UNIQUENESS AND DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS SHARING ONE VALUE

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Abstract

In this paper, we study the uniqueness of differential polynomials of meromorphic functions sharing one value, and obtain some results, which improve and generalize the related results due to Fang, Zhang-Lin, and Bhoosnurmath-Dyavanal, etc.

1 Introduction

In this paper, the term “meromorphic” will always mean meromorphic in the complex plane $\mathbb{C}$. Let $f$ and $g$ be two nonconstant meromorphic functions, and let $a$ be a complex number. We say that $f$ and $g$ share $a$ CM (counting multiplicity) when $f - a$ and $g - a$ have the same zeros with the same multiplicity.

It is assumed that the reader is familiar with the standard notations of value distribution theory that can be found, for instance, in [3, 7, 9]. We use $I$ to denote any set of positive real numbers of infinite linear measure and use $E$ to denote any set of positive real numbers of finite linear measure, not necessary the same at each occurrence. We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o(T(r, f))$$

as $r \to \infty$, $r \not\in E$.

In addition, we shall also use the following notations.

Let $k$ be a positive integer. We denote by $N_k(\frac{1}{f-a})$ the counting function for zeros of $f - a$ with multiplicities at least $k$, and by $N_{\tilde{k}}(\frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Similarly, we denote by $N_k(\frac{1}{f-a})$ the

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counting function for zeros of \( f - a \) with multiplicities at most \( k \), and by \( \bar{N}_k(r, \frac{1}{f - a}) \) the corresponding one for which multiplicity is not counted. Set

\[
N_k(r, \frac{1}{f - a}) = \bar{N}(r, \frac{1}{f - a}) + \bar{N}_2(r, \frac{1}{f - a}) + \bar{N}_3(r, \frac{1}{f - a}) + \ldots + \bar{N}_k(r, \frac{1}{f - a}).
\]

Clearly, \( N_k(r, \frac{1}{f - a}) \) can be viewed as the counting function corresponding to the zeros of \( f - a \) where a \( n \)-fold zero is counted min \( \{n, k\} \) times.

Let

\[
\Theta(a, f) = 1 - \lim_{{r \to \infty}} \frac{\bar{N}(r, 1/(f - a))}{T(r, f)},
\]

\[
\Theta_k(a, f) = 1 - \lim_{{r \to \infty}} \frac{\bar{N}_k(r, 1/(f - a))}{T(r, f)}.
\]


**Theorem A.** Let \( f(z) \) and \( g(z) \) be two nonconstant entire functions, and let \( n, k \) be two positive integers with \( n \geq 2k + 8 \). If \([f^n(z)(f(z) - 1)]^{(k)}\) and \([g^n(z)(g(z) - 1)]^{(k)}\) share 1 CM, then \( f(z) \equiv g(z) \).

Recently, Zhang and Lin [10] proved the following result, which generalizes and improves the above theorem.

**Theorem B.** Let \( f(z) \) and \( g(z) \) be two nonconstant entire functions, and let \( n, m, k \) be three positive integers with \( n \geq 2k + 4 \). If \([f^n(z)(f(z) - 1)^m]^{(k)}\) and \([g^n(z)(g(z) - 1)^m]^{(k)}\) share 1 CM, then either \( f(z) \equiv g(z) \), or \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) \equiv 0 \), where \( R(w_1, w_2) = w_1^n(w_1 - 1)^m - w_2^n(w_2 - 1)^m \).

**Remark 1.** Theorem A is a special case of Theorem B. Indeed, let \( m = 1 \), and suppose that \( f(z) \equiv g(z) \) and \( f^n(f - 1) = g^n(g - 1) \), then \( g \) is constant by Picard’s theorem (for details, see [10]).

In this paper, we extend Theorem A and B to meromorphic functions. For the case \( m > k \), we prove the following result.

