ON SOME VECTOR VALUED GENERALIZED DIFFERENCE MODULAR SEQUENCE SPACES

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Abstract

In this paper we generalize the modular sequence space $\ell\{M_k\}$ by introducing the sequence space $\ell\{M_k, p, q, s, \Delta^s_{(vm)}\}$. We give various properties relevant to algebraic and topological structures of this space and derived some other spaces.

1 Introduction

By $w(X)$, we shall denote the space of all $X$-valued sequences spaces, where $(X, q)$ is a seminormed space, seminormed by $q$. For $X = C$, the space of complex numbers, these represent the corresponding scalar valued sequence spaces. The zero sequence is denoted by $\theta = (\theta, \theta, \theta, \ldots)$ where $\theta$ is the zero element of $X$.

The notion of difference sequence space was introduced by Kizmaz [5], who studied the difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [3] by introducing the spaces $\ell_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$.

Let $r, s$ be non-negative integers and $v = v_k$ be a sequence of non-zero scalars. Also let $Z = \{\ell_\infty, c, c_0\}$. Dutta [2] define the following sequence spaces

$$Z(\Delta^s_{(vr)}) = \left\{ x = (x_k) \in w : (\Delta^s_{(vr)} x_k) \in Z \right\},$$

where $(\Delta^s_{(vr)} x_k) = (\Delta^s_{(vr)} x_k - \Delta^s_{(vr)} x_{k-r})$ and $\Delta^0_{(vr)} x_k = v_k x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta^s_{(vr)} = \sum_{i=0}^{s} (-1)^i \binom{s}{i} v_{k-ri} x_{k-ri}.$$
In this expansion we take $v_k = 0$ and $x_k = 0$ for non-positive values of $k \in \mathbb{N}$. Dutta [2] shown that these spaces can be made $BK$-spaces under the norm

$$
\|x\| = \sup_k |\Delta_{(er)}x_k|.
$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$. If convexity of Orlicz function $M$ is replaced by

$$
M(x + y) \leq M(x) + M(y),
$$

then this function is called a modulus function introduced by Nakano [9].

Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to construct sequence space

$$
\ell_M = \{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0\},
$$

The space $\ell_M$ becomes a Banach space, with the norm

$$
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}
$$

which is called an Orlicz space. The space $\ell_M$ is closely related to the space $\ell_p$ which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$. Another generalization of Orlicz sequence spaces due to Woo [13]. Let $\{M_k\}$ be a sequence of Orlicz functions. Define the vector space $\ell\{M_k\}$ by

$$
\ell\{M_k\} = \{x \in w : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho}\right) \leq \infty, \text{for some } \rho > 0\}
$$

and this space has a norm defined by

$$
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}
$$

Then $\ell\{M_k\}$ becomes a Banach space and is called a modular sequence space. The space $\ell\{M_k\}$ also generalizes the concept of modularized sequence space introduced by Nakano [10], who considered the space $\ell\{M_k\}$ when $M_k(x) = x^{\alpha_k}$, where $1 \leq \alpha_k < \infty$ for $k \geq 1$.

An Orlicz function $M$ is said to satisfy the $\Delta_2$-condition for all values of $u$, if there exists a constant $K > 0$ such that $M(2u) \leq kM(u)$ ($u \geq 0$). The $\Delta_2$-condition is equivalent to the satisfaction of inequality $M(lu) \leq klM(u)$ for all values of $u$ and for $l > 1$ (see [7]). The $\Delta_2$-condition implies $M(lu) \leq K^{\log_2 K} M(u)$ for all values $u > 0$, $l > 1$. 
Karakaya [6], Bekta¸s and Altin [1], Parasar and Choudhary [11], Mursaleen , Khan and Qamaruddin [8], Tripathy and Dutta [12] and many others have studied sequence spaces using Orlicz functions.

In [14], it is shown that a BK-spaces is a Banach space of complex sequences $x = (x_k)$ in which the co-ordinate maps are continuous , that is, $|x_n^k - x_k^k| \to 0$, whenever $\|x^n - x\| \to 0$ as $n \to \infty$, where $x^n = (x^n_k)$ for all $n \in N$ and $x = (x_k)$.

