LACUNARY SERIES IN MIXED NORM SPACES ON THE BALL AND THE POLYDISK

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Abstract

We characterize lacunary series in mixed norm spaces on the unit ball $B^n$ in $\mathbb{C}^n$ and on the unit polydisk $D^n$ in $\mathbb{C}^n$.

Introduction and main results

Let $n$ be a positive integer. Two domains will be used in the paper: the open unit ball $B^n$ in $\mathbb{C}^n$,

$$B^n = \{ z \in \mathbb{C}^n : |z| < 1 \},$$

and the open unit polydisk $D^n$ in $\mathbb{C}^n$,

$$D^n = \{ z = (z_1, ..., z_n) \in \mathbb{C}^n : |z_1| < 1, ..., |z_n| < 1 \}.$$

We write $D = D^1 = B^1$.

Denote by $T^n$ the Shilov boundary of $D^n$, by $\partial B^n$ the boundary of $B^n$, by $d\sigma_n$ the normalized surface measure on $\partial B^n$, and define the measure $d\mu_n$ on $T^n$ by

$$d\mu_n(e^{i\theta_1}, ..., e^{i\theta_n}) = d\theta_1 \cdots d\theta_n.$$

Lacunary series on the unit ball $B^n$

The mixed norm space $H^{p,q,\alpha}(B^n)$, $0 < p, q \leq \infty$ $0 < \alpha < \infty$, consists of all functions $f$ holomorphic in $B^n$, $f \in H(B^n)$, such that

$$\| f \|^q_{p,q,\alpha} = \int_0^1 (1 - r)^{q\alpha - 1} M_p(r, f)^q dr < \infty, \quad \text{if} \quad 0 < q < \infty,$$

and

$$\| f \|_{p,\infty,\alpha} = \sup_{0 < r < 1} (1 - r)^\alpha M_p(r, f) < \infty.$$
Here, as usual, 
\[ M_p(r, f) = \left( \int_{\partial B^n} |f(r\xi)|^p d\sigma_n(\xi) \right)^{1/p}, \quad 0 < p < \infty, \]
and 
\[ M_\infty(r, f) = \sup_{|\xi|=1} |f(r\xi)|. \]
We write \( ||f||_p = \sup_{0<r<1} M_p(r, f) \).

Note that when \( 0 < p = q < \infty \), then \( H^{p,p,(\alpha+1)/p}(\mathbb{B}^n) \), where \( \alpha > -1 \), coincides, as a topological linear space, with the weighted Bergman space \( A^{p,\alpha}(\mathbb{B}^n) \), consisting of those \( f \in H(\mathbb{B}^n) \) for which 
\[ \int_{\mathbb{B}^n} |f(z)|^p (1 - |z|^2)^\alpha dV_n(z) < \infty, \]
where \( dV_n \) is the normalized volume measure on \( \mathbb{B}^n \).

We say that a holomorphic function \( f \) on \( \mathbb{B}^n \) has a lacunary expansion if its homogeneous expansion is of the form 
\[ f(z) = \sum_{k=1}^{\infty} f_{m_k}(z), \]
where \( m_k \) satisfies the condition 
\[ \inf_{1 \leq k < \infty} \frac{m_{k+1}}{m_k} = \lambda > 1. \]
The series \( \sum_{k=1}^{\infty} f_{m_k}(z) \) as well as the sequence \( \{m_k\} \) are then said to be lacunary.