**Theorem 1.** Let \( n, m, k \) be three positive integers, and \( f(z) \) and \( g(z) \) be two nonconstant meromorphic functions such that \([f^n(z)(f(z) - 1)^m]^{(k)}\) and \([g^n(z)(g(z) - 1)^m]^{(k)}\) share 1 CM. If \( m > k \) and \( n > 3k + m + 8 \), and \( \Theta(\infty, f) > 2m(m + n)/[(n + m)^2 - 4k^2] \) or \( \Theta(\infty, g) > 2m(m + n)/[(n + m)^2 - 4k^2] \), then either \( f(z) \equiv g(z) \), or \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) \equiv 0 \), where \( R(w_1, w_2) = w_1^n(w_1 - 1)^m - w_2^n(w_2 - 1)^m \).

For the case \( m \leq k \), we have
Theorem 2. Let \( n, m, k \) be three positive integers, and \( f(z) \) and \( g(z) \) be two nonconstant meromorphic functions such that \([f^n(z)(f(z)−1)^m]^k\) and \([g^n(z)(g(z)−1)^m]^k\) share 1 CM. If \( m \leq k \) and \( n > 3k + m + 8 \), and
\[
\Theta(\infty, f) + \Theta(1, f) > 1 + \frac{2m(m+n)}{(n+m)^2 - 4k^2}
\]
or
\[
\Theta(\infty, g) + \Theta(1, g) > 2\frac{m(m+n)}{(n+m)^2 - 4k^2},
\]
then the conclusion of Theorem 1 holds.

If \( f \) and \( g \) have the same poles (not necessary with the same multiplicity), we prove that the above theorems still hold without the condition about the deficiency of \( f \) or \( g \), as follows.

Theorem 3. Let \( n, m, k \) be three positive integers, and \( f(z) \) and \( g(z) \) be two nonconstant meromorphic functions such that \([f^n(z)(f(z)−1)^m]^k\) and \([g^n(z)(g(z)−1)^m]^k\) share 1 CM. If \( f \) and \( g \) have the same poles (not necessary with the same multiplicity), and \( n > 3k + m + 8 \), then the conclusion of Theorem 1 holds.

Remark 2. In [1], Bhoosnurmath and Dyavanal proved that: Let \( f(z) \) and \( g(z) \) be two nonconstant meromorphic functions such that \([f^n(z)(f(z)−1)^m]^k\) and \([g^n(z)(g(z)−1)^m]^k\) share 1 CM, and let \( n, k \) be two positive integers with \( n \geq 3k + 13 \). If \( \Theta(\infty, f) > 3/(n+1) \), then \( f(z) \equiv g(z) \). However, the proof given there seems to contain some gaps (for example, see [1, p.1203,(6.9) etc.]).

2 Some Lemmas

For the proof of our results, we need the following lemmas.

Lemma 1. (see [7]) Let \( f \) be a nonconstant meromorphic function, and \( a_0, a_1, \ldots, a_n \) be finite complex numbers such that \( a_n \neq 0 \). Then
\[
T(r, a_nf^n + \cdots + a_1f + a_0) = nT(r, f) + S(r, f).
\]

The next lemma is due to Milloux (see [3, 6, 9]).

Lemma 2. Let \( f \) be a nonconstant meromorphic function, \( k \) be a positive integer. Then
\[
T(r, f) \leq N(r, f) + N(r, \frac{1}{f}) + N(r, \frac{1}{f^{k+1}}) - N(r, \frac{1}{f^{k+1}}) + S(r, f) \\
\leq N(r, f) + N_{k+1}(r, \frac{1}{f}) + N(r, \frac{1}{f^{k+1}}) - N_0(r, \frac{1}{f^{k+1}}) + S(r, f),
\]
where $N_0(r, 1/f^{(k+1)})$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - 1) \neq 0$.