Let $A$ denotes the set of all complex sequences which have only a finite number of non-zero coordinates, $\lambda$ denotes a BK−space of sequences $x = (x_k)$ which contains $A$. An element $x = (x_k)$ of $\lambda$ will be called sectionally convergent if

$$x^n = \sum_{k=1}^{n} x_k e_k \to x$$

as $n \to \infty$, where $e_k = (\delta_{ki})$, where $\delta_{kk} = 1$, $\delta_{ki} = 0$, for $k \neq i$.

The space $\lambda$ will be called AK-space if and only if each of its elements is sectionally convergent. Let $M_k = (M_k)$ be a sequence of Orlicz functions, $X$ be a semi-normed space with seminorm $q$, $p = (p_k)$ be a sequence of positive real numbers and $v = (v_k)$ be a fixed sequence of non-zero scalars. Then for non-negative real numbers $s, m$ and $n$, we define

$$\ell \left\{ M_k, p, q, s, \Delta^n_{(vm)} \right\} = \left\{ x \in w(X) : \sum_{k=1}^{\infty} k^{-s} \left[ M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right) \right]^{p_k} < \infty \text{ for some } \rho > 0 \right\}$$

Considering $X = C$, $q(x) = |x|$, $p_k = l$, $v_k = 1$ for all $k \in N$, $s = 0$ and $n = 0$, we get the modular space $\ell \{ M_k \}$ introduced and studied by Woo [13].

2 Main Results

In this section, we give the theorems that characterize the structure of the class of sequences $\ell \left\{ M_k, p, q, s, \Delta^n_{(vm)} \right\}$ and some other spaces which can be derived from this space.

**Theorem 1.** Let $p = (p_k)$ be bounded sequence of positive reals, then $\ell \left\{ M_k, p, q, s, \Delta^n_{(vm)} \right\}$ is a linear space over the field $C$.

**Proof.** Let $x, y \in \ell \left\{ M_k, p, q, s, \Delta^n_{(vm)} \right\}$ and $\alpha, \beta \in C$. Then there exist some $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{k=1}^{\infty} k^{-s} \left[ M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\rho_1} \right) \right) \right]^{p_k} < \infty$$

and

$$\sum_{k=1}^{\infty} k^{-s} \left[ M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\rho_2} \right) \right) \right]^{p_k} < \infty.$$
We consider $\rho_3 = \max (2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since each $M_k$ is non-decreasing and convex, and since $q$ is a seminorm,

$$\sum_{k=1}^{\infty} k^{-s} \left[ M_k \left( q \left( \frac{\Delta_{nvm}^n (\alpha x_k + \beta y_k)}{\rho_3} \right) \right) \right]^{p_k}$$

$$\leq \sum_{k=1}^{\infty} k^{-s} \left[ M_k \left( q \left( \frac{\Delta_{nvm}^n (\alpha x_k)}{\rho_3} \right) + q \left( \frac{\Delta_{nvm}^n (\beta y_k)}{\rho_3} \right) \right) \right]^{p_k}$$

$$\leq \sum_{k=1}^{\infty} k^{-s} \left[ M_k \left( q \left( \frac{\Delta_{nvm}^n x_k}{\rho_1} \right) + q \left( \frac{\Delta_{nvm}^n y_k}{\rho_2} \right) \right) \right]^{p_k}$$

$$\leq D \sum_{k=1}^{\infty} k^{-s} \left[ M_k \left( q \left( \frac{\Delta_{nvm}^n x_k}{\rho_1} \right) \right) \right]^{p_k}$$

$$+ D \sum_{k=1}^{\infty} k^{-s} \left[ M_k \left( q \left( \frac{\Delta_{nvm}^n y_k}{\rho_2} \right) \right) \right]^{p_k}$$

$$< \infty$$

where, $D = \max \{1, 2H^{-1}\}$ and $H = \sup_k p_k$. Hence this completes the proof. \(\blacksquare\)

**Theorem 2.** $\ell \left\{ M_k, p, q, s, \Delta_{nvm}^n \right\}$ is a paranormed space (need not total paranorm) space with paranorm $g$, defined as follows.

$$g(x) = \inf \left\{ \rho : \sum_{k=1}^{\infty} k^{-s} M_k \left( q \left( \frac{\Delta_{nvm}^n x_k}{\rho} \right) \right) \leq 1, \ n = 1, 2, ... \right\}$$

where $H = \sup_k p_k$.

