UNIVALENCE CRITERION FOR TWO INTEGRAL OPERATORS

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Abstract. In this paper we extend some results obtained by Breaz et al. in [3], for a general integral operators $G_{n,\alpha}(z)$, $G_{\alpha_1,\ldots,\alpha_n}(z)$ and $J_{\alpha_1,\alpha_2,\ldots,\alpha_n,\gamma}$.

1 Introduction and preliminaries

Let $\mathbb{U} = \{z : |z| < 1\}$ the unit disc and $\mathcal{A}$ the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in $\mathbb{U}$. We denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Lemma 1.1. (Schwarz Lemma) [5] Let the analytic function $f$ be regular in the open unit disk $\mathbb{U}$ and let $f(0) = 0$. If $|f(z)| \leq 1$, for all $z \in \mathbb{U}$, then

$$|f(z)| \leq |z| \quad (z \in \mathbb{U}), \quad (2)$$

where the equality holds true only if

$$|f(z)| = Kz \quad (z \in \mathbb{U}) \quad \text{and} \quad K = 1. \quad (3)$$

Pescar has proved the following univalent condition:

Theorem 1.1. [7] Let $\alpha \in \mathbb{C}$ (Re($\alpha$) > 0) and $c \in \mathbb{C}$ (|c| ≤ 1; c ≠ -1). Suppose also that the function $f(z)$ given by (1) is analytic in $\mathbb{U}$. If

$$|ez^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)}| \leq 1 \quad (z \in \mathbb{U}),$$

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then the function $F_\alpha(z)$ defined by

$$F_\alpha(z) := \left( \alpha \int_0^z t^{\alpha-1} f'(t) dt \right)^{\frac{1}{\alpha}} = z + \ldots$$

is analytic and univalent in $U$.

The following theorem is another univalent condition which was proved by Ozaki and Nunokawa [6]:

**Theorem 1.2.** [6] Let $f \in A$ satisfy the following inequality:

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \leq 1 \quad (z \in U) \quad (4)$$

then $f$ is univalent in $U$.

D. Breaz et al. in [3] proved next theorem:

**Theorem 1.3.** [3] Let $M \geq 1$ and suppose that each of the functions $g_j \in A$, $(j \in \{1, \ldots, n\})$ satisfies the inequality (4). Also let $\alpha \in \mathbb{R}$, $\left( \alpha \in \left[ 1, \frac{(2M+1)n}{(2M+1)n-1} \right] \right)$ and $c \in \mathbb{C}$. If

$$|c| \leq 1 + \left( \frac{1 - \alpha}{\alpha} \right) (2M + 1)n$$

and

$$|g_j(z)| \leq M \quad (z \in U; j \in \{1, \ldots, n\}),$$

then the function

$$G_{n,\alpha}(z) = \left( (n\alpha - 1) + 1 \int_0^z (g_1(t))^{\alpha-1} \ldots (g_n(t))^{\alpha-1} dt \right)^{\frac{1}{n(n-1)+1}} \quad (5)$$

is in the univalent function class $S$.

## 2 Main results

**Theorem 2.1.** Let $g_i \in A$ for $i \in \{1, \ldots, n\}$ all the functions which satisfies the inequality (4) and $M_i \geq 1$.

We consider $\alpha_i \in \mathbb{R}$ $\left( \alpha_i \in \left[ \frac{1}{n}, \max \left\{ \frac{(2M_i+1)n}{(2M_i+1)n-1} \right\} \right] \right)$ and $c \in \mathbb{C}$. If

$$|c| \leq 1 + \frac{n}{n \left( \sum_{i=1}^n \alpha_i - 1 \right) + 1} \cdot \max_{1 \leq i \leq n} (\alpha_i - 1)(2M_i + 1) \quad (6)$$
We consider the function
\[ |g_i(z)| \leq M_i \quad (z \in U, i \in \{1, \ldots, n\}), \tag{7} \]
then the function
\[ G_{\alpha_1, \ldots, \alpha_n, n}(z) = \left( \left( n \left( \sum_{i=1}^{n} \alpha_i - 1 \right) + 1 \right) \int_0^z (g_1(t))^{\alpha_1 - 1} \cdots (g_n(t))^{\alpha_n - 1} dt \right)^{- \frac{1}{n(\sum_{i=1}^{n} \alpha_i - 1) + 1}} \tag{8} \]
belongs to the univalent function class \( \mathcal{S} \).

