ESTIMATION OF A CONDITION NUMBER RELATED TO $A_{T,S}^{(2)}$

Dijana Mosić

Abstract

In this paper we get estimation of the absolute condition number a Hilbert space operator, which is related with the outer generalized inverse of a given operator.

1 Introduction

In this paper $X$ and $Y$ denote arbitrary Hilbert spaces. We use $B(X, Y)$ to denote the set of all linear bounded operators from $X$ to $Y$. Set $B(X) = B(X, X)$.

Let $A \in B(X, Y)$. We use $R(A)$ and $N(A)$, respectively, to denote the range and the null-space of $A$. If there exists some operator $A' \in B(Y, X)$ satisfying $A'AA' = A'$, then $A'$ is called the outer inverse of $A$ [1]. If $T = R(A')$ and $S = N(A')$, then $A'$ is well-known as the $A_{T,S}^{(2)}$ generalized inverse of $A$. It can easily be deduced that for given subspaces $T$ of $X$ and $S$ of $Y$, there exists the generalized inverse $A_{T,S}^{(2)}$ of $A$ if and only if the following is satisfied: $T$, $S$ and $A(T)$ are closed complemented subspaces of $X$, $Y$ and $Y$ respectively, the reduction $A_1 = A|_T : T \to A(T)$ is invertible and $A(T) \oplus S = Y$. In this case the generalized inverse $A_{T,S}^{(2)}$ is unique and the notation is justified. Moreover, the following holds $T = R(A_{T,S}^{(2)}) = R(A_{T,S}^{(2)}A)$. Hence, we denote $T_1 = N(A_{T,S}^{(2)}A) \subset X$ and $S_1 = A(T) \subset Y$. Now we have $X = T \oplus T_1$ and $Y = S_1 \oplus S$.

The matrix form of $A$ is as follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \longrightarrow \begin{bmatrix} S_1 \\ S \end{bmatrix},$$ (1)
where $A_1 \in B(T, S_1)$ is invertible. Now it is easy to verify that

$$A^{(2)}_{T,S} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} S_1 \\ S \end{bmatrix} \longrightarrow \begin{bmatrix} T \\ T_1 \end{bmatrix}. \quad (2)$$

We use $B(X, Y)_{T,S}$ to denote the set of all $A \in B(X, Y)$, such that $A^{(2)}_{T,S}$ exists. Here we assume that $T$ and $S$, respectively, are closed subsets of $X$ and $Y$.

Let $X$ and $Y$ be equipped with the norms $\| \cdot \|_X$ and $\| \cdot \|_Y$. The $P$-norm for a vector $x \in X$, the $Q$-norm for a vector $y \in Y$ and the $QP$-norm for an operator $A \in B(X, Y)$, respectively, are defined by (see [10]):

$$\|x\|_P = \sqrt{\|x_1\|_X^2 + \|x_2\|_X^2},$$

$$\|y\|_Q = \sqrt{\|y_1\|_Y^2 + \|y_2\|_Y^2},$$

$$\|A\|_{QP} = \sup_{\|x\|_P \leq 1} \|Ax\|_Q$$

where

$$x = x_1 + x_2, \quad x_1 \in T, \ x_2 \in T_1,$$

$$y = y_1 + y_2, \quad y_1 \in S_1, \ y_2 \in S.$$

Notice that we can also change the inner product in $X$ in the following way:

$$\langle x, y \rangle_P = \langle x_1, y_1 \rangle_X + \langle x_2, y_2 \rangle_X$$

where

$$x = x_1 + x_2, \ y = y_1 + y_2, \ x_1, y_1 \in T, \ x_2, y_2 \in T_1.$$

Now, $\| \cdot \|_P$ is induced by $\langle \cdot, \cdot \rangle_P$. Similarly for $\langle \cdot, \cdot \rangle_Q$ and $\| \cdot \|_Q$ in $Y$.