The following lemma is the second fundamental theorem for small functions, which is due to Yamanoi [8]

\textbf{Lemma 3} Let $f(z)$ be a non-constant meromorphic function on $\mathbb{C}$, and let $a_1(z), a_2(z), \cdots, a_q(z)$ be distinct meromorphic functions on $\mathbb{C}$. Assuming that $a_i(z)$ are small functions with respect to $f$ for all $i = 1, \cdots, q$. Then

$$(q - 2)T(r, f) \leq \sum_{i=1}^{q} \bar{N}(r, \frac{1}{f - a_i}) + S(r, f).$$

\textbf{Lemma 4} (see [3, 6, 9]). Let $f(z)$ be a non-constant meromorphic function, and let $k$ be a positive integer. Suppose that $f^{(k)} \neq 0$, then

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + kN(r, f) + S(r, f).$$

\textbf{Lemma 5}. Let $f(z)$ be a nonconstant meromorphic function, and let $k$ be a positive integer. Suppose that $f^{(k)} \neq 0$, then

$$N(r, \frac{1}{f^{(k)}}) \leq N_{k+1}(r, \frac{1}{f}) + kN(r, f) + S(r, f).$$

\textbf{Proof}. This can be found in [5]. For details, see the inequality (12) in the proof of Theorem 2 [5, p.1296](cf.[4]). Here we omit the details.

\textbf{Lemma 6}. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and $n, m, k$ be three positive integers with $n > k + 2$, and let $F = [f^n(z)(f(z) - 1)^m]^{(k)}$ and $G = [g^n(z)(g(z) - 1)^m]^{(k)}$ such that $F$ and $G$ share 1 CM. Set

$$H = \left[ \frac{F''}{F'} - \frac{2F'''}{F''} \right] - \left[ \frac{G''}{G'} - \frac{2G'''}{G''} \right].$$

If $H \neq 0$, then

$$m(r, \frac{1}{F'}) \leq N(r, \frac{1}{F'}) + 2\bar{N}(r, f) + \bar{N}(r, g) - (n - k - 2)[N(r, \frac{1}{F'}) + N(r, \frac{1}{F})] + S(r, f) + S(r, g),$$

$$m(r, \frac{1}{G'}) \leq N(r, \frac{1}{G'}) + 2\bar{N}(r, f) + \bar{N}(r, g) - (n - k - 2)[N(r, \frac{1}{G'}) + N(r, \frac{1}{G})] + S(r, f) + S(r, g).$$

\textbf{Proof}. Since $F$ and $G$ share 1 CM, by a local expansions, we see from (1) that $H(z_0) = 0$ if $z_0$ is a simple zero of both $F - 1$ and $G - 1$. It follows that