**Proof.** Clearly $g(x) = g(-x)$. Since $M_k(0) = 0$, for all $k \in N$, we get $\inf \left\{ \rho \right\} = 0$ for $x = \theta$. Now let $x, y \in \ell \left\{ M_k, p, q, s, \Delta_{nvm}^n \right\}$ and let us choose $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{k=1}^{\infty} k^{-s} M_k \left( q \left( \frac{\Delta_{nvm}^n x_k}{\rho_1} \right) \right) \leq 1$$

and

$$\sum_{k=1}^{\infty} k^{-s} M_k \left( q \left( \frac{\Delta_{nvm}^n y_k}{\rho_2} \right) \right) \leq 1$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\sum_{k=1}^{\infty} k^{-s} M_k \left( q \left( \frac{\Delta_{nvm}^n (x_k + y_k)}{\rho} \right) \right) \leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \sum_{k=1}^{\infty} k^{-s} M_k \left( q \left( \frac{\Delta_{nvm}^n x_k}{\rho_1} \right) \right)$$

$$+ \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \sum_{k=1}^{\infty} k^{-s} M_k \left( q \left( \frac{\Delta_{nvm}^n y_k}{\rho_2} \right) \right)$$

$$\leq 1$$
Hence \( g(x + y) \leq g(x) + g(y) \).

Finally, let \( \lambda \) be a given non-zero scalar, then the continuity of the scalar multiplication follows from the following equality

\[
g(\lambda x) = \inf \left\{ \rho_{\mathbb{N}} \mathcal{P} : \sum_{k=1}^{\infty} k^{-s} M_k \left( g \left( \frac{\Delta_{\mathbb{N}}(\lambda x_k)}{\rho} \right) \right) \leq 1, n = 1, 2, \ldots \right\}
\]

\[
= \inf \left\{ (|\lambda| s)_{\mathbb{N}} \mathcal{P} : \sum_{k=1}^{\infty} k^{-s} M_k \left( g \left( \frac{\Delta_{\mathbb{N}}(\lambda x_k)}{s} \right) \right) \leq 1, n = 1, 2, \ldots \right\},
\]

where \( s = \frac{\rho}{|\lambda|} \). This completes the proof. \( \blacksquare \)

The proof of the following Theorem is easy, so it is omitted.

**Theorem 3.** Let \( M = (M_k) \) and \( T = (T_k) \) be sequences of Orlicz functions. For any two sequences \( p = (p_k) \) and \( t = (t_k) \) of bounded positive real numbers and for any two seminorms \( q_1 \) and \( q_2 \) we have

1. If \( q_1 \) is stronger than \( q_2 \), then \( \ell \left\{ M_k, p, q_1, s, \Delta_{\mathbb{N}}^n \right\} \subset \ell \left\{ M_k, p, q_2, s, \Delta_{\mathbb{N}}^n \right\} \).
2. \( \ell \left\{ M_k, p, q_1, s, \Delta_{\mathbb{N}}^n \right\} \cap \ell \left\{ M_k, p, q_2, s, \Delta_{\mathbb{N}}^n \right\} \subset \ell \left\{ M_k, p, q_1 + q_2, s, \Delta_{\mathbb{N}}^n \right\} \).
3. \( \ell \left\{ M_k, p, q_1, s, \Delta_{\mathbb{N}}^n \right\} \cap \ell \left\{ T_k, p, q_1, s, \Delta_{\mathbb{N}}^n \right\} \subset \ell \left\{ M_k + T_k, p, q_1, s, \Delta_{\mathbb{N}}^n \right\} \).
4. If \( s_1 \leq s_2 \), then \( \ell \left\{ M_k, p, q_1, s_1, \Delta_{\mathbb{N}}^n \right\} \subset \ell \left\{ M_k, p, q_2, s_2, \Delta_{\mathbb{N}}^n \right\} \).
5. The inclusions \( \ell \left\{ M_k, p, q_1, s, \Delta_{\mathbb{N}}^{n-1} \right\} \subset \ell \left\{ M_k, p, q_2, s, \Delta_{\mathbb{N}}^n \right\} \) are strict.

