Abstract

In this paper we introduce \((I, \gamma)\)-generalized semi-closed sets in topological spaces and also introduce \(\gamma S - T \gamma\)-spaces and investigate some of their properties.

1. Introduction

Recently Julian Dontchev et al. [1] introduce \((I, \gamma)\)-generalized closed sets via topological ideals. In this paper we introduce \((I, \gamma)\)-generalized semi-closed sets and investigate some of their properties.

An ideal \(I\) on a topological space \((X, \tau)\) is a non-empty collection of subsets of \(X\) satisfying the following two properties:

1. \(A \in I\) and \(B \subset A\) implies \(B \in I\)
2. \(A \in I\) and \(B \in I\) implies \(A \cup B \in I\)

For a subset \(A \subset X\), \(A^{*}(I) = \{x \in X/ U \cap A \notin I \text{ for each neighbourhood } U \text{ of } x\}\) is called the local function of \(A\) with respect to \(I\) and \(\tau\). Recall that \(A \subseteq (X, \tau, I)\) is called \(\tau^{*}\)-closed [2] if \(A^{*} \subseteq A\). It is well known that \(Cl^{*}(A) = A \cup A^{*}\) defines a Kuratowski closure operator for a topology \(\tau^{*}(I)\), finer than \(\tau\). An operation \(\gamma\) [3, 6] on the topology \(\tau\) on a given topological space \((X, \tau)\) is a function from the topology itself into the power set \(P(X)\) of \(X\) such that \(V \subseteq V^{*}\) for each \(V \in \tau\), where \(V^{*}\) denotes the value of \(\gamma\) at \(V\).

The following operators are examples of the operation \(\gamma\): the closure operator \(\gamma_{cl}\) defined by \(\gamma(U) = cl(U)\), the identity operator \(\gamma_{id}\) defined by \(\gamma(U) = U\). Another example of the operation \(\gamma\) is the \(\gamma_{f}\)-operator defined by \(U^{*} = (FrU)^{c} = X/ FrU\) [7]. Two operators \(\gamma_{1}\) and \(\gamma_{2}\) are called mutually dual [7] if \(U^{*1} \cap U^{*2} = U\) for each \(U \in \tau\). For example the identity operator is mutually dual to any other operator, while the \(\gamma_{f}\)-operator is mutually dual to the closure operator [7].

Definition: A subset \(A\) of a space \((X, \tau)\) is called

(a) an \(\alpha\)-open set [5] if \(A \subseteq Int(cl(int(A)))\).

2010 Mathematics Subject Classifications. 54B05, 54C08, 54D05.
Key words and Phrases. \((I, \gamma)\)-\(g\)-closed set, \((I, \gamma)\)-\(gs\)-closed sets, \(\gamma S - T \gamma\)-spaces.
Received: August 17, 2009
Communicated by Ljubisa Kocinac
(b) a generalized closed (briefly g-closed) set \[4\] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

(c) a \((I, \gamma)\)-generalized closed (briefly \((I, \gamma)\)-g-closed) set \([1]\) if \( A^* \subseteq U^\gamma \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

We denote the family of all \((I, \gamma)\)-generalized semi-closed subsets (briefly \((I, \gamma)\)-gs-closed) of a space \((X, \tau, I, \gamma)\) by \( \text{IGS}(X) \) and simply write \( I\text{-gs-closed}\) in case when \( \gamma \) is an identity operator. Throughout this paper the operator \( \gamma \) is defined as \( \gamma: \tau^* \rightarrow P(X) \), where \( \tau^* \) denotes the set of all semi-open sets of \((X, \tau)\).

2. Basic properties of \((I, \gamma)\)-generalized semi-closed sets

Definition 2.1. A subset \( A \) of a topological space \((X, \tau)\) is called \((I, \gamma)\)-generalized semi closed (briefly \((I, \gamma)\)-gs-closed) if \( A^* \subseteq U^\gamma \), whenever \( A \subseteq U \) and \( U \) is semi-open in \((X, \tau)\).

Example 2.2. Let \( X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} \) and \( I = \{\{a\}, \{a, b\}\} \). Here \((I, \gamma)\)-gs-closed sets are \( X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\).

Theorem 2.3. Every \((I, \gamma)\)-gs-closed set is \((I, \gamma)\)-g-closed set.

Proof. Let \( A \subseteq U, U \) is open and hence it is semi-open. Since \( A \) is \((I, \gamma)\)-gs-closed, \( A^* \subseteq U^\gamma \). Hence \( A \) is \((I, \gamma)\)-g-closed.

Remark 2.4. The converse of the above theorem need not be true by the following example.

Example 2.5. Let \( X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\} \) and \( I = \{a\} \). Let \( \gamma_1: \tau^* \rightarrow P(X) \) and \( \gamma_2: \tau \rightarrow P(X) \) be defined by \( U^\gamma_1 = \text{cl} \ U \) and \( U^\gamma_2 = \text{cl} \ U \) respectively. Therefore \( A = \{b, c\} \) is \((I, \gamma)\)-g-closed but not \((I, \gamma)\)-gs-closed.