Generalized inverses are frequently related with the system of equations

$$Ax = b,$$

with $A$ and $b$ given, and $x$ is unknown. If $A$ is invertible, then the condition number of $A$ is defined as $\|A\|\|A^{-1}\|$. If $A$ is singular, then we can use some generalized inverse of $A$ instead of $A^{-1}$. Thus, the generalized condition number of $A$ related with the generalized inverse $A^{(2)}_{T,S}$ (in the case when it exists), is denoted by $\kappa(A) = \|A\|\|A^{(2)}_{T,S}\|$. The other approach to define the condition number of a linear system $Ax = b$, is connected with differentiable functions. Let $A \in B(X, Y)_{T,S}$ and $b \in Y$. Define the mapping

$$F : B(X, Y)_{T,S} \times Y \rightarrow X$$

as follows:

$$F(A, b) = A^{(2)}_{T,S}b.$$
Estimation of a condition number related to $A^{(2)}_{T,S}$

The mapping $F$ is differentiable, if the limit
\[
\lim_{\epsilon \to 0} \frac{F(A + \epsilon E, b + \epsilon f) - F(A, b)}{\epsilon} = F'(A, b)\big|_{(E, f)}
\]
exists for some perturbations $E \in B(X, Y)$ of $A$ and $f \in Y$ of $b$. We assume that $A + \epsilon E \in B(X, Y)_{T,S}$ for small values of $\epsilon \in \mathbb{C}$. If we have this kind of differentiability, then
\[
C(A, b) = \|F'(A, b)\|_{(E, f)}
\]
is the absolute condition number of the linear system $Ax = b$, related with the generalized inverse $A^{(2)}_{T,S}$ and perturbations $E$ of $A$ and $f$ of $b$.

We can get easy the following useful result.

**Theorem 1.1.** Suppose that for $A \in B(X, Y)$ and for closed subspaces $T \subset X$ and $S \subset Y$, there exists the generalized inverse $A^{(2)}_{T,S} \in B(Y, X)$. Let $B = A + E$, $R(E) \subseteq A(T)$ and $N(E) \supseteq T_1$. If $\|A^{(2)}_{T,S}\|_{PQ} \|E\|_{QP} < 1$, then $B^{(2)}_{T,S}$ exists and
\[
B^{(2)}_{T,S} = [I + A^{(2)}_{T,S}E]^{-1}A^{(2)}_{T,S} = A^{(2)}_{T,S}[I + EA^{(2)}_{T,S}]^{-1}.
\]  

**Proof.** This result is analogy with the results in [18] for complex matrices. □

Higham [8] discussed different condition numbers of regular inverses and nonsingular linear systems. Concerning generalized inverses and singular linear systems there are similar results on these problems. Papers [3, 7, 15, 16, 9, 2, 17] have some results when the generalized inverse is a Moore-Penrose inverse, Drazin inverse and generalized Bott-Duffin inverse, respectively. In [13], Y. Wei and H. Diao considered the condition number for the Drazin inverse and the Drazin inverse solution of singular linear system. X. Cui and H. Diao generalized the results of [13] and get the results of the condition number for the $W$-weighted Drazin inverse and the $W$-weighted Drazin inverse solution of a linear system in paper [4]. In [10], we extend the result obtained in [4] to linear bounded operators between Hilbert spaces. In [11], the authors established the condition number of the $W$-weighted Drazin inverse of a rectangular matrix by the Schur decomposition and the spectral norm. Because all generalized inverses belong to outer inverse $A^{(2)}_{T,S}$ with the prescribed range $T$ and null space $S$, we are more interested in the condition numbers connected with the outer inverse $A^{(2)}_{T,S}$. In [5], H. Diao, M. Qin and Y. Wei investigated the condition number of the outer inverse $A^{(2)}_{T,S}$ and the outer inverse $A^{(2)}_{T,S}$ solution of a constrained linear system which extends the results in [13, 4]. They gave the explicit formula of the condition number for the outer inverse $A^{(2)}_{T,S}$ solution of a constrained linear system. The results obtained in [5] are generalized in [12] using the Schur decomposition and the spectral norm. In this paper we extend the result obtained in [5] to linear bounded operators between Hilbert spaces.

