WARPED PRODUCT CR-SUBMANIFOLDS 
OF LP-COSYMPLECTIC MANIFOLDS

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Abstract

In this paper, we study warped product CR-submanifolds of LP-cosymplectic manifolds. We have shown that the warped product of the type $M = N_T \times f N_\perp$ does not exist, where $N_T$ and $N_\perp$ are invariant and anti-invariant submanifolds of an LP-cosymplectic manifold $\tilde{M}$, respectively. Also, we have obtained a characterization result for a CR-submanifold to be locally a CR-warped product.

1 Introduction

The geometry of warped product was introduced by Bishop and O’Neill [1]. These manifolds appear in differential geometric studies in natural way and these are generalization of Riemannian product manifolds and then it was studied by many geometers in different known spaces [2, 5]. Recently, B.Y. Chen has introduced the notion of CR-warped product in Kaehler manifolds and showed that there exist no proper warped product CR-submanifolds in the form $M = N_\perp \times f N_T$ in a Kaehler manifold [3]. Later on, Hasegawa and Mihai proved that warped product CR-submanifolds $N_\perp \times f N_T$ in Sasakian manifolds are trivial where $N_T$ and $N_\perp$ are $\phi$-invariant and anti-invariant submanifolds of Sasakian manifold respectively [5].

Matsumoto [7] introduced the notion of a Lorentzian almost paracontact manifold. Then Mihai and Rosca [8] introduced the same notion and obtained several results in this manifold. Submanifolds of a Lorentzian almost paracontact manifold have been studied by Prasad and Ojha and defined a class of Lorentzian almost paracontact manifold as an LP-cosymplectic manifold in [9].

In view of the physical applications of these manifolds, the question of existence or non existence of warped product submanifolds assumes significance. In the
present paper, we have shown that the warped product in the form $M = N_T \times f N_{\perp}$ is trivial where $N_T$ is an invariant submanifold tangent to $\xi$ and $N_{\perp}$ is an anti-invariant submanifold of an LP-cosymplectic manifold $\bar{M}$. On the other hand we have obtained a characterization result for the warped product of the type $M = N_{\perp} \times f N_T$ when $\xi$ is tangent to $N_{\perp}$. Also, we have shown that there is no warped product $M = N_1 \times f N_2$ when $\xi$ is tangent to $N_2$, where $N_1$ and $N_2$ are submanifolds of an LP-cosymplectic manifold.

2 Preliminaries

Let $\bar{M}$ be a $n$-dimensional Lorentzian almost paracontact manifold with the almost paracontact metric structure $(\phi, \xi, \eta, g)$, that is, $\phi$ is a $(1, 1)$ tensor field, $\xi$ is a contravariant vector field, $\eta$ is a $1-$form and $g$ is a Lorentzian metric with signature $(-, +, +, \cdots, +)$ on $\bar{M}$, satisfying [7]:

$$\phi^2 = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \text{rank}(\phi) = n - 1$$

(2.1)

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi),$$

(2.2)

$$\Phi(X, Y) = g(\phi X, Y) = g(X, \phi Y) = \Phi(Y, X),$$

(2.3)

for all $X, Y \in T\bar{M}$, where $\Phi$ is the fundamental $2-$form defined as above.

A Lorentzian almost contact metric structure on $\bar{M}$ is called a Lorentzian para-cosymplectic structure if $\nabla \phi = 0$, where $\nabla$ denotes the Riemannian connection with respect to $g$. The manifold $\bar{M}$ in this case is called a Lorentzian para-cosymplectic (in brief, an LP-cosymplectic) manifold. From formula $\nabla \phi = 0$, it follows that $\nabla X \xi = 0$.