Theorem 2.6. If \( A \) is \( I\text{-gs-closed} \) and semi-open, then \( A \) is \( \tau^*\)-closed.

Proof. Since \( A \) is \( I\text{-gs-closed} \), then \( A^* \subseteq U \), \( U \) is semi-open. It is given that \( A \) is semi-open implies \( A^* \subseteq U = A \), this implies that \( A^* \subseteq A \). Hence \( A \) is \( \tau^*\)-closed.

Theorem 2.7. Let \((X, \tau, I, \gamma)\) be a topological space.

(i) If \((A_i)_{i \in I}\) is a locally finite family of sets and each \( A_i \in \text{IGS}(X) \), then \( \bigcup_{i \in I} A_i \in \text{IGS}(X) \)

(ii) Finite intersection of \((I, \gamma)\)-gs-closed sets need not be \((I, \gamma)\)-gs-closed.

Proof:

(i) Let \( \bigcup_{i \in I} A_i \subseteq U \), where \( U \in \tau^* \). Since \( A_i \in \text{IGS}(X) \) for each \( i \in I \), then \( A_i^* \subseteq U^\gamma_i \). Hence \( \bigcup_{i \in I} A_i^* \subseteq U^\gamma \). But we know that \( \bigcup_{i \in I} A_i^* = \bigcup_{i \in I} A_i^* \); therefore \( \bigcup_{i \in I} A_i \in \text{IGS}(X) \)

(ii) Let \( X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( I = \{\{a\}, \{a, b\}\} \). Set \( A = \{a, b\} \) and \( B = \{b, c\} \), clearly \( A, B \in \text{IGS}(X) \) but \( A \cap B = \{b\} \notin \text{IGS}(X) \).

Theorem 2.8. Let \((X, \tau, I, \gamma_{id})\) be a space. If \( A \subseteq X \) is \( I\text{-gs-closed} \) and \( B \) is closed and \( \tau^*\)-closed, then \( A \cap B \) is \( I\text{-gs-closed} \).

Proof. Let \( U \in \tau^* \) be such that \( A \cap B \subseteq U \). Then \( A \subseteq U \cup (X/B) \). Since \( A \) is \( I\text{-gs-closed} \), then \( A^* \subseteq U \cup (X/B) \). Hence \( B \cap A^* \subseteq U \cap B \subseteq U \), But we know that
$B^* \subseteq B$. Therefore $(A \cap B)^* \subseteq A^* \cap B^* \subseteq A^* \cap B \subseteq U$, since $B$ is $\tau^*$-closed. Hence $A \cap B$ is $I$-gs-closed.

**Result 2.9.** A subset $S$ of a space $(X, \tau, I)$ is a topological space with an ideal $I = \{I \cap S : I \in I\}$ on $S$.

**Theorem 2.10.** Let $A \subseteq S \subseteq (X, \tau, I, \gamma_{id})$. If $A$ is $I_{gs}$-closed in $(S, \tau/s, I_s, \gamma_{id})$ and $S$ is closed in $(X, \tau)$, then $A$ is $I$-gs-closed in $(X, \tau, I, \gamma_{id})$.

**Proof:** Let $A \subseteq U$, where $U \in \tau^*$. Let $x \notin U$. We consider the following two cases.

- **Case (i)** $x \in S$. By assumption, $A^*(I_s, \tau/s) \subseteq U \cap S \subseteq U$. We show that $A^*(I) \subseteq A^*(I_s, \tau/s)$. Let $x \notin A^*(I_s, \tau/s)$. Since $x \in S$, then for some open subset $V_s$ of $(S, \tau/s)$ containing $x$, we have $V_s \cap A \in I_s$. Hence $V_s = V \cap S$ for some $V \in \tau$, then $(S \cap V) \cap A = V \cap A \in I_s$ is disjoint from $A$. This shows that $x \notin A^*(I)$. Hence $A^*(I) \subseteq U$.

- **Case (ii)** $x \notin S$. Then $X/S$ is an open neighbourhood of $x$ disjoint from $A$. Hence $x \notin A^*(I)$. Consequently $A^*(I) \subseteq U$.

Both cases show that the local function of $A$ with respect to $I$ and $\tau$ is in $U$. Hence $A$ is $I$-gs-closed in $(X, \tau, I, \gamma_{id})$.

**Theorem 2.11.** Let $A \subseteq S \subseteq (X, \tau, I, \gamma)$, if $A \in (IGS(X))$ and $S \in \tau$, then $A \in IGS(S)$.