2 Absolute condition number of a linear system

First, we prove that the mapping $F$ is differentiable if we assume some conditions.
Lemma 2.1. The mapping \( F : B(X, Y) \times Y \to X \) is a differentiable function, if the perturbation \((E, f)\) of \((A, b)\) fulfils the following condition:

\[
AA^{(2)}_{T,S}E = E, \quad EA^{(2)}_{T,S}A = E, \quad \|A^{(2)}_{T,S}\|_{PQ} \|E\|_{QP} < 1. \tag{4}
\]

Proof. From Theorem 1.1 follows that \((A + \epsilon E)^{(2)}_{T,S}\) exists and

\[
(A + \epsilon E)^{(2)}_{T,S} = [I + A^{(2)}_{T,S} \epsilon E]^{-1} A^{(2)}_{T,S} = [I - \epsilon A^{(2)}_{T,S} E + \epsilon^2 (A^{(2)}_{T,S})^2 - ...]^{-1} A^{(2)}_{T,S} - \epsilon A^{(2)}_{T,S} EA^{(2)}_{T,S} + O(\epsilon^2)
\]

Consider the existence of the limit

\[
\lim_{\epsilon \to 0} \frac{F(A + \epsilon E, b + \epsilon f) - F(A, b)}{\epsilon} = \lim_{\epsilon \to 0} \frac{(A + \epsilon E)^{(2)}_{T,S}(b + \epsilon f) - A^{(2)}_{T,S}b}{\epsilon} = \lim_{\epsilon \to 0} \frac{(A^{(2)}_{T,S} - \epsilon A^{(2)}_{T,S} EA^{(2)}_{T,S} + O(\epsilon^2))(b + \epsilon f) - A^{(2)}_{T,S}b}{\epsilon} = \lim_{\epsilon \to 0} \frac{\epsilon A^{(2)}_{T,S} f - \epsilon A^{(2)}_{T,S} EA^{(2)}_{T,S} (b + \epsilon f)}{\epsilon} = \lim_{\epsilon \to 0} (A^{(2)}_{T,S} f - A^{(2)}_{T,S} EA^{(2)}_{T,S} b - \epsilon A^{(2)}_{T,S} EA^{(2)}_{T,S} f) = -A^{(2)}_{T,S} Ex + A^{(2)}_{T,S} f.
\]

Hence,

\[
F'(A, b)_{|(E, f)} = -A^{(2)}_{T,S} Ex + A^{(2)}_{T,S} f. \quad \square
\]

Let \( A \in B(X, Y), b \in A(T) \) and let us consider the equation

\[
Ax = b, \quad x \in T. \tag{5}
\]

If \( A \in B(X, Y)_{T,S} \), then the equation (5) have a unique solution if and only if \( b \in A(T) \) and \( T \cap N(A) = \{0\} \). Then the unique solution of the equation (5) is given by

\[
x = A^{(2)}_{T,S} b. \tag{6}
\]

The norm on the data is the norm in \( B(X, Y) \times Y \) defined as

\[
(A, b) \mapsto \|\alpha A, \beta b\| = \sqrt{\alpha^2\|A\|^2_{QP} + \beta^2\|b\|^2_{Q}}.
\]
Now, we prove the estimation of the absolute condition number of a linear system related to the generalized inverse $A_{T,S}^{(2)}$. The following result is a generalization of results from [5] and [10].