Let $M$ be a submanifold of a Lorentzian almost paracontact manifold $\bar{M}$ with Lorentzian almost paracontact structure $(\phi, \xi, \eta, g)$. Let the induced metric on $M$ also be denoted by $g$. Then Gauss and Weingarten formulae are given by

$$\nabla_X Y = \nabla_X Y + h(X, Y)$$

(2.4)

$$\nabla_X N = -A_N X + \nabla_X N,$$

(2.5)

for any $X, Y \in TM$ and $N \in T^\perp M$, where $TM$ is the Lie algebra of vector field in $M$ and $T^\perp M$ is the set of all vector fields normal to $M$. $\nabla^\perp$ is the connection in the normal bundle, $h$ the second fundamental form and $A_N$ is the Weingarten endomorphism associated with $N$. It is easy to see that

$$g(A_N X, Y) = g(h(X, Y), N).$$

(2.6)

For any $X \in TM$, we write

$$\phi X = P X + F X,$$

(2.7)

where $P X$ is the tangential component and $F X$ is the normal component of $\phi X$. Similarly for $N \in T^\perp M$, we write

$$\phi N = B N + C N,$$

(2.8)
where $BN$ is the tangential component and $CN$ is the normal component of $\phi N$.

The covariant derivatives of the tensor fields $\phi$, $P$ and $F$ are defined as

\[
(\bar{\nabla}_X \phi)Y = \bar{\nabla}^T_X \phi Y - \phi \bar{\nabla}^T_X Y, \quad \forall \, X, Y \in T\bar{M} \tag{2.9}
\]

\[
(\bar{\nabla}_X P)Y = \nabla^T_X P Y - P \nabla^T_X Y, \quad \forall \, X, Y \in TM \tag{2.10}
\]

\[
(\bar{\nabla}_X F)Y = \nabla^\perp_X FY - F \nabla^T_X Y, \quad \forall \, X, Y \in TM. \tag{2.11}
\]

Moreover, for an LP-cosymplectic manifold we have

\[
(\bar{\nabla}_X P)Y = A FYX + Bh(X, Y), \tag{2.12}
\]

\[
(\bar{\nabla}_X F)Y = Ch(X, Y) - h(X, PY). \tag{2.13}
\]

For submanifolds tangent to the structure vector field $\xi$, there are different classes of submanifolds. We mention the following.

(i) A submanifold $M$ tangent to $\xi$ is called an invariant submanifold if $F$ is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand $M$ is said to be an anti-invariant submanifold if $P$ is identically zero, that is, $\phi X \in T^\perp M$, for any $X \in TM$.

(ii) A submanifold $M$ tangent to $\xi$ is called a contact CR-submanifold if it admits a pair of differentiable distributions $D$ and $D^\perp$ such that $D$ is invariant and its orthogonal complementary distribution $D^\perp$ is anti-invariant i.e., $TM = D \oplus D^\perp \oplus \langle \xi \rangle$ with $\phi(D_x) \subseteq D_x$ and $\phi(D^\perp_x) \subset T^\perp_x M$, for every $x \in M$.

Let $M$ be an $m-$dimensional CR-submanifold of an LP-cosymplectic manifold $\bar{M}$. Then, $F(T_xM)$ is a subspace of $T^\perp_x M$. Thus it follows that $T_xM \oplus F(T_xM)$ is invariant with respect to $\phi$. Then for every $x \in M$, there exists an invariant subspace $\nu_x$ of $T_x\bar{M}$ such that

\[T_x\bar{M} = T_xM \oplus F(T_xM) \oplus \nu_x.\]

### 3 Warped and Doubly Warped Product Submanifolds

Let $(N_1, g_1)$ and $(N_2, g_2)$ be two semi-Riemannian manifolds and $f$, a positive differentiable function on $N_1$. The warped product of $N_1$ and $N_2$ is the manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

\[g = g_1 + f^2 g_2. \tag{3.1}\]

We recall the following general formula on a warped product [1].

\[\nabla_X V = \nabla_X X = (X \ln f)V, \tag{3.2}\]

where $X$ is tangent to $N_1$ and $V$ is tangent to $N_2$. 

Let $M = N_1 \times f N_2$ be a warped product manifold, this means that $N_1$ is totally geodesic and $N_2$ is totally umbilical submanifold of $M$, respectively.

Doubly warped product manifolds were introduced as a generalization of warped product manifolds by B. Ünal [10]. A doubly warped product manifold of $N_1$ and $N_2$, denoted as $f_2 N_1 \times f_1 N_2$ is endowed with a metric $g$ defined as

$$g = f_2^2 g_1 + f_1^2 g_2$$

where $f_1$ and $f_2$ are positive differentiable functions on $N_1$ and $N_2$ respectively.