In this paper we characterize holomorphic functions with lacunary expansions in mixed norm spaces \( H^{p,q,\alpha}(\mathbb{B}^n) \). More precisely, we prove

**Theorem 1.** Let \( 0 < p, q \leq \infty \), \( 0 < \alpha < \infty \) and let \( f(z) = \sum_{k=1}^{\infty} f_{m_k}(z) \) be a holomorphic function on \( \mathbb{B}^n \) with a lacunary expansion. Then \( f \in H^{p,q,\alpha}(\mathbb{B}^n) \) if and only if
\[ \sum_{k=1}^{\infty} ||f_{m_k}||_p^q m_k^\alpha < \infty \quad \text{if} \quad 0 < q < \infty, \]
or
\[ \sup_{1 \leq k < \infty} m_k^{-\alpha} ||f_{m_k}||_p < \infty, \quad \text{if} \quad q = \infty. \]

Lacunary series in \( H^{p,q,\alpha}(\mathbb{B}^n) \) are characterized in [MP]. (See also [JP]).

Our work was motivated by characterizations of lacunary series in weighted Bergman spaces \( A^{p,\alpha}(\mathbb{B}^n) \), see [Ch], [YO], and [St]. Case \( q = \infty \) of Theorem 1 also follows from [ZZ, Proposition 63]. We note that in [St] lacunary series in mixed norm spaces \( H^{p,q,\alpha}(\mathbb{B}^n) \) are considered and some partial results have been obtained.
Lacunary series on the unit polydisk in $\mathbb{C}^n$

For any Lebesgue measurable function $f$ in $\mathbb{D}^n$, we define

$$M_p(r, f) = \left( \int_{\mathbb{D}^n} |f(r\xi)|^p d\mu_n(\xi) \right)^{1/p}, \quad 0 < p < \infty,$$

and

$$M_\infty(r, f) = \sup_{\xi \in \mathbb{T}^n} |f(r\xi)|,$$

where $r = (r_1, ..., r_n)$.

If $0 < p \leq \infty$, $0 < q < \infty$, and $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_j > 0$, $j = 1, ..., n$, let

$$| | f | |_{p, q, \alpha} = \int_{\mathbb{D}^n} \left( \prod_{j=1}^n (1 - r_j)^{q\alpha_j - 1} M_p(r, f)^q \right) dr,$$

where $I^n = [0, 1)^n$ and $dr = dr_1 \cdots dr_n$. The mixed norm space $H^{p, q, \alpha}(\mathbb{D}^n)$ is then defined to be the space of functions $f$ holomorphic in $\mathbb{D}^n$, $f \in H(\mathbb{D}^n)$, such that $| | f | |_{p, q, \alpha} < \infty$.

The mixed norm space $H^{p, \infty, \alpha}(\mathbb{D}^n)$, $0 < p \leq \infty$, $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_1 > 0$, ..., $\alpha_n > 0$, is the set of those functions $f \in H(\mathbb{D}^n)$ for which

$$| | f | |_{p, \infty, \alpha} = \sup_{r \in I^n} \prod_{j=1}^n (1 - r_j)^{\alpha_j} M_p(r, f)$$

is finite.

Our second result is a characterization of lacunary series in mixed norm spaces $H^{p, q, \alpha}(\mathbb{D}^n)$.

**Theorem 2.** Let $0 < p \leq \infty$, $0 < q \leq \infty$, $\alpha_j > 0$, $j = 1, ..., n$, and

$$f(z) = \sum_{k_1, ..., k_n \geq 1} a_{k_1, ..., k_n} z_1^{m_{1, k_1}} \cdots z_n^{m_{k_n}}$$

be a holomorphic function on $\mathbb{D}^n$ such that there is $\lambda > 1$ satisfying the condition

$$m_{j, k_j+1}/m_{j, k_j} \geq \lambda \quad \text{for all} \quad k_j \in \mathbb{N}, \ j = 1, ..., n.$$

If $0 < q < \infty$, then the following statements are equivalent:

1. $f \in H^{p, q, \alpha}(\mathbb{D}^n)$;
2. $\sum_{k_1, ..., k_n \geq 1} \left| a_{k_1, ..., k_n} \right|^q \prod_{j=1}^n m_{j, k_j}^{\alpha_j} < \infty$. 