$$N_1(r, \frac{1}{F - 1}) = N_1(r, \frac{1}{G - 1}) \leq N(r, \frac{1}{F}) \leq T(r, H) + O(1) \leq N(r, H) + S(r, f) + S(r, g).$$
Noting that $F$ and $G$ share 1 CM, we deduce from (1) that
\[
N(r, H) \leq \hat{N}(r, F) + \hat{N}(r, G) + \hat{N}(2(r, \frac{1}{F})) + \hat{N}(2(r, \frac{1}{G})) + N_0(r, \frac{1}{F}) + N_0(r, \frac{1}{G}),
\]  
where $N_0(r, \frac{1}{F})$ denotes the counting function corresponding to the zeros of $F'$ that are not zeros of $F(F - 1)$, and $N_0(r, \frac{1}{G})$ is defined similarly. Since
\[
N(r, \frac{1}{F}) = \hat{N}_1(r, \frac{1}{F}) + \hat{N}_2(r, \frac{1}{F}),
\]
by the second fundamental theorem, we obtain from (4) and (5) that
\[
T(r, F) \leq \hat{N}(r, F) + \hat{N}(r, \frac{1}{F}) + \hat{N}(r, \frac{1}{F}) - N_0(r, \frac{1}{F}) + S(r, f)
\leq 2\hat{N}(r, F) + \hat{N}(r, G) + \hat{N}(r, \frac{1}{F}) + \hat{N}(2(r, \frac{1}{F})) + \hat{N}(2(r, \frac{1}{G})) + N_0(r, \frac{1}{F}) + N_0(r, \frac{1}{G}) + S(r, f) + S(r, g).
\]  
From the definition of $N_0(r, \frac{1}{F})$, we see that
\[
N_0(r, \frac{1}{F}) + \hat{N}_2(r, \frac{1}{F}) + \hat{N}_2(r, \frac{1}{F}) - \hat{N}_2(r, \frac{1}{F}) \leq N(r, \frac{1}{F}).
\]  
The above inequality and Lemma 4 give
\[
N_0(r, \frac{1}{G}) + \hat{N}_2(r, \frac{1}{G}) + \hat{N}_2(r, \frac{1}{G}) - \hat{N}_2(r, \frac{1}{G}) \leq N(r, \frac{1}{G}).
\]  
Substituting (7) in (6), we get
\[
T(r, F) \leq 2[\hat{N}(r, F) + \hat{N}(r, G)] + \hat{N}(r, \frac{1}{F}) + \hat{N}(r, \frac{1}{G}) + \hat{N}_2(r, \frac{1}{F}) + \hat{N}_2(r, \frac{1}{G}) + S(r, f) + S(r, g).
\]  
It is not difficult to see that
\[
\hat{N}(r, \frac{1}{F}) + \hat{N}_2(r, \frac{1}{F}) = N(r, \frac{1}{F}) - [N_0(r, \frac{1}{F}) - 2\hat{N}_2(r, \frac{1}{F})].
\]  
Next we give the estimate of $N_0(r, \frac{1}{F}) - 2\hat{N}_2(r, \frac{1}{F})$. If $z_0$ is a zero of $f$ with multiplicity $l \geq 1$, then $z_0$ is a zero of $[f^n(z)(f(z) - 1)^m]^{(k)}$ with multiplicity $nl - k$. Since $n > k + 2$, we know that $nl - k > (k + 2)l - k = (l - 1)k + 2l \geq 2$. Thus
\[
N_0(r, \frac{1}{F}) - 2\hat{N}_2(r, \frac{1}{F}) \geq (n - k - 2)N(r, \frac{1}{F}).
\]
It follows from (9) that
\[
\hat{N}(r, \frac{1}{F}) + \hat{N}_2(r, \frac{1}{F}) \leq N(r, \frac{1}{F}) - (n - k - 2)N(r, \frac{1}{F}).
\]  
Similarly, we have
\[
\hat{N}(r, \frac{1}{G}) + \hat{N}_2(r, \frac{1}{G}) \leq N(r, \frac{1}{G}) - (n - k - 2)N(r, \frac{1}{G}).
\]
Substituting (10) and (11) in (8), we obtain
\[ T(r, F) \leq 2[\bar{N}(r, F) + \bar{N}(r, G)] + N(r, \frac{1}{F}) + N(r, \frac{1}{G}) - (n - k - 2)[N(r, \frac{1}{F}) + N(r, \frac{1}{G})] + S(r, f) + S(r, g). \] (12)

Then, noting that \( \bar{N}(r, F) = \bar{N}(r, f) \) and \( \bar{N}(r, G) = \bar{N}(r, g) \), and using the first fundamental theorem, we obtain (2). Similarly, we can prove (3). Lemma 6 is proved.

3 Proof of Theorems

Proof of Theorem 1. Let
\[ F = [f^n(z)(f(z) - 1)^m]^k, \quad G = [g^n(z)(g(z) - 1)^m]^k, \] (13) and
\[ F_1 = f^n(z)(f(z) - 1)^m, \quad G_1 = g^n(z)(g(z) - 1)^m. \] (14)

Then \( F \) and \( G \) share 1 CM, \( F_1 = F, G_1 = G \), and
\[ m(r, \frac{1}{F_1}) \leq m(r, \frac{1}{F}) + S(r, f). \] (15)