In this case formula (3.2) is generalized as

$$\nabla_X Z = (X \ln f_1) Z + (Z \ln f_2) X$$

for each $X \in TN_1$ and $Z \in TN_2$ [10].

If neither $f_1$ nor $f_2$ is constant we have a non trivial doubly warped product $M = f_2 N_1 \times f_1 N_2$. Obviously in this case both $N_1$ and $N_2$ are totally umbilical submanifolds of $M$.

We now consider a doubly warped product of two semi-Riemannian manifolds $N_1$ and $N_2$ embedded into an LP-cosymplectic manifold $\bar{M}$ such that the structure vector field $\xi$ is tangential to the submanifold $M = f_2 N_1 \times f_1 N_2$.

**Theorem 3.1.** There does not exist a proper doubly warped product submanifold in LP-cosymplectic manifolds.

**Proof.** Let $M = f_2 N_1 \times f_1 N_2$ be a doubly warped product submanifold of an LP-cosymplectic manifold $\bar{M}$, where $N_1$ and $N_2$ are submanifolds of $M$. We have using Gauss formula and the fact that $\bar{M}$ is LP-cosymplectic, for any $U \in TM$

$$\nabla_U \xi = 0.$$  \hspace{1cm} (3.5)

Thus in case $\xi \in TN_1$ and $U \in TN_2$ equation (3.4) and (3.5) imply that $(\xi \ln f_1) U + (U \ln f_2) \xi = 0$, which shows that $f_2$ is constant. Similarly, for $\xi \in TN_2$ and $U \in TN_1$, we have $(\xi \ln f_2) U + (U \ln f_1) \xi = 0$, showing that $f_1$ is constant. This completes the proof. \hfill \Box

In above theorem we see that $f_2$ is constant if the structure vector field $\xi$ is tangent to $N_1$ and $f_1$ is constant if the structure vector field $\xi$ is tangent to $N_2$. The following corollary is an immediate consequence of the above theorem.

**Corollary 3.1.** There does not exist a warped product submanifold $N_1 \times f N_2$ of an LP-cosymplectic manifold $\bar{M}$ such that $\xi$ is tangent to $N_2$.

Thus the only remaining case to study is the warped product submanifold $N_1 \times f N_2$ with structure vector field $\xi$ tangential to $N_1$, we first obtain some useful formulae for later use.

**Lemma 3.1.** Let $M = N_1 \times f N_2$ be a proper warped product submanifold of an LP-cosymplectic manifold $\bar{M}$ such that $\xi$ is tangent to $N_1$, where $N_1$ and $N_2$ are submanifolds of $M$. Then
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(i) $\xi \ln f = 0$,
(ii) $A_{FZ}X = -Bh(X, Z)$,
(iii) $g(h(X, Y), FZ) = -g(h(X, Z), FY)$,
(iv) $g(h(X, Z), FW) = -g(h(X, W), FZ)$

for any $X, Y \in TN_1$ and $Z, W \in TN_2$.

Proof. The first part of the lemma is an immediate consequence of the fact that $\bar{\nabla}_U \xi = 0$, for $U \in TM$ and using formula (2.4) and separating the tangential and normal parts. Now, for any $X \in TN_1$ and $Z \in TN_2$, then formula (2.12) gives

$$\overline{\nabla}_X P Z = A_{FZ}X + Bh(X, Z). \quad (3.6)$$

Also, we have

$$\overline{\nabla}_X P Z = \nabla_X P Z - P \nabla_X Z = (X \ln f)PZ - P(X \ln f)Z = 0, \quad (3.7)$$

for any $X \in TN_1$ and $Z \in TN_2$. Part (ii) follows by equations (3.6) and (3.7). Parts (iii) and (iv) follow by taking the product in (ii) by $Y$ and $W$ respectively. \qed

4 CR-Warped Product Submanifolds

Throughout this section the structure vector field $\xi$ is either tangent to the invariant submanifold $N_T$ or tangent to the anti-invariant submanifold $N_\perp$. There are two types of warped product in an LP-cosymplectic manifold $\bar{M}$, namely $N_T \times_f N_\perp$ and $N_\perp \times_f N_T$ are called CR-warped product submanifolds with $\xi$ tangent to $N_T$ and $N_\perp$, respectively. The following theorem is dealt with the case when $\xi$ is tangent to $N_T$.