In general, \( \ell \left\{ M_k, p, q, s, \Delta_{\mathbb{N}}^n \right\} \subset \ell \left\{ M_k, p, q, s, \Delta_{\mathbb{N}}^n \right\} \) for \( i = 1, 2, 3, \ldots, n - 1 \) and the inclusion is strict.

**Theorem 4.** Let \( M = (M_k) \) and \( T = (T_k) \) be sequences of Orlicz functions which satisfy \( \Delta_2 \)-condition and \( s > 1 \), then

\[
\ell \left\{ M_k, p, q, s, \Delta_{\mathbb{N}}^n \right\} \subset \ell \left\{ T_k \circ M_k, p, q, s, \Delta_{\mathbb{N}}^n \right\}.
\]

**Proof.** Let \( x \in \ell \left\{ M_k, p, q, s, \Delta_{\mathbb{N}}^n \right\} \) and \( \varepsilon > 0 \). We choose \( 0 < \delta < 1 \) such that \( M(u) < \varepsilon \) for \( 0 \leq u \leq \delta \). We write \( y_k = M_k \left( q \left( \frac{\Delta_{\mathbb{N}}^n x_k}{\rho} \right) \right) \) and consider

\[
\sum_{k=1}^{\infty} k^{-s} [T_k(y_k)]^p = \sum_{k=1}^{\infty} k^{-s} [T_k(y_k)]^p + \sum_{k=2}^{\infty} k^{-s} [T_k(y_k)]^p
\]

where the first summation is over \( y_k \leq \delta \) and the second over \( y_k > \delta \). Since \( s > 1 \), we have

\[
\sum_{k=1}^{\infty} k^{-s} [T_k(y_k)]^p < \max (1, \varepsilon^H) \sum_{k=1}^{\infty} k^{-s} < \infty.
\]
For \( y_k > \delta \), we use the fact that \( y_k < \frac{y_k}{\delta} \leq 1 + \left( \frac{y_k}{\delta} \right) \).

Since each \( T_k \) is non-decreasing and convex, it follows that, for each \( k \in N \),
\[
T_k (y_k) < T_k \left( 1 + \frac{y_k}{\delta} \right) < \frac{1}{2} T_k (2) + \frac{1}{2} T_k \left( 2 \frac{y_k}{\delta} \right).
\]

Since each \( T_k \) is satsfy \( \Delta_2 \)-condition, we have
\[
T_k (y_k) < \frac{1}{2} K \frac{y_k}{\delta} T_k (2) + \frac{1}{2} K \frac{y_k}{\delta} T_k (2) = Ky_k \delta^{-1} T_k (2).
\]

Hence
\[
\sum_{k} k^{-s} [T_k (y_k)]^{p_k} \leq \max \left( 1, \left( K \delta^{-1} M (2) \right)^H \right) \sum_{k=1}^{\infty} k^{-s} (y_k)^{p_k} < \infty.
\]

Thus
\[
\sum_{k=1}^{\infty} k^{-s} [T_k (y_k)]^{p_k} = \sum_{k=1}^{\infty} k^{-s} [T_k (y_k)]^{p_k} + \sum_{k=2}^{\infty} k^{-s} [T_k (y_k)]^{p_k}
\]
\[
\leq \max \left( 1, \epsilon^H \right) \sum_{k=1}^{\infty} k^{-s} + \max \left( 1, \left( K \delta^{-1} M (2) \right)^H \right) \sum_{k=1}^{\infty} k^{-s} (y_k)^{p_k}
\]
\[
< \infty.
\]

Hence \( x \in \ell \left\{ T_k \circ M_k, p, q, s, \Delta_{(vm)}^{n} \right\} \). This completes the proof.

Taking \( M_k (x) = x \), for all \( k \) in \( N \), in the Theorem 4, we get the next Corollary.

**Corollary 5.** Let \( M = (M_k) \) be any sequence of Orlicz functions which satisfy \( \Delta_2 \)-condition and \( s > 1 \), then
\[
\ell \left\{ p, q, s, \Delta_{(vm)}^{n} \right\} \subseteq \ell \left\{ M_k, p, q, s, \Delta_{(vm)}^{n} \right\}.
\]

We will write \( f \approx g \) for non-negative functions \( f \) and \( g \) whenever \( C_1 f \leq g \leq C_2 f \) for some \( C_j > 0 \), \( j = 1, 2 \).