By Lemma 1, we have
\[ T(r, F_1) = (n + m)T(r, f) + S(r, f), \] (16) \[ T(r, G_1) = (n + m)T(r, g) + S(r, g). \] (17)

We first prove that \( H \equiv 0 \), where \( H \) is defined as (1). Suppose that \( H \neq 0 \), by Lemma 6, we have
\[ m(r, \frac{1}{F}) \leq N(r, \frac{1}{F}) + 2[\bar{N}(r, f) + \bar{N}(r, g)] - (n - k - 2)[N(r, \frac{1}{F}) + N(r, \frac{1}{G})] + S(r, f) + S(r, g). \] (18)

Combining (15), (16) and (18), we get
\[ (n + m)T(r, f) = T(r, F_1) + S(r, f) = N(r, \frac{1}{F_1}) + m(r, \frac{1}{F_1}) + S(r, f) \leq N(r, \frac{1}{F}) + m(r, \frac{1}{F}) + S(r, f) \leq N(r, \frac{1}{F}) + N(r, \frac{1}{G}) + 2[\bar{N}(r, f) + \bar{N}(r, g)] -(n - k - 2)[N(r, \frac{1}{F}) + N(r, \frac{1}{G})] + S(r, f) + S(r, g). \] (19)

It follows from Lemma 4 that
\[ N(r, \frac{1}{F}) \leq N(r, \frac{1}{F_1}) + k\bar{N}(r, G_1) + S(r, G_1) \leq N(r, \frac{1}{F_1}) + k\bar{N}(r, g) + S(r, g). \] (20)
Substituting (20) in (19), and nothing that
\[ N(r, 1/F_1) = nN(r, 1/f) + mN(r, 1/(f - 1)) \]
and
\[ N(r, 1/G_1) = nN(r, 1/g) + mN(r, 1/(g - 1)), \]
we have
\[ (n + m)T(r, f) \leq 2\bar{N}(r, f) + (k + 2)\bar{N}(r, g) + m[N(r, \frac{1}{f-1}) + N(r, \frac{1}{g-1})] + S(r, f) + S(r, g). \]  
(21)

Similarly, we have
\[ (n + m)T(r, g) \leq 2\bar{N}(r, g) + (k + 2)\bar{N}(r, f) + m[N(r, \frac{1}{f-1}) + N(r, \frac{1}{g-1})] + S(r, f) + S(r, g). \]  
(22)

Combining (21) and (22), we obtain
\[ [n - (3k + 8 + m)]T(r, f) + T(r, g)] \leq S(r, f) + S(r, g), \]  
(23)

which is impossible since \( n > 3k + m + 8 \). Therefore, \( H \equiv 0 \), and we deduce from (1) that
\[ F = \frac{aG + b}{cG + d}. \]  
(24)

where \( a, b, c \) and \( d \) are finite complex numbers satisfying \( ad - bc \neq 0 \).

Next we prove that either \( F_1 \equiv G_1 \) or \( FG \equiv 1 \). Without loss of generality, we may suppose that there exists a set \( I \) with infinite linear measure such that \( T(r, g) \leq T(r, f) \) for \( r \in I \). We consider three cases.

**Case 1.** \( ac \neq 0 \). From (24), we see that
\[ \bar{N}(r, \frac{1}{F_1(k) - a/c}) = \bar{N}(r, \frac{1}{F - a/c}) = \bar{N}(r, G) = \bar{N}(r, g). \]
Using Lemma 2 for \( F_1 \), we have
\[ T(r, F_1) \leq \bar{N}(r, F_1) + N_{k+1}(r, \frac{1}{F_1}) + \bar{N}(r, \frac{1}{F_1(a/c) - a/c}) + S(r, F_1) \leq \bar{N}(r, f) + (k + 1)[\bar{N}(r, \frac{1}{f-1}) + \bar{N}(r, \frac{1}{g-1})] + \bar{N}(r, g) + S(r, f). \]

Then
\[ (n + m - 2k - 4)T(r, f) \leq S(r, f), \quad r \in I, \]
which is impossible since \( n > 3k + m + 8 \).