**Theorem 4.1.** There does not exist a proper warped product submanifold $N_T \times_f N_\perp$ where $N_T$ is an invariant and $N_\perp$ is an anti-invariant submanifolds of an LP-cosymplectic manifold $\bar{M}$ such that $\xi$ is tangent to $N_T$.

Proof. Let $M = N_T \times_f N_\perp$ be a warped product CR-submanifold of an LP-cosymplectic manifold $\bar{M}$ with $\xi \in TN_T$ then from equations (2.2), (2.4) and the fact that $M$ is an LP-cosymplectic, we have

$$g(\nabla_X Z, W) = g(\overline{\nabla}_Z X, W) = g(\overline{\nabla}_Z \phi X, \phi W) = g(\phi \overline{\nabla}_Z X, \phi W)$$

for any $X \in TN_T$ and $Z \in TN_\perp$. Using (3.2), we get

$$(X \ln f)g(Z, W) = g(\overline{\nabla}_Z \phi X, \phi W) = g(\overline{\nabla}_Z \phi X + h(Z, \phi X), \phi W),$$

or

$$(X \ln f)g(Z, W) = g(h(Z, \phi X), \phi W) + (\phi X \ln f)g(Z, \phi W) = g(h(Z, \phi X), \phi W).$$
That is,
\[(X \ln f)g(Z, W) = g(h(Z, \phi X), \phi W). \quad (4.1)\]
Again, we have
\[g(h(Z, \phi X), \phi W) = g(\nabla_{\phi X} Z, \phi W). \quad (4.2)\]
Making use of equations (2.3), (2.5), (2.6) and (2.10) we deduce from (4.2) that
\[g(h(Z, \phi X), \phi W) = -g(h(\phi X, W), \phi Z). \quad (4.3)\]
Interchanging \(Z\) and \(W\) in (4.1) and then adding the resulting equation in (4.1), we get
\[2(X \ln f)g(Z, W) = g(h(Z, \phi X), \phi W) + g(h(\phi X, W), \phi Z). \quad (4.4)\]
Using (4.3), we obtain
\[(X \ln f)g(Z, W) = 0, \quad (4.5)\]
for all \(X \in TN_T\) and \(Z, W \in TN_{\perp}\). As \(N_{\perp} \neq \{0\}\) anti-invariant submanifold then equation (4.4) and Lemma 3.1 (i) imply that \(f\) is constant on\(N_T\), proving the result. \(\square\)

Now, the other case i.e., \(N_{\perp} \times fN_T \) with \(\xi\) is tangent to \(N_{\perp}\).

**Lemma 4.1.** Let \(M = N_{\perp} \times fN_T\) be a warped product submanifold of an LP-cosymplectic manifold \(\bar{M}\). Then
\[g(h(X, \phi Y), \phi Z) = -(Z \ln f)g(X, Y), \quad (4.6)\]
for any \(X, Y \in TN_T\) and \(Z \in TN_{\perp}\).

**Proof.** For any \(X, Y \in TN_T\) and \(Z \in TN_{\perp}\), by formula (3.2) we have
\[g(\nabla_X Y, Z) = g(\nabla_X Y, Z) = -(Z \ln f)g(X, Y). \quad (4.7)\]
Now, for any \(X, Y \in TN_T\) and \(Z \in TN_{\perp}\), consider
\[g(\nabla_X Y, Z) = g(\phi \nabla_X Y, \phi Z) = g(\nabla_X \phi Y, \phi Z) = g(h(X, \phi Y), \phi Z), \quad (4.8)\]
i.e.,
\[g(\nabla_X Y, Z) = g(h(X, \phi Y), \phi Z). \quad (4.9)\]
Thus equation (4.5) follows by (4.6) and (4.7). This completes the proof of the lemma. \(\square\)

**Theorem 4.2.** Let \(M\) be a CR-submanifold of an LP-cosymplectic manifold \(\bar{M}\). Then \(M\) is locally a contact CR-warped product if and only if
\[A_{\phi Z} X = -Z(\mu)\phi X, \quad X \in D, \quad Z \in D^\perp \oplus \{\xi\}. \quad (4.10)\]
for some function $\mu$ on $M$ satisfying $W'(\mu) = 0$ for each $W' \in \mathcal{D}$.