**Theorem 2.1.** If the perturbation $E$ in $A$ fulfills the condition (4), then the absolute condition number $C(A, b)$ of the generalized inverse $A_{T,S}^{(2)}$ solution of the constrained linear system, with the norm

$$
\|\|\alpha A, \beta b\|\| = \sqrt{\alpha^2 \|A\|_Q^2 + \beta^2 \|b\|_Q^2}
$$
on the data $(A, b)$, and the norm $\|x\|_P$ on the solution, satisfies

$$
C(A, b) \leq \|A_{T,S}^{(2)}\|_P Q \sqrt{\frac{1}{\beta^2} + \frac{\|x\|_P^2}{\alpha^2}}.
$$

Let $(E_n)_n$ be a sequence of perturbations of $A$ fulfilling the condition (4), and let $(f_n)_n$ be a sequence of perturbations of $b$. If $C(E_n, f_n)$ is the corresponding absolute condition number and $\|A_{T,S}^{(2)}\|_P Q < \alpha$, then

$$
C(E_n, f_n) \to \|A_{T,S}^{(2)}\|_P Q \sqrt{\frac{1}{\beta^2} + \frac{\|x\|_P^2}{\alpha^2}}, \quad n \to \infty.
$$

Hence, $\|A_{T,S}^{(2)}\|_P Q \sqrt{\frac{1}{\beta^2} + \frac{\|x\|_P^2}{\alpha^2}}$ is a sharp bound.

**Proof.** We know that $F(A, b) = A_{T,S}^{(2)} b$. Under the condition (4), $F$ is a differentiable function and $F'$ is defined as follows

$$
F'(A, b)|_{(E, f)} = \lim_{\epsilon \to 0} \frac{(A + \epsilon E)_{T,S}^{(2)} (b + \epsilon f) - A_{T,S}^{(2)} b}{\epsilon},
$$
where $E$ is the perturbation of $A$ and $f$ is the perturbation of $b$. Since $E$ satisfies the condition (4), we have

$$
(A + \epsilon E)_{T,S}^{(2)} = A_{T,S}^{(2)} - \epsilon A_{T,S}^{(2)} E A_{T,S}^{(2)} + O(\epsilon^2),
$$
and then we can easily get that

$$
F'(A, b)|_{(E, f)} = -A_{T,S}^{(2)} E A_{T,S}^{(2)} b + A_{T,S}^{(2)} f = -A_{T,S}^{(2)} E x + A_{T,S}^{(2)} f.
$$

Then

$$
\|F'(A, b)|_{(E, f)}\|_P = \|A_{T,S}^{(2)}(E x - f)\|_P \leq \|A_{T,S}^{(2)}\|_P Q (\|E\|_Q P \|x\|_P + \|f\|_Q).
$$

The norm of a linear map $(E, f) \mapsto F'(A, b)|_{(E, f)}$ is the supermum of $\|F'(A, b)|_{(E, f)}\|_P$ on the unit ball of $B(X, Y) \times Y$. Since

$$
\|\|\alpha E, \beta f\|\|^2 = \alpha^2 \|E\|_Q^2 + \beta^2 \|f\|_Q^2
$$
we get

\[
\|F'(A, b)_{(E, f)}\| \\
\leq \sup_{\alpha^2\|E\|_P^2 + \beta^2\|f\|_Q^2 \leq 1} \|A_{T,S}^{(2)}\|_{PQ}(\|E\|_P\|x\|_P + \|f\|_Q) \\
= \sup_{\alpha^2\|E\|_P^2 + \beta^2\|f\|_Q^2 \leq 1} \|A_{T,S}^{(2)}\|_{PQ}\left(\alpha \|E\|_P\|x\|_P + \beta \|f\|_Q \frac{1}{\beta}\right) \\
= \|A_{T,S}^{(2)}\|_{PQ} \sup_{\alpha^2\|E\|_P^2 + \beta^2\|f\|_Q^2 \leq 1} (\alpha \|E\|_P\|x\|_P + \beta \|f\|_Q) \cdot \left(\frac{\|x\|_P}{\alpha}, \frac{1}{\beta}\right)
\]

where \((\alpha \|E\|_P, \beta \|f\|_Q)\) and \(\left(\frac{\|x\|_P}{\alpha}, \frac{1}{\beta}\right)\) can be consider as vectors in \(\mathbb{R}^2\), and the previous line contains the inner product in \(\mathbb{R}^2\).