**Case 2.** \( a \neq 0, c = 0 \). Then \( d \neq 0 \) and \( F = (aG + b)/d \). It follows that
\[ F_1 = \frac{a}{d}G_1 + p(z), \]  
(25)
where \( p(z) \) is a polynomial with \( \deg p(z) \leq k \).

If \( p(z) \neq 0 \), then

\[
\bar{N}(r, \frac{1}{F_1 - p(z)}) = \bar{N}(r, \frac{1}{G_1}) = \bar{N}(r, \frac{1}{g}) + \bar{N}(r, \frac{1}{g - 1}).
\]

Clearly, \( p(z) \) is a small function of \( f \). Then by Lemma 3, we have

\[
(n + m)T(r, f) = T(r, F_1) \leq \bar{N}(r, F_1) + \bar{N}(r, \frac{1}{F_1}) + \bar{N}(r, \frac{1}{G_1 - p(z)}) + S(r, F_1)
\]

\[
\leq \bar{N}(r, f) + \bar{N}(r, \frac{1}{T}) + \bar{N}(r, \frac{1}{g}) + \bar{N}(r, \frac{1}{g - 1}) + S(r, f),
\]

that is, \((n + m - 5)T(r, f) \leq S(r, f)\) for \( r \in I \), a contradiction.

From (26), we have

\[
F_1 = \frac{a}{d} G_1. \tag{26}
\]

If \( a/d \neq 1 \), then \( F \neq 1 \) since \( F \) and \( G \) share 1 CM. By Lemma 2, we have

\[
(n + m)T(r, f) = T(r, F_1) \leq \bar{N}(r, F_1) + N_{k+1}(r, \frac{1}{F_1}) + \bar{N}(r, \frac{1}{F_1 - p(z)}) + S(r, F_1)
\]

\[
\leq \bar{N}(r, f) + (k + 1)[\bar{N}(r, \frac{1}{T}) + \bar{N}(r, \frac{1}{g - 1})] + S(r, f),
\]

that is, \((n + m - 2k - 3)T(r, f) \leq S(r, f)\), a contradiction. Thus \( a/d = 1 \), and \( F_1 \equiv G_1 \).

**Case 3.** \( a = 0, c \neq 0 \). Then \( b \neq 0 \) and \( F = b/(cG + d) \). Using almost the same argument as in Case 2, we can prove that \( FG \equiv 1 \).

We thus proved that either \( F_1 \equiv G_1 \) or \( FG \equiv 1 \).

If \( FG \equiv 1 \), then

\[
[f^n(z)(f(z) - 1)^m]^{(k)} [g^n(z)(g(z) - 1)^m]^{(k)} \equiv 1. \tag{27}
\]

Let \( z_0 \) be a zero of \( f \) of order \( p \), we see from (27) that \( z_0 \) must be a pole of \( g \).

Suppose that \( z_0 \) is a pole of \( g \) of order \( q \). Again by (27), we obtain

\[
np - k = nq + mq + k,
\]

that is,

\[
n(p - q) = mq + 2k,
\]

which implies that \( p \geq q + 1 \) and \( mq + 2k \geq n \). Then

\[
p \geq \frac{n - 2k}{m} + 1. \tag{28}
\]

Let \( z_1 \) be a zero of \( f - 1 \) of order \( p_1 \). Noting that \( m > k \), \( z_1 \) is also the zero of \( [f^n(z)(f(z) - 1)^m]^{(6)} \) of order \( mp_1 - k \). From (27), we see that \( z_1 \) must be a pole of \( g \) of order, say \( q_1 \). Again by (27), we have \( mp_1 - k = nq_1 + mq_1 + k \). This gives