**Theorem 6.** Let \( M = (M_k) \) and \( T = (T_k) \) be a sequence of Orlicz functions. If \( M_k \approx T_k \) for each \( k \in N \), then \( \ell \left\{ M_k, p, q, s, \Delta_{(vm)}^{n} \right\} = \ell \left\{ T_k, p, q, s, \Delta_{(vm)}^{n} \right\} \).

**Proof.** Proof is obvious.

**Theorem 7.** Let \( M = (M_k) \) be a sequence of Orlicz functions. If \( \lim_{t \to 0} \frac{M_k(t)}{t} > 0 \) and \( \lim_{t \to 0} \frac{M_k(t)}{t} < \infty \) for each \( k \in N \), then
\[
\ell \left\{ M_k, p, q, s, \Delta_{(vm)}^{n} \right\} = \ell \left\{ p, q, s, \Delta_{(vm)}^{n} \right\}.
\]
Proof. If the given conditions are satisfied, we have $M_k(t) \approx t$ for each $k$ and the proof follows from Theorem 5.

If we take $s=0$, the sequence space $\ell\left\{M_k, p, q, \Delta_n^{(vm)}\right\}$ reduce to the following sequence space:

$$\ell\left\{M_k, p, q, \Delta_n^{(vm)}\right\} = \left\{x \in w(X) : \sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta_n^{(vm)}x_k}{\rho}\right)\right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$ 

**Theorem 8.** Let $p = (p_k)$ be bounded sequence of positive reals and $(X, q)$ be a complete seminormed space , then $\ell\left\{M_k, p, q, \Delta_n^{(vm)}\right\}$ is a complete paranormed space paranormed by $h$, defined by

$$h(x) = \inf \left\{\rho^{\frac{p_k}{p}} : \sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta_n^{(vm)}x_k}{\rho}\right)\right)^{p_k} \leq 1, n = 1, 2, \ldots \right\},$$

where $H = \sup_k p_k$.

**Proof.** Let $(x^i)$ be a Cauchy sequence in $\ell\left\{M_k, p, q, \Delta_n^{(vm)}\right\}$. Let $\delta > 0$ be fixed and $r > 0$ be such that for a given $0 < \epsilon < 1$, $\frac{\epsilon}{r^2} > 0$, and $r\delta \geq 1$. Then there exists a positive integer $n_0$ such that

$$h(x^i - x^j) < \frac{\epsilon}{r^2}$$

for all $i, j \geq n_0$

$$h(x^i - x^j) = \inf \left\{\rho^{\frac{p_k}{p}} : \sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta_n^{(vm)}x^i_k - \Delta_n^{(vm)}x^j_k}{\rho}\right)\right)^{p_k} \leq 1 \right\} < \frac{\epsilon}{r^2}$$

for all $i, j \geq n_0$. Hence we have

$$\sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta_n^{(vm)}x^i_k - \Delta_n^{(vm)}x^j_k}{h(x^i - x^j)}\right)\right) \leq 1$$

for all $i, j \geq n_0$. It follows that

$$M_k\left(q\left(\frac{\Delta_n^{(vm)}x^i_k - \Delta_n^{(vm)}x^j_k}{h(x^i - x^j)}\right)\right) \leq 1$$

for all $i, j \geq n_0$ and $k \in N$. For $r > 0$ with $M_k\left(\frac{\epsilon}{2}\right) \geq 1$, we have

$$M_k\left(q\left(\frac{\Delta_n^{(vm)}x^i_k - \Delta_n^{(vm)}x^j_k}{h(x^i - x^j)}\right)\right) \leq M_k\left(\frac{r\delta}{2}\right).$$
for all \( i, j \geq n_0 \) and \( k \in N \). Since \( M_k \) is non-decreasing for each \( k \in N \), we have

\[
q \left( \Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k^j \right) \leq \frac{r \delta}{2} \frac{\epsilon}{r \delta} = \frac{\epsilon}{2}.
\]

Hence \( \left( \Delta_{(vm)}^n x_k^i \right) \) is a Cauchy sequence in \((X, q)\) for each \( k \in N \). But \((X, q)\) is complete and so \( \left( \Delta_{(vm)}^n x_k^i \right) \) is convergent in \((X, q)\) for each \( k \in N \).