Proof. We consider the function
\[ f(z) = \int_0^z \prod_{i=1}^{n} \left( \frac{g_i(t)}{t} \right)^{\alpha_i - 1} dt. \]
From here we have that
\[ \frac{zf''(z)}{f'(z)} = \sum_{i=1}^{n} (\alpha_i - 1) \left( \frac{zg_i'(z) - g_i(z)}{g_i(z)} - 1 \right) \]
So
\[ \left| c|z|^{2(n(\sum_{i=1}^{n} \alpha_i - 1) + 1)} + (1 - |z|)^{2(n(\sum_{i=1}^{n} \alpha_i - 1) + 1)} \right| \frac{zg''(z)}{(f'(z))^2} \]
\[ = \left| c|z|^{2(n(\sum_{i=1}^{n} \alpha_i - 1) + 1)} + (1 - |z|)^{2(n(\sum_{i=1}^{n} \alpha_i - 1) + 1)} \right| \frac{1}{(n(\sum_{i=1}^{n} \alpha_i - 1) + 1)} \sum_{i=1}^{n} (\alpha_i - 1) \left( \frac{zg_i'(z) - g_i(z)}{g_i(z)} - 1 \right) \]
\[ \leq |c| + \left( \frac{1}{n(\sum_{i=1}^{n} \alpha_i - 1) + 1} \right) \sum_{i=1}^{n} \left( \frac{|zg_i'(z)|}{|g_i(z)|} + 1 \right). \]
From (7), because \( |g_i(z)| \leq M_i \) for \( i \in \{1, \ldots, n\} \), and (4) we obtain that
\[ \left| c|z|^{2(n(\sum_{i=1}^{n} \alpha_i - 1) + 1)} + (1 - |z|)^{2(n(\sum_{i=1}^{n} \alpha_i - 1) + 1)} \right| \frac{zg''(z)}{(f'(z))^2} \]
\[ \leq |c| + \frac{1}{n(\sum_{i=1}^{n} \alpha_i - 1) + 1} \sum_{i=1}^{n} (\alpha_i - 1)(2M_i + 1) \]
\[ \leq |c| + \frac{n}{n(\sum_{i=1}^{n} \alpha_i - 1) + 1} \max_{1 \leq i \leq n} (\alpha_i - 1)(2M_i + 1). \]
According with (6), we have
\[ \left| c|z|^{2(n(\sum_{i=1}^{n} \alpha_i - 1) + 1)} + (1 - |z|)^{2(n(\sum_{i=1}^{n} \alpha_i - 1) + 1)} \right| \frac{zg''(z)}{(n(\sum_{i=1}^{n} \alpha_i - 1) + 1)f'(z)} \leq 1 \quad (z \in U) \]
Now, applying Theorem 1.1 we obtain that the function \( G_{\alpha_1, \ldots, \alpha_n, n}(z) \) defined by (8) is in \( \mathcal{S} \).
For \( n = 1 \) in Theorem 2.1 we obtain:

**Corollary 2.1.** Let \( g \in A \) a function that satisfies the inequality (4) and \( M \geq 1 \). We consider \( \alpha \in \mathbb{R} (\alpha \in [1, \frac{2M+1}{2M} + 1]) \) and \( c \in \mathbb{C} \). If

\[
|c| \leq 1 + \frac{\alpha - 1}{\alpha} (2M + 1)
\]

and \( |g(z)| \leq M \) for all \( z \in U \), then the function

\[
G_\alpha(z) = \left( \alpha \int_0^z (g(t))^{\alpha-1} dt \right)^\frac{1}{\alpha}
\]

is in the univalent function class \( S \).

For \( \alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha \) we obtain

**Corollary 2.2.** Let \( g_i \in A \) for \( i \in \{1, \ldots, n\} \) all the functions which satisfies the inequality (4) and \( M_i \geq 1 \).

We consider \( \alpha \in \mathbb{R} \left( \alpha \in \left[ 1, \frac{1}{n} \max \left\{ \frac{(2M_i+1)n}{M_i} - 1 \right\} \right] \right) \) and \( c \in \mathbb{C} \).

If

\[
|c| \leq 1 + \frac{n}{n(\alpha - 1) + 1} \cdot \max_{1 \leq i \leq n} (\alpha - 1)(2M_i + 1)
\]

and

\[
|g_i(z)| \leq M_i \quad (z \in U, i \in \{1, \ldots, n\}),
\]

then the function \( G_{n, \alpha}(z) \) defined by (5) is in \( S \).