If $q = \infty$, then the following statements are equivalent:

(iii) $f \in H^{p,\infty,\alpha}(\mathbb{D}^n)$;

(iv) $\sup_{k_1, \ldots, k_n \geq 1} \left| a_{k_1, \ldots, k_n} \right| \prod_{j=1}^n m_{j,k_j}^{-\alpha} < \infty$.

We note that the equivalence (iii) and (iv) also follows from [Av, Theorem 3]. The equivalence (i) $\iff$ (ii) for $0 < p = q < \infty$ was proved in [St].

1 Preliminaries

In this section we gather several well-known lemmas that will be used in the proofs of our results.

Lemma 1. [P] Let $\alpha > -1, 0 < q < \infty$ and $I_n = \{k \in \mathbb{N} : 2^n \leq k < 2^{n+1}\}$ for $n \geq 1$, $I_0 = \{0, 1\}$. If $\{a_n\}_0^\infty$ is a sequence of non-negative numbers such that the series $G(r) = \sum_{n=0}^\infty a_n r^n$ converges for every $r \in (0, 1)$, then the following two conditions are equivalent and the corresponding quantities are “proportional”:

(i) $\int_0^1 (1-r)^\alpha G(r)^q dr < \infty$;

(ii) $\sum_{n=0}^\infty 2^{-n(\alpha+1)} \left( \sum_{k \in I_n} a_k \right)^q < \infty$.

In the case of the function $G(r) = \sup_{n \geq 0} a_n r^n$ in (i) the expression $\sum_{k \in I_n} a_k$ in (ii) should be replaced by $\sup_{k \in I_n} a_k$.

Lemma 2. If $\{n_k\}$ is a lacunary sequence of positive integers, that is $\inf_k \frac{n_{k+1}}{n_k} = \lambda > 1$, and $\{a_k\}$ is a sequence of nonnegative real numbers, then the following conditions are equivalent and the corresponding quantities are “proportional”:

(i) $\int_0^1 (1-r)^\alpha \left( \sum_{k=1}^\infty a_k r^{n_k} \right)^q dr < \infty$;

(ii) $\int_0^1 (1-r)^\alpha \left( \sup_{k \geq 1} a_k r^{n_k} \right)^q dr < \infty$;

(iii) $\sum_{k=1}^\infty \frac{|a_k|^q}{n_k^\alpha} < \infty$.

Proof. By Lemma 1,

$$\int_0^1 (1-r)^\alpha \left( \sum_{k=1}^\infty a_k r^{n_k} \right)^q dr \equiv \sum_{k=1}^\infty \frac{2^{-k(\alpha+1)} \left( \sum_{n_j \in I_k} a_j \right)^q}{n_k^\alpha}.$$
Since \( \frac{n_{j+1}}{n_j} \geq \lambda > 1 \), for all \( j \in N \), the number of \( a_j \) when \( n_j \in I_k \) is at most \( \log_2 \lambda + 1 \). Using this and the fact that \( n_j \sim 2^k \) when \( n_j \in I_k \), we see that

\[
\sum_{k=1}^{\infty} 2^{-k(\alpha+1)} \left( \sum_{n_j \in I_k} a_j \right)^q \leq \sum_{k=1}^{\infty} \frac{a_k^q}{n_k^{\alpha+1}}.
\]

**Lemma 3.** [Zy, Du, P] Let \( 0 < p < \infty \). If \( \{n_k\} \) is an increasing sequence of positive integers satisfying \( \frac{n_{k+1}}{n_k} \geq \lambda > 1 \) for all \( k \), then there is a positive constant \( C \) depending only on \( p \) and \( \lambda \) such that

\[
C^{-1} \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} a_k e^{i n_k \theta} \right|^p d\theta \right)^{1/p} \leq C \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}.
\]

These Paley’s inequalities were extended to the unit polydisk \( D^n \) in [Av]:

**Lemma 4.** Let \( \{m_{j,k}\}_{j=1}^{\infty}, j = 1, \ldots, n, \) be arbitrary lacunary sequences and \( f(z) \) be a holomorphic function in \( D^n \) given by

\[
f(z) = \sum_{k_1, \ldots, k_n \geq 1} a_{k_1, \ldots, k_n} z_1^{m_1, k_1} \cdots z_n^{m_n, k_n}, \quad z = (z_1, \ldots, z_n) \in D^n.
\]

Then for any \( p, 0 < p < \infty \), \( f \) is in the Hardy space \( H^p(D^n) \), i.e. \( ||f||_p = \sup_{r \in \mathbb{R}} M_p(r, f) < \infty \), if and only if \( \sum_{k_1, \ldots, k_n \geq 1} |a_{k_1, \ldots, k_n}|^2 < \infty \). Moreover,

\[
C^{-1} ||f||_p \leq \left( \sum_{k_1, \ldots, k_n \geq 1} |a_{k_1, \ldots, k_n}|^2 \right)^{1/2} \leq C ||f||_p,
\]

where \( C \) is a constant independent of \( f \).

## 2 Proof of Theorem 1

Let

\[
\sum_{k=1}^{\infty} \frac{||f_{n_k}||_p^q}{n_k^{\alpha q}} < \infty, \quad 0 < p \leq \infty, \quad 0 < q < \infty.
\]

If \( 1 \leq p < \infty \), then by using Minkowski’s inequality we obtain

\[
M_p(r, f) \leq \sum_{k=1}^{\infty} ||f_{n_k}||_p r^{n_k}.
\]  \hspace{1cm} (1)

If \( p = \infty \), then

\[
M_\infty(r, f) \leq \sum_{k=1}^{\infty} ||f_{n_k}||_\infty r^{n_k}.
\]  \hspace{1cm} (2)
An application of Lemma 2 gives

\[
\|f\|^q_{p,q,\alpha} \leq \int_0^1 (1-r)^{\alpha q-1} \left( \sum_{k=1}^{\infty} \|f_{n_k}\|_p r^{n_k} \right)^q dr
\]

\[
\leq C \sum_{k=1}^{\infty} \frac{\|f_{n_k}\|^q_p}{n_k^{\alpha q}}.
\]

If \(0 < p < 1\), then

\[
M_p^p(r, f) \leq \sum_{k=1}^{\infty} \|f_{n_k}\|_{p^p r^{n_k}}.
\]

Hence,

\[
\|f\|^q_{p,q,\alpha} \leq \int_0^1 (1-r)^{\alpha q-1} \left( \sum_{k=1}^{\infty} \|f_{n_k}\|_{p^p r^{n_k}} \right)^q dr
\]

\[
\leq C \int_0^1 (1-r)^{\alpha q-1} \left( \sum_{k=1}^{\infty} \|f_{n_k}\|_{p^p r^{n_k}} \right)^{q/p} dr
\]

\[
\leq C \sum_{k=1}^{\infty} \frac{\|f_{n_k}\|^q_p}{n_k^{\alpha q}}.
\]

by Lemma 2.

If \(\alpha > 0\) and \(\{n_k\}\) is a lacunary sequence of positive integers, then

\[
\sum_{k=1}^{\infty} n_k^\alpha r^{n_k} = O\left(\frac{1}{(1-r)^\alpha}\right), \text{ see [Du]}.
\]

Using this, (1), (2), and (3) we find that

\[
\|f\|_{p,\infty,\alpha} = \sup_{0 < r < 1} (1-r)^{\alpha} M_p(r, f) \leq C \sup_{k \geq 1} \frac{\|f_{n_k}\|_p}{n_k^\alpha}.
\]

Conversely, let \(\|f\|_{p,\infty,\alpha} < \infty\).