Let \( \lim_{i \to \infty} \Delta_{(vm)}^n x_k^i = y_k \) for all \( k \geq 1 \). Let \( k = 1 \), then we have

\[
\lim_{i \to \infty} \Delta_{(vm)}^n x_1^i = \lim_{i \to \infty} \sum_{v=0}^{n} (-1)^v \binom{n}{v} v_{1-mv} x_1^1 = \lim_{i \to \infty} v_{1} x_1^1 = y_1
\]

(1)

Simaly we have,

\[
\lim_{i \to \infty} \Delta_{(vm)}^n x_k^i = \lim_{i \to \infty} v_k x_k^i = y_k \text{ for } k = 1, \ldots, nm
\]

(2)

Thus from (2.1) and (2.2), we have \( \lim x_1^i + nm \) exists. Let \( \lim_{i \to \infty} x_1^i + nm = x_{1+nm} \).

Proceeding in this way inductively, we have \( \lim_{i \to \infty} x_k^i = x_k \) for each \( k \in N \). Now we have for all \( i, j \geq n_0 \),

\[
\inf \left\{ \rho^M : \sum_{k=1}^{\infty} M_k \left( q \left( \frac{\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k^j}{\rho} \right) \right) \leq 1 \} < \epsilon.
\]

Then we have

\[
\lim_{j \to \infty} \left\{ \inf \left\{ \rho^M : \sum_{k=1}^{\infty} M_k \left( q \left( \frac{\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k^j}{\rho} \right) \right) \leq 1 \} \right\} < \epsilon
\]

for all \( i \geq n_0 \). Using the continuity of Orlicz functions, we have

\[
\inf \left\{ \rho^M : \sum_{k=1}^{\infty} M_k \left( q \left( \frac{\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k^j}{\rho} \right) \right) \leq 1 \} < \epsilon
\]

for all \( i \geq n_0 \). This implies

\[
\inf \left\{ \rho^M : \sum_{k=1}^{\infty} M_k \left( q \left( \frac{\Delta_{(vm)}^n x_k^i - \Delta_{(vm)}^n x_k^j}{\rho} \right) \right) \leq 1 \} < \epsilon
\]

for all \( i \geq n_0 \). It follows that \((x^i - x) \in \ell \left\{ M_k, p, q, \Delta_{(vm)}^n \right\} \). Since \((x^i) \in \ell \left\{ M_k, p, q, \Delta_{(vm)}^n \right\} \) and \( \ell \left\{ M_k, p, q, \Delta_{(vm)}^n \right\} \) is a linear space, so we have \( x = x^i - (x^i - x) \in \ell \left\{ M_k, p, q, \Delta_{(vm)}^n \right\} \). This completes the proof.
If we take $s = 0$ and $p_k = l$, the sequence space $\ell \left\{ M_k, p, q, s, \Delta^p_{(vm)} \right\}$ reduce to the following sequence space:

$$\ell \left\{ M_k, q, \Delta^p_{(vm)} \right\} = \left\{ x \in w(X) : \sum_{k=1}^{\infty} M_k \left( q \left( \frac{\Delta^p_{(vm)} x_k}{\rho} \right) \right) < \infty, \text{for some } \rho > 0 \right\}.$$ 

**Theorem 9.** Let $(X, q)$ be a complete normed space, then $\ell \left\{ M_k, q, \Delta^p_{(vm)} \right\}$ is a Banach space normed by $\| \cdot \|$, defined by

$$\| x \| = \inf \left\{ \rho : \sum_{k=1}^{\infty} M_k \left( q \left( \frac{\Delta^p_{(vm)} x_k}{\rho} \right) \right) \leq 1 \right\}.$$ 

**Proof.** We prove that $\| \cdot \|$ is a norm on $\ell \left\{ M_k, q, \Delta^p_{(vm)} \right\}$. The completeness part can be proved using similar arguments as applied to prove above Theorem.