The integral operator

\[
J_{\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma} = \left( \gamma \int_0^z t^{\gamma-1} \prod_{j=1}^n \left( \frac{f_j(t)}{t} \right)^{\alpha_j} dt \right)^\frac{1}{\gamma}
\]

was introduced and studied by D. Breaz and N. Breaz in [1].

**Theorem 2.2.** Let \( f_j \in A \) for \( j \in \{1, \ldots, n\} \) all the functions which satisfies the inequality (4) and \( M_j \geq 1 \).

We consider \( \alpha_j \in \mathbb{R} \left( \alpha_j \in \left[ 1, \max \left\{ \frac{(2M_j+1)n}{M_j} - 1 \right\} \right] \right) \) and \( \gamma, c \in \mathbb{C} \).

If

\[
|c| \leq 1 + \frac{n}{|\gamma|} \cdot \max_{1 \leq j \leq n} \alpha_j (2M_j + 1)
\]

and

\[
|f_j(z)| \leq M_j \quad (z \in U, j \in \{1, \ldots, n\}),
\]

then the function \( J_{\alpha_1, \ldots, \alpha_n, \gamma}(z) \) given by (9) is in the univalent function class \( S \).
Proof. We define the function

$$h(z) = \int_0^z \prod_{j=1}^n \left( \frac{f_j(t)}{t} \right)^{\alpha_j} dt.$$ 

Then from here we have that

$$\frac{h''(z)}{h'(z)} = \sum_{j=1}^n \alpha_j \left( \frac{zf_j'(z) - f_j(z)}{f_j(z)} \right)$$

So

$$\left| c |z|^{2\gamma} + (1 - |z|^{2\gamma}) \frac{zh''(z)}{h'(z)} \right| = \left| c |z|^{2\gamma} + (1 - |z|^{2\gamma}) \frac{1}{\gamma} \sum_{j=1}^n \alpha_j \left( \frac{zf_j'(z) - f_j(z)}{f_j(z)} \right) \right|$$

$$\leq |c| + \frac{1}{\gamma} \sum_{j=1}^n \alpha_j \left( \frac{zf_j'(z) - f_j(z)}{|f_j(z)|} + 1 \right)$$

Because from (11), $|f_j(z)| \leq M_j$ for $z \in U$ and $j \in \{1, \ldots, n\}$, using the inequality (4), we obtain that

$$\left| c |z|^{2\gamma} + (1 - |z|^{2\gamma}) \frac{zh''(z)}{h'(z)} \right| \leq |c| + \frac{1}{\gamma} \sum_{j=1}^n \alpha_j (2M_j + 1)$$

$$\leq |c| + \frac{n}{\gamma} \max_{1 \leq j \leq n} \alpha_j (2M_j + 1)$$

Now, using the hypothesis (10) results

$$\left| c |z|^{2\gamma} + (1 - |z|^{2\gamma}) \frac{zh''(z)}{h'(z)} \right| \leq 1, \quad z \in U$$

Applying the Theorem 1.1 we obtain that the function $J_{\alpha_1, \ldots, \alpha_n, \gamma}(z)$ is in the univalent functions class $S$. \qed

Remark 2.1. For $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ and $\frac{1}{\gamma} = \frac{1}{n(\alpha - 1) + 1}$ in Theorem 2.2 we obtain the theorem proved by Breaz et al. in [3] for the function $G_{n, \alpha}(z)$.

Corollary 2.3. Let $f \in A$ the function that satisfies the inequality (4) and $M \geq 1$. We suppose that $\alpha \in \mathbb{R}$, $(\alpha \in [1, \frac{2M+1}{2M}])$ and $\gamma, c \in \mathbb{C}$.

If

$$|c| \leq 1 + \frac{\alpha}{\gamma} (2M + 1)$$

and

$$|f(z)| \leq M, \quad z \in U$$

then the function $J_{\alpha, \gamma}(z) = \left( \frac{\gamma}{\alpha} t^{\gamma-1} \left( \frac{t^\alpha}{\alpha} \right) dt \right)^{\frac{1}{\gamma}}$ is in the univalent functions class $S$. 


Proof. In Theorem 2.2 we consider $n = 1$. 

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References


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