If \(0 < p < \infty\), then by using the slice integration formula [Ru2, Proposition 1.4.7] and Lemma 3 we find that

\[
M_p(r, f) = \left( \int_{\partial B^n} \left| \sum_{k=1}^{\infty} f_{n_k}(r\xi) \right|^p d\sigma(\xi) \right)^{1/p}
\]

\[
= \left( \int_{\partial B^n} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} f_{n_k}(\xi) r^{n_k} e^{i\alpha_k} \right|^p d\theta \right) d\sigma(\xi) \right)^{1/p}
\]

\[
= \left( \int_{\partial B^n} \left( \sum_{k=1}^{\infty} |f_{n_k}(\xi)|^2 r^{2n_k} \right)^{p/2} d\sigma(\xi) \right)^{1/p},
\]
and consequently
\[ M_p(r, f) \geq C ||f_{nk}||_p r^{nk}, \quad \text{for all} \quad k \geq 1. \]

If \( p = \infty \), also we have \( M_\infty(r, f) \geq ||f_{nk}||_\infty r^{nk}, \) for all \( k \geq 1 \).

Thus, if \( 0 < q < \infty \), then
\[ ||f||_{p,q,\alpha}^q \geq C \int_0^1 (1 - r)^{q\alpha - 1} \left( \sup_{k \geq 1} ||f_{nk}||_p r^{nk} \right)^q dr \]
\[ \geq C \sum_{k=1}^\infty \frac{||f_{nk}||_p^q}{n_k^q}, \]
by Lemma 2.

If \( q = \infty \), then
\[ ||f||_{p,\infty,\alpha} \geq \sup_{0 < r < 1} (1 - r)^\alpha \sup_{k \geq 1} ||f_{nk}||_p r^{nk} \]
\[ \geq \sup_{k \geq 1} ||f_{nk}||_p \frac{1}{n_k^n} (1 - \frac{1}{n_k})^{n_k} \]
\[ \geq \frac{1}{e} \sup_{k \geq 1} ||f_{nk}||_p n_k. \]
This finishes the proof of Theorem 1.

### 3 Proof of Theorem 2

In order to avoid too much calculations we will assume that \( n = 2 \).

**Proof of implications** (ii) \( \implies \) (i) and (iv) \( \implies \) (iii)

Let \( 0 < p \leq \infty, r = (r_1, r_2) \) and \( \alpha = (\alpha_1, \alpha_2), \alpha_1 > 0, \alpha_2 > 0 \). Then
\[ M_p(r, f) \leq \sum_{k_1, k_2 \geq 1} |a_{k_1, k_2}| r_1^{m_1, k_1} r_2^{m_2, k_2}. \]

If \( 0 < q < \infty \) then by applying Lemma 2 twice we obtain
\[ ||f||_{p,q,\alpha}^q \leq \int_0^1 (1 - r_2)^{q \alpha_2 - 1} dr_2 \int_0^1 (1 - r_1)^{q \alpha_1 - 1} M_p(r, f)^q dr_1 \]
\[ \leq \int_0^1 (1 - r_2)^{q \alpha_2 - 1} dr_2 \int_0^1 (1 - r_1)^{q \alpha_1 - 1} \]
\[ \times \left( \sum_{k_1 \geq 1} \left( \sum_{k_2 \geq 1} |a_{k_1, k_2}| r_2^{m_2, k_2} r_1^{m_1, k_1} \right)^q \right) dr_1 \]
\[ \leq C \int_0^1 (1 - r_2)^{q \alpha_2 - 1} \left( \sum_{k_1 \geq 1} \frac{1}{m_1, k_1} \left( \sum_{k_2 \geq 1} |a_{k_1, k_2}| r_2^{m_2, k_2} \right) \right) dr_2 \]
This holds also for