If $x = 0$, then it is obvious that $\| x \| = 0$. Conversely assume $\| x \| = 0$. Then using the definition of norm, we have

$$\inf \left\{ \rho : \sum_{k=1}^{\infty} M_k \left( q \left( \frac{\Delta^p_{(vm)} x_k}{\rho} \right) \right) \leq 1 \right\} = 0.$$ 

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon \left( 0 < \rho_\epsilon < \epsilon \right)$ such that

$$\sum_{k=1}^{\infty} M_k \left( q \left( \frac{\Delta^p_{(vm)} x_k}{\rho_\epsilon} \right) \right) \leq 1.$$ 

Thus

$$M_k \left( q \left( \frac{\Delta^p_{(vm)} x_k}{\epsilon} \right) \right) \leq M_k \left( q \left( \frac{\Delta^p_{(vm)} x_k}{\rho_\epsilon} \right) \right) \leq 1, \forall k \in N.$$ 

Suppose that $\Delta^p_{(vm)} x_{n_i} \neq 0$ for some $i$. Let $\epsilon \to 0$, then $\left| \Delta^p_{(vm)} x_{n_i} \right| / \epsilon \to \infty$. It follows that $M_k \left( \left| \Delta^p_{(vm)} x_{n_i} \right| / \epsilon \right) \to \infty$ as $\epsilon \to 0$ for some $n_i \in N$. This is a contradiction. Therefore $\Delta^p_{(vm)} x_k = 0$ for all $k \in N$. Let $k = 1$, then $\Delta^p_{(vm)} x_1 = \sum_{i=0}^{n} (-1)^i \binom{n}{i} v_{1-m_i} x_{1-m_i} = 0$, and so $v_1 x_1 = 0$, by putting $v_{1-m_i} = 0$ and $x_{1-m_i} = 0$ for $i = 1, 2, ..., n$.

Hence $x_1 = 0$, since $(\lambda_k)$ is a sequence of non-zero scalars. Similarly taking $k = 2, \ldots, mn$, we have $x_2 = \cdots = x_{mn} = 0$. Next let $k = mn + 1$, then $\Delta^p_{(vm)} x_{mn+1} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} v_{mn-m_i} x_{mn-m_i} = 0$. Since $x_1 = x_2 = \cdots = x_{mn} = 0$, we must have $v_{mn+1} x_{mn+1} = 0$ and thus $x_{mn+1} = 0$. Proceeding in this way we can conclude that $x_k = 0$ for all $k \geq 1$. Hence $x = 0$. Again proof of the properties $\| x + y \| \leq \| x \| + \| y \|$ and for any scalar $\alpha$, $\| \alpha x \| = |\alpha| \| x \|$ are similar to that Theorem 2. It is easy to see that $\| x^i \| \to 0$ implies that $x^i_k \to 0$ for each $i \geq 1$. \qed
Proposition 10. $\ell \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ is a $BK$-space.

Now we study the $AK$-characteristic of the space $\ell \left\{ M_k, q, s, \Delta^n_{(vm)} \right\}$. Before that we give a new definition and prove some results those will be required.

Definition 1. For any sequence of Orlicz functions $M = (M_k)$,

$$h \left\{ M_k, q, \Delta^n_{(vm)} \right\} = \left\{ x \in w(X) : \sum_{k=1}^{\infty} M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right) < \infty, \text{ for every } \rho > 0 \right\}.$$ 

Clearly $h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ is a subspace of $\ell \left\{ M_k, q, \Delta^n_{(vm)} \right\}$. The topology of $h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ is the one it inherits from $\| \|$. 

Proposition 11. Let $M = (M_k)$ be a sequence of Orlicz functions which satisfy $\Delta_2$-condition. Then

$$\ell \left\{ M_k, q, \Delta^n_{(vm)} \right\} = h \left\{ M_k, q, \Delta^n_{(vm)} \right\}.$$ 

Proof. It is enough to prove that $\ell \left\{ M_k, q, \Delta^n_{(vm)} \right\} \subseteq h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$. Let $x \in h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$, then for some $\rho > 0$,

$$\sum_{k=1}^{\infty} M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right) < \infty$$

Therefore

$$M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right) < \infty \text{ for every } k \geq 1.$$ 

Choose an arbitrary $\eta > 0$. If $\rho \leq \eta$, then $M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\eta} \right) \right) < \infty$ for every $k \geq 1$ and so

$$\sum_{k=1}^{\infty} M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\eta} \right) \right) < \infty$$