Therefore, from the Cauchy–Schwarz inequality, we get:

\[
\|F'(A, b)_{(E, f)}\| \leq \|A_{T,S}^{(2)}\|_{PQ} \sqrt{\|x\|_P^2 + \frac{1}{\beta^2}}.
\]

Next, we show the other part of the theorem. Recall the matrix forms (1) and (2). There exists a sequence \((u_n)_n\) in \(S_1\) satisfying \(\|u_n\| = 1\) and \(\lim_{n \to \infty} \|A_1^{-1} u_n\| = \|A_1^{-1}\|\). So, there exists a sequence \((v_n)_n\) in \(T\), \(v_n = \frac{A_1^{-1}}{\|A_1^{-1}\|} u_n\), such that \(\|v_n\| \leq 1\), \(\lim_{n \to \infty} \|v_n\| = 1\) and, for all \(n \in \mathbb{N}\),

\[
A_1^{-1} u_n = \|A_1^{-1}\| v_n = \|A_{T,S}^{(2)}\|_{PQ} v_n.
\]

The last equality follows from

\[
\|A_{T,S}^{(2)}\|_{PQ} = \sup_{\|x\|_Q \leq 1} \|A_{T,S}^{(2)} x\|_P \\
= \sup_{\sqrt{\|x_1\|^2 + \|x_2\|^2} \leq 1} \left\| \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_P \\
= \sup_{\|x_1\| \leq 1} \|A_1^{-1} x_1\|_P \\
= \sup_{\|x_1\| \leq 1} \|A_1^{-1} x_1\| \\
= \|A_1^{-1}\|
\]

Taking, for all \(n \in \mathbb{N}\),

\[
\hat{u}_n = \begin{bmatrix} u_n \\ 0 \end{bmatrix} \in \begin{bmatrix} S_1 \\ S \end{bmatrix}, \quad \hat{v}_n = \begin{bmatrix} v_n \\ 0 \end{bmatrix} \in \begin{bmatrix} T \\ T_1 \end{bmatrix},
\]
Estimation of a condition number related to $A_{T,S}^{(2)}$

we obtain

$$\begin{align*}
A_{T,S}^{(2)} \hat{u}_n &= \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ 0 \end{bmatrix} = \begin{bmatrix} A_1^{-1}u_n \\ 0 \end{bmatrix} \\
\|A_1^{-1}v_n\| &= \|A_1^{-1}\| \begin{bmatrix} v_n \\ 0 \end{bmatrix} \\
\|A_{T,S}^{(2)}\|_{PQ} \hat{v}_n.
\end{align*}$$

It is easy to check that $\|\hat{u}_n\|_Q = 1$ and $\|\hat{v}_n\|_P \leq 1$, for all $n \in N$.

Let $u \in S_1$ and $v \in T$. Define $S_{u,v} \in \mathcal{B}(T, S_1)$ as follows: if $x \in T$, then

$$S_{u,v}(x) \overset{\text{def}}{=} \langle x, v \rangle u.$$ 

For all $T \in \mathcal{B}(S_1, T)$ we have

$$TS_{u,v}(x) = T(u)\langle x, v \rangle.$$ 

Now we choose, for $n = 1, 2, 3, \ldots$,

$$\eta = \sqrt{\frac{\|x\|_T^2}{\alpha^2} + \frac{1}{\beta^2}}, \quad f_n = \frac{1}{\beta^2\eta^2} \hat{u}_n,$$

$$E_n = -\frac{1}{\alpha^2\eta^2} \begin{bmatrix} S_{u_n, x} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$ 