Thus

By Lemma 4 we have

If \( q = \infty \), then we have

\[
||f||_{p,\infty,\alpha} = \sup_{0<r_1<1} \sup_{0<r_2<1} (1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2} M_p(r, f)
\]

\[
\leq \sup_{0<r_1<1} \sup_{0<r_2<1} (1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2} \sum_{k_1, k_2 \geq 1} |a_{k_1, k_2}| r_1^{m_{1,k_1}} r_2^{m_{2,k_2}}
\]

\[
\leq \sup_{0<r_1<1} (1-r_1)^{\alpha_1} \sum_{k_1 \geq 1} \sup_{0<r_2<1} (1-r_2)^{\alpha_2} \sum_{k_2 \geq 1} |a_{k_1, k_2}| r_1^{m_{1,k_1}} r_2^{m_{2,k_2}}
\]

\[
C \sup_{0<r_1<1} (1-r_1)^{\alpha_1} \sum_{k_1 \geq 1} |a_{k_1, k_2}| r_1^{m_{1,k_1}}
\]

\[
C \sup_{k_1 \geq 1} |a_{k_1, k_2}|^{\alpha_1} r_1^{m_{1,k_1}} r_2^{m_{2,k_2}}
\]

**Proof of implications** (i) \( \implies \) (ii) and (iii) \( \implies \) (iv)

By Lemma 4 we have

\[
M_p(r, f) \equiv \left( \sum_{k_1, k_2 \geq 1} |a_{k_1, k_2}|^2 r_1^{2m_{1,k_1}} r_2^{2m_{2,k_2}} \right)^{1/2}.
\]

Thus

\[
M_p(r, f) \geq \sup_{k_1, k_2 \geq 1} |a_{k_1, k_2}| r_1^{m_{1,k_1}} r_2^{m_{2,k_2}}, \quad 0 < p < \infty.
\]

This holds also for \( p = \infty \). Hence, if \( 0 < q < \infty \), by applying Lemma 2 twice we get

\[
||f||_{p,q,\alpha}^q \geq \int_0^1 (1-r_1)^{\alpha_1-1} dr_1 \int_0^1 (1-r_2)^{\alpha_2-1} dr_2 
\]

\[
\times \left( \sup_{k_2 \geq 1} \left( \sup_{k_1 \geq 1} |a_{k_1, k_2}| r_1^{m_{1,k_1}} r_2^{m_{2,k_2}} \right) q \right) dr_2
\]

\[
\geq C \int_0^1 (1-r_1)^{\alpha_1-1} \sum_{k_2 \geq 1} m_{2,k_2}^{\alpha_2} \left( \sup_{k_1 \geq 1} |a_{k_1, k_2}| r_1^{m_{1,k_1}} \right) q dr_1
\]

\[
= C \sum_{k_2 \geq 1} m_{2,k_2}^{\alpha_2} \int_0^1 (1-r_1)^{\alpha_1-1} \left( \sup_{k_1 \geq 1} |a_{k_1, k_2}| r_1^{m_{1,k_1}} \right) q dr_1
\]

\[
\geq C \sum_{k_1 \geq 1} \sum_{k_2 \geq 1} m_{2,k_2}^{\alpha_2} m_{1,k_1}^{\alpha_1} |a_{k_1, k_2}| q.
\]
If \( q = \infty \), then
\[
\|f\|_{p, \infty, \alpha} = \sup_{0 < r_1 < 1} \sup_{0 < r_2 < 1} (1 - r_1)^{\alpha_1} (1 - r_2)^{\alpha_2} M_p(r, f) \\
\geq \sup_{0 < r_1 < 1} \sup_{0 < r_2 < 1} (1 - r_1)^{\alpha_1} (1 - r_2)^{\alpha_2} \sup_{k_1, k_2 \geq 1} |a_{k_1, k_2}| \left( \frac{r_1}{m_1^{\alpha_1}} \right)^{k_1} \left( \frac{r_2}{m_2^{\alpha_2}} \right)^{k_2} \\
\geq C \sup_{k_1, k_2 \geq 1} |a_{k_1, k_2}| \left( \frac{r_1}{m_1^{\alpha_1}} \right)^{k_1} \left( \frac{r_2}{m_2^{\alpha_2}} \right)^{k_2}.
\]
This finishes the proof of Theorem 2.

References


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