$ be convex univalent in $U$. Also let

$$zQ_{\alpha,\beta}^{\lambda}f(z) \in \mathcal{H}[q(0),1] \cap Q$$

and $\psi$ be defined by (3.13) is univalent in $U$. If $q$ satisfies

$$\Re \left( \frac{\sigma q'(z)}{\gamma} \right) > 0,$$  
(3.25)

and

$$\sigma q(z) + \delta + \gamma zq'(z) \prec \psi(z),$$

then

$$q(z) \prec \left( \frac{azQ_{\alpha,\beta}^{\lambda+1}f(z) + bq_{n+1,\beta}f(z)}{a+b} \right)^\mu,$$

and $q$ is the best subordinant.

Corollary 15. Let $q$ be convex univalent in $U$. Also let

$$zQ_{\alpha,\beta}^{\lambda}f(z) \in \mathcal{H}[q(0),1] \cap Q$$

and $\varphi$ be defined by (3.16) is univalent in $U$. If $q$ satisfies (3.25) and

$$\sigma q(z) + \delta + \gamma zq'(z) \prec \varphi(z),$$

then

$$q(z) \prec \left( \frac{azQ_{\alpha,\beta}^{\lambda}f(z) + bq_{n+1,\beta}f(z)}{a+b} \right)^\mu,$$

and $q$ is the best subordinant.

Finally, combining the above mentioned subordination and superordination results, we get the following sandwich-type results.
Corollary 16. Let \( q_1 \) and \( q_2 \) be convex univalent in \( U \), and \( \Re(\eta) > 0 \). Suppose that \( q_2 \) satisfies \((3.1)\) and \( zQ_{\alpha,\beta}^\lambda f(z) \in \mathcal{H}[q(0), 1] \cap Q \). Let
\[
\eta zQ_{\alpha,\beta}^{\lambda+1} f(z) + (1 - \eta)zQ_{\alpha,\beta}^\lambda f(z)
\]
is univalent in \( U \). If
\[
q_1(z) + \frac{\eta z q_1'(z)}{\lambda} < \eta zQ_{\alpha,\beta}^{\lambda+1} f(z) + (1 - \eta)zQ_{\alpha,\beta}^\lambda f(z) < q_2(z) + \frac{\eta z q_2'(z)}{\lambda},
\]
then
\[
q_1(z) < zQ_{\alpha,\beta}^\lambda f(z) < q_2(z)
\]
and \( q_1 \) and \( q_2 \) are, respectively, the best subordinant and the best dominant.

Corollary 17. Let \( q_3 \) and \( q_4 \) be convex univalent in \( U \), and \( \Re(\eta) > 0 \). Suppose that \( q_4 \) satisfies \((3.6)\) and \( zQ_{\alpha+1,\beta}^\lambda f(z) \in \mathcal{H}[q(0), 1] \cap Q \). Let
\[
\eta zQ_{\alpha,\beta}^\lambda f(z) + (1 - \eta)zQ_{\alpha+1,\beta}^\lambda f(z)
\]
is univalent in \( U \). If
\[
q_3(z) + \frac{\eta z q_3'(z)}{\beta + \alpha} < \eta zQ_{\alpha,\beta}^\lambda f(z) + (1 - \eta)zQ_{\alpha+1,\beta}^\lambda f(z) < q_4(z) + \frac{\eta z q_4'(z)}{\beta + \alpha},
\]
then
\[
q_3(z) < zQ_{\alpha+1,\beta}^\lambda f(z) < q_4(z)
\]
and \( q_3 \) and \( q_4 \) are, respectively, the best subordinant and the best dominant.

Corollary 18. Let \( q_5 \) be convex univalent and \( q_6 \) be univalent in \( U \). Suppose that \( q_6 \) satisfies \((3.7)\), and
\[
\left(\frac{azQ_{\alpha,\beta}^{\lambda+1} f(z) + b zQ_{\alpha,\beta}^\lambda f(z)}{a + b}\right)^\mu \in \mathcal{H}[q(0), 1] \cap Q.
\]
Let \( q \) be defined by \((3.8)\) is univalent in \( U \). If
\[
1 + \gamma \frac{z q_5'(z)}{q_5(z)} < q(z) < 1 + \gamma \frac{z q_6'(z)}{q_6(z)},
\]
then
\[
q_5(z) < \left(\frac{azQ_{\alpha,\beta}^{\lambda+1} f(z) + b zQ_{\alpha,\beta}^\lambda f(z)}{a + b}\right)^\mu < q_6(z),
\]
and \( q_5 \) and \( q_6 \) are, respectively, the best subordinant and the best dominant.
Corollary 19. Let $q_7$ be convex univalent and $q_8$ be univalent in $U$. Suppose that $q_8$ satisfies (3.7), and

$$
\left( \frac{azQ_{\alpha,\beta}^{\lambda}f(z) + bzQ_{\alpha+1,\beta}^{\lambda}f(z)}{a+b} \right)^{\mu} \in \mathcal{H}[q(0), 1] \cap Q.
$$

Let $\chi$ be defined by (3.11) is univalent in $U$. If

$$
1 + \gamma \frac{q_7'(z)}{q_7(z)} < \chi(z) < 1 + \gamma \frac{q_8'(z)}{q_8(z)},
$$

then

$$
q_7(z) \prec \left( \frac{azQ_{\alpha,\beta}^{\lambda}f(z) + bzQ_{\alpha+1,\beta}^{\lambda}f(z)}{a+b} \right)^{\mu} \prec q_8(z),
$$

and $q_7$ and $q_8$ are, respectively, the best subordinant and the best dominant.

Corollary 20. Let $q_9$ be convex univalent and $q_{10}$ be univalent in $U$. Suppose that $q_9$ satisfies (3.25), $q_{10}$ satisfies (3.12), and

$$
zQ_{\alpha,\beta}^{\lambda}f(z) \in \mathcal{H}[q(0), 1] \cap Q.
$$

Let $\psi$ be defined by (3.13) is univalent in $U$. If

$$
\sigma q_9(z) + \delta + \gamma z q_9'(z) \prec \psi(z) \prec \sigma q_{10}(z) + \delta + \gamma z q_{10}'(z),
$$

then

$$
q_9(z) \prec \left( \frac{azQ_{\alpha,\beta}^{\lambda+1}f(z) + bzQ_{\alpha+1,\beta}^{\lambda}f(z)}{a+b} \right)^{\mu} \prec q_{10}(z),
$$

and $q_9$ and $q_{10}$ are, respectively, the best subordinant and the best dominant.

Corollary 21. Let $q_{11}$ be convex univalent and $q_{12}$ be univalent in $U$. Suppose that $q_{11}$ satisfies (3.25), $q_{12}$ satisfies (3.12), and

$$
zQ_{\alpha+1,\beta}^{\lambda}f(z) \in \mathcal{H}[q(0), 1] \cap Q.
$$

Let $\varphi$ be defined by (3.16) is univalent in $U$. If

$$
\sigma q_{11}(z) + \delta + \gamma z q_{11}'(z) \prec \varphi(z) \prec \sigma q_{12}(z) + \delta + \gamma z q_{12}'(z),
$$

then

$$
q_{11}(z) \prec \left( \frac{azQ_{\alpha,\beta}^{\lambda}f(z) + bzQ_{\alpha+1,\beta}^{\lambda}f(z)}{a+b} \right)^{\mu} \prec q_{12}(z),
$$

and $q_{11}$ and $q_{12}$ are, respectively, the best subordinant and the best dominant.

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