Let now $\eta < \rho$ and put $l = \frac{\rho}{\eta} > 1$. Since $M$ satisfied the $\Delta_2$-condition, there exists a constant $K$ such that

$$M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\eta} \right) \right) \leq K \left( \frac{\rho}{\eta} \right)^{\log_2 K} M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right)$$

for every $k \geq 1$. Now we can find $U > 0$ with $s > 1$ such that

$$M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right) < U k^{-s}$$
for the fixed $\rho > 0$ and for every $k \geq 1$. Then it follows that for every $\eta > 0$, we have
\[ \sum_{k=1}^{\infty} M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\eta} \right) \right) < K \left( \frac{\rho}{\eta} \right) \log_2 K M \sum_{k=1}^{\infty} k^{-s} < \infty. \]
This completes the proof.

**Proposition 12.** Let $(X, q)$ be a complete normed space, then $h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ is an $AK$-space.

**Proof.** Let $x \in h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$. Then for each $\epsilon$, $0 < \epsilon < 1$, we can find an $s_0$ such that
\[ \sum_{k \geq s_0} M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\epsilon} \right) \right) \leq 1. \]
Hence for $s \geq s_0$,
\[
\left\| x - x^{[s]} \right\| = \inf \left\{ \rho > 0 : \sum_{k \geq s+1} M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\epsilon} \right) \right) \leq 1 \right\} 
\leq \inf \left\{ \rho > 0 : \sum_{k \geq s} M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right) \leq 1 \right\} \leq \epsilon.
\]
Thus we can conclude that $h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ is an $AK$ space.

**Theorem 13.** Let $\mathbf{M} = (M_k)$ be a sequence of Orlicz functions which satisfy $\Delta_2$-condition, then $\ell \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ is an $AK$-space.

**Proposition 14.** $h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ is a closed subspace of $\ell \left\{ M_k, q, \Delta^n_{(vm)} \right\}$.

**Proof.** Let $\{x^s\}$ be a sequence in $h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ such that $\|x^s - x\| \to 0$, where $x \in h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$. To complete the proof we need to show that $x \in h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$, i.e.,
\[ \sum_{k \geq 1} M_k \left( q \left( \frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right) < \infty \text{ for every } \rho > 0. \]
For $\rho > 0$, there corresponds an $l$ such that $\|x^l - x\| \leq \frac{\rho}{2}$. Then using convexity of
each $M_k$,
\[
\sum_{k \geq 1} M_k \left( q \left( \frac{\| \Delta_{vm}^n x_k \|}{\rho} \right) \right) = \sum_{k \geq 1} M_k \left( q \left( \frac{2 \left| \Delta_{vm}^n x_k \right| - 2 \left( \left| \Delta_{vm}^n x_k \right| - \left| \Delta_{vm}^n x_l \right| \right)}{2\rho} \right) \right)
\leq \frac{1}{2} \sum_{k \geq 1} M_k \left( q \left( \frac{2 \left| \Delta_{vm}^n x_k \right|}{\rho} \right) \right) + \frac{1}{2} \sum_{k \geq 1} M_k \left( q \left( \frac{2 \left| \Delta_{vm}^n x_l - x_k \right|}{\rho} \right) \right)
\leq \frac{1}{2} \sum_{k \geq 1} M_k \left( q \left( \frac{2 \left| \Delta_{vm}^n x_l - x_k \right|}{\| x_l - x \|} \right) \right)
\]

Now from Theorem 8, using the definition of norm $\| . \|$, we have
\[
\sum_{k \geq 1} M_k \left( q \left( \frac{2 \left| \Delta_{vm}^n (x_l - x_k) \right|}{\| x_l - x \|} \right) \right) \leq 1
\]

It follows that
\[
\sum_{k \geq 1} M_k \left( q \left( \frac{\left| \Delta_{vm}^n x_k \right|}{\rho} \right) \right) < \infty \quad \text{for every } \rho > 0
\]

Thus $x \in h \left\{ M_k, q, \Delta_{vm}^n \right\}$

Hence we have the following Corollary

**Corollary 15.** $h \left\{ M_k, q, \Delta_{vm}^n \right\}$ is a BK-space.

**References**


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