**Proof:** Let $U$ be a semi-open subset of $(S, \tau/s)$ such that $A \subseteq U$. Since $S \in \tau$, then $U \in \tau^*$. Then $A^*(I) \subseteq U^=\tau^*$, since $A \in IGS(X)$. We show that $A^*(I_s, \tau/s) \subseteq A^*(I)$. Let $x \notin A^*(I)$. We assume that $x \in S$, since otherwise we are done. Now, for some $V \in \tau$ containing $x$, $V \cap S \subseteq I$. Moreover, $V \cap A \in I_s$, since $A \subseteq S$. Then $V \cap S$ is an open neighbourhood of $x$ in $(S, \tau/s)$ such that $(V \cap S) \cap A = V \cap A \in I_s$. This shows that $x \notin A^*(I_s, \tau/s)$. Hence $A^*(I_s, \tau/s) \subseteq U^=\tau^*$, where $U^=\tau^*$ means the image of the operation $\gamma/s : \tau^* \to P(S)$ defined by, $(\gamma/s)(U) = \gamma(U) \cap S$ for each $U \in \tau^*$. Hence $A \in IGS(S)$.

**Theorem 2.12.** Let $A$ be a subset of $(X, \tau, I, \gamma_{id})$. If $A$ is $I$-gs-closed, then $A^*/A$ does not contain any non-empty semi-closed subset.

**Proof:** Assume that $F$ is semi-closed subset of $A^*/A$. Clearly $A \subseteq X/F$, where $A$ is $I$-gs-closed and $X/F \in \tau^*$. This $A^* \subseteq X/F$, that is $F \subseteq X/A^*$. Since due to our assumption $F \subseteq A^*$, $F \subseteq (X/A^*) \cap A^* = \emptyset$.

**Theorem 2.13.** If the set $A \subseteq (X, \tau, I)$ is both $(I, \gamma_1)$-gs-closed and $(I, \gamma_2)$-gs-closed, then it is $I$-gs-closed, granted the operators $\gamma_1$ and $\gamma_2$ are mutually dual.

**Proof:** Let $A \subseteq U$, where $U \in \tau^*$. Since $A^* \subseteq U^\gamma_{\gamma_1}$ and $A^* \subseteq U^\gamma_{\gamma_2}$, then $A^* \subseteq U^\gamma_{\gamma_1} \cap U^\gamma_{\gamma_2} = U$. Since $\gamma_1$ and $\gamma_2$ are mutually dual. Hence $A$ is $I$-gs-closed.

**Theorem 2.14.** Every set $A \subseteq (X, \tau, I)$ is $(I, \gamma_{id})$-gs-closed.

**Proof:** Let $A \subseteq U, U$ is semi-open. We know that $A \cup A^* = cl^*(A) \subseteq cl(A) \subseteq cl(U)$. This implies that $A^* \subseteq cl(U)$. Hence $A$ is $(I, \gamma_{id})$-gs-closed.

**Corollary 2.15.** For a set $A \subseteq (X, \tau, I)$, the following conditions are equivalent.

(i) $A$ is $(I, \gamma_{id})$-gs-closed.

(ii) $A$ is $I$-gs-closed.

**Proof:**

(i) $\Rightarrow$ (ii), By the above theorem, $A$ is $(I, \gamma_{id})$-gs-closed. Since $\gamma_{id}$ and $\gamma_{id}$ are mutually dual due to [7], then $\gamma_{id}(U) \cap \gamma_{id}(U) = U$. This implies that $A^* \subseteq U$, that is, $A$ is $I$-gs-closed.
(ii) $\Rightarrow$ (i), Let $A \subseteq U$, $U$ is semi-open. Since $A$ is $I$-gs-closed, $A^* \subseteq U$. But we know that $U \subseteq U^{\gamma f}$, we have $A^* \subseteq U \subseteq U^{\gamma f}$. This implies that $A^* \subseteq U^{\gamma f}$. Therefore $A$ is $(I, \gamma_f)$-gs-closed.

3. $\gamma S - T_I$-space

**Definition 3.1.** A space $(X, \tau, I, \gamma)$ is called an $\gamma S - T_I$-space if every $(I, \gamma)$-gs-closed subset of $X$ is $\tau^*$-closed. We use the simple notation $ST_I$-space, in case $\gamma$ is the identity operator.

**Theorem 3.2.** For a space $(X, \tau, I)$, the following conditions are equivalent.

(i) $X$ is a $ST_I$-space

(ii) Each singleton of $X$ is either semi-closed or $\tau^*$-open.

**Proof:** (i) $\Rightarrow$ (ii), Let $x \in X$. If $\{x\}$ is not semi-closed, then $A = X \setminus \{x\} \notin \tau^*$ and then $A$ is trivially $I$-gs-closed. By (i) $A$ is $\tau^*$-closed and $\{x\}$ is $\tau^*$-open.

(ii) $\Rightarrow$ (i), Let $A$ be $I$-gs-closed and let $x \in \text{cl}^* A$. We have the following two cases.

case(i): $\{x\}$ is semi-closed. By theorem 2.12, $A^*/A$ does not contain a non-empty semi-closed subset. This shows that $x \in A$.

case(ii): $\{x\}$ is $\tau^*$-open. Then $\{x\} \cap A \neq \emptyset$. Hence $x \in A$. Thus in both cases $x$ is in $A$ and so $A = \text{cl}^* A$. that is $A$ is $\tau^*$-closed, which shows that $X$ is a $ST_I$-space.

**References**


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