\[
p_1 \geq \frac{n + 2k}{m} + 1. \tag{29}
\]
From (28) and (29), by the second fundamental theorem, we obtain
\[
T(r, f) \leq \hat{N}(r, f) + \hat{N}(r, \frac{1}{f}) + \hat{N}(r, \frac{1}{f^2}) + S(r, f)
\]
\[
\leq \hat{N}(r, f) + \frac{m}{n+m-2k}N(r, \frac{1}{f}) + \frac{m}{n+m+2k}N(r, \frac{1}{f^2}) + S(r, f).
\]
Then
\[
[\Theta(\infty, f) - \frac{2m(m+n)}{(n+m)^2 - 4k^2}]T(r, f) \leq S(r, f), \quad r \not\in E.
\]
Similarly, we have
\[
[\Theta(\infty, g) - \frac{2m(m+n)}{(n+m)^2 - 4k^2}]T(r, g) \leq S(r, g), \quad r \not\in E.
\]
But these contradicts the assumption \(\Theta(\infty, f) > 2m(m+n)/[(n+m)^2 - 4k^2]\) or \(\Theta(\infty, g) > 2m(m+n)/[(n+m)^2 - 4k^2]\).

Hence \(F_1 \equiv G_1\). Using the argument as in [10, p.950], we can obtain the conclusion of Theorem 1. This finally completes the proof of Theorem 1.

**Proof of Theorem 2.** Theorem 2 can be proved by using almost the same argument as in Theorem 1. The only one difference is in the proof of \(FG \not\equiv 1\). Suppose that \(FG \equiv 1\). Then we have (27). Similarly, we have (28).

Let \(z_1\) be a zero of \((f - 1)\) of order \(p_1\) with \(p_1 \geq \frac{m}{m} + 1\). Obviously, \(mp_1 \geq k + 1\). Then \(z_1\) is a zero of \([f^m(z)(f(z) - 1)/m]^k\) of order \(mp_1 - k\). From (27), we conclude that \(z_1\) is a pole of \(g\) of order, say \(q_1 \geq 1\). Similarly as in the proof of Theorem 1, we obtain \(p_1 \geq (n+2k)/m + 1\). Thus
\[
\hat{N}(r, \frac{1}{f}) \leq \hat{N}(\frac{1}{f}) + \hat{N}(\frac{1}{f^2}) + \hat{N}(\frac{1}{f^3}) + \frac{m}{n+m+2k}N(r, \frac{1}{f^2}) + \frac{m}{n+m+2k}N(r, \frac{1}{f^3}).
\]
Then, from (28) and (30), using the second fundamental theorem, we have
\[
T(r, f) \leq \hat{N}(r, f) + \hat{N}(r, \frac{1}{f}) + \frac{m}{n+m+2k}N(r, \frac{1}{f^2}) + S(r, f)
\]
\[
\leq \hat{N}(r, f) + \frac{m}{n+m+2k}N(r, \frac{1}{f^2}) + \hat{N}(r, \frac{1}{f^3}) + \frac{m}{n+m+2k}N(r, \frac{1}{f^3}) + S(r, f),
\]
that is,
\[
[\Theta(\infty, f) + \Theta(\frac{1}{f})](1, f) - 1 - \frac{2m(m+n)}{(n+m)^2 - 4k^2}]T(r, f) \leq S(r, f), \quad r \not\in E,
\]
which contradicts the assumption \(\Theta(\infty, f) + \Theta(\frac{1}{f})](1, f) > 1 + \frac{2m(m+n)}{(n+m)^2 - 4k^2}\). Theorem 2 is thus proved.

**Proof of Theorem 3.** The proof is also almost the same argument as the proof of Theorem 1. The only one difference is also in the proof of \(FG \not\equiv 1\). Suppose that \(FG \equiv 1\). Then we have (27). Since \(f\) and \(g\) have same poles (not necessary with
the same multiplicity), we conclude from (27) that $f$ and $g$ have not poles, and then they are entire functions. Next we can use the same argument as in [10, p.948] to arrive at a contradiction. Here we omit the details. Theorem 3 is proved. □

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References


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