**Proof.** If $M = N_\perp \times fN_T$ is CR-warped product submanifold, then on applying Lemma 4.1, we obtain (4.8). In this case $\mu = \ln f$.

Conversely, suppose $M$ is CR-submanifold of $\tilde{M}$ and satisfying

$$A_{\phi Z}X = -Z(\mu)\phi X,$$

then

$$g(h(X, X), \phi Z) = g(A_{\phi Z}X, X) = -Z(\mu)g(\phi X, X) = 0$$

i.e., $h(X, Y) \in \nu$ the orthogonal complementary distribution of $\phi(\mathcal{D}^\perp \oplus \langle \xi \rangle)$. On the other hand, for any $X \in TN_T$ and $Z, W \in TN_\perp$ we have

$$g(\nabla_W Z, \phi X) = g(\nabla_Z W, \phi X).$$

As $g$ is Lorentzian and $\tilde{M}$ is LP-cosymplectic, the above equation takes the form

$$g(\nabla_W Z, \phi X) = -g(\nabla_W \phi Z, X).$$

Thus, on using (2.5) and (2.6) we get

$$g(\nabla_W Z, \phi X) = g(A_{\phi Z}W, X) = g(h(X, W), \phi Z).$$

Also, by (2.4) we have

$$g(h(X, W), \phi Z) = g(\nabla_X W, \phi Z)$$

$$= -g(\nabla_X \phi Z, W)$$

$$= g(A_{\phi Z}X, W).$$

Using (4.8) in above, we get

$$g(\nabla_W Z, \phi X) = -(Z\mu)g(\phi X, W) = 0.$$ 

This means that $\mathcal{D}^\perp \oplus \langle \xi \rangle$ is integrable and its leaves are totally geodesic in $M$.

Also, we have

$$g(\nabla_{XY}, Z) = g(\nabla_X Y, Z) = -g(\nabla_X Z, Y) = -g(\nabla_X \phi Z, \phi Y)$$

$$= g(A_{\phi Z}X, \phi Y) = -Z(\mu)g(\phi X, \phi Y) = -Z(\mu)g(X, Y)$$

i.e.,

$$g(\nabla_{XY}, Z) = -Z(\mu)g(X, Y)$$ \hspace{1cm} (4.9)

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp \oplus \langle \xi \rangle$. Now, by Gauss formula

$$g(h'(X, Y), Z) = g(\nabla_{XY}, Z)$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp \oplus \langle \xi \rangle$. Now, by Gauss formula
where $h'$ denotes the second fundamental form of the immersion of $N_T$ into $M$. On using (4.9), the last equation gives

$$g(h'(X,Y),Z) = -Z(\mu)g(X,Y)$$

which shows that each leaf of $N_T$ of $D$ is totally umbilical in $M$. Moreover the fact that $W'\mu = 0$ for all $W' \in D$, implies that the mean curvature vector on $N_T$ is parallel along $N_T$ i.e., each leaf of $D$ is an extrinsic sphere in $M$. Hence by virtue of a result in [6] which states that "If the tangent bundle of a Riemannian manifold $M$ splits into an orthogonal sum $TM = E_0 \oplus E_1$ of non trivial vector sub bundles such that $E_1$ is spherical and its orthogonal complement $E_0$ is auto parallel, then the manifold $M$ is locally isometric to a warped product $M_0 \times_f M_1^1$", we get that, $M$ is locally a warped $N_\perp \times_f N_T$ of a holomorphic submanifold $N_T$ and a totally real submanifold $N_\perp$ of $M$. Here $N_T$ is a leaf of $D$ and $N_\perp$ is a leaf of $D^\perp \oplus \langle \xi \rangle$ and $f$ is a warping function. □

Acknowledgement. The author is grateful to referees and professor Angelina Chin Yan Mui for their valuable suggestions and comments.

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