Then, for a fixed $n$, we can verify that $E_n$ fulfills the first equation of the condition (4):

$$\begin{align*}
AA_{T,S}^{(2)}E_n &= -\frac{1}{\alpha^2\eta^2} \begin{bmatrix} A_1 \\ 0 \\ A_2 \end{bmatrix} \begin{bmatrix} A_1^{-1} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} S_{u_n, x} \\ 0 \\ 0 \end{bmatrix} \\
&= -\frac{1}{\alpha^2\eta^2} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} S_{u_n, x} \\ 0 \\ 0 \end{bmatrix} \\
&= -\frac{1}{\alpha^2\eta^2} \begin{bmatrix} S_{u_n, x} \\ 0 \\ 0 \end{bmatrix} \\
&= E_n.
\end{align*}$$

In the same way, we have

$$\begin{align*}
E_nA_{T,S}^{(2)}A &= -\frac{1}{\alpha^2\eta^2} \begin{bmatrix} S_{u_n, x} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} A_1^{-1} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} A_1 \\ 0 \\ A_2 \end{bmatrix} \\
&= -\frac{1}{\alpha^2\eta^2} \begin{bmatrix} S_{u_n, x} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \\
&= -\frac{1}{\alpha^2\eta^2} \begin{bmatrix} S_{u_n, x} \\ 0 \\ 0 \end{bmatrix} \\
&= E_n.
\end{align*}$$
\[ \|A^{(2)}_{T,S}\|_{PQ}\|E_n\|_{QP} = \frac{1}{\alpha^2\eta^2} \left\| \left[ \begin{array}{ccc} A_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ S_{u_n,x} & 0 & 0 \end{array} \right] \right\|_{PQ} + \frac{1}{\beta^2\eta^2} \|\hat{u}_n\|_Q^2 \]

\[ = \frac{1}{\alpha^2\eta^2} \|A_1^{-1}\| \left\| S_{u_n,x} \right\|_{QP} + \frac{1}{\beta^2\eta^2} \|\hat{u}_n\|_Q^2 \]

\[ \leq \frac{1}{\alpha^2\eta^2} \|A_1^{-1}\| \|u_n\|_Q \]

\[ = \left\| A_1^{-1} \right\|_{PQ} \frac{1}{\alpha} \]

\[ = \|A^{(2)}_{T,S}\|_{PQ} \frac{1}{\alpha} \]

\[ < 1. \]

Thus \( E_n \) fulfills the condition (4), for all \( n \in \mathbb{N} \). Now we want to verify that the perturbation \( (E_n, f_n) \) satisfies \( \alpha^2\|E_n\|_{QP}^2 + \beta^2\|f_n\|_Q^2 \leq 1 \).

\[ \alpha^2\|E_n\|_{QP}^2 + \beta^2\|f_n\|_Q^2 = \frac{1}{\alpha^2\eta^2} \left\| \left[ \begin{array}{ccc} S_{u_n,x} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \right\|_{QP}^2 + \frac{1}{\beta^2\eta^2} \|\hat{u}_n\|_Q^2 \]

\[ = \frac{1}{\alpha^2\eta^2} \left\| S_{u_n,x} \right\|_{QP}^2 + \frac{1}{\beta^2\eta^2} \|\hat{u}_n\|_Q^2 \]

\[ \leq \frac{1}{\alpha^2\eta^2} \|u_n\|_P \|x\|_P \frac{1}{\beta^2\eta^2} \]

\[ = \frac{1}{\eta^2} \left( \frac{\|x\|_P^2}{\alpha^2} + \frac{1}{\beta^2} \right) \]

\[ = 1. \]

The inner product \( \langle \cdot, \cdot \rangle_P \) in \( T \) is the same as the inner product \( \langle \cdot, \cdot \rangle \). Thus, we have,
for \( x = A_{T,S}^{(2)} b, \)

\[
F'(A, b)|_{(E_n, f_n)} = -A_{T,S}^{(2)} E_n x + A_{T,S}^{(2)} f_n
\]

\[
= \frac{1}{\alpha^2} \left[ A_{T,S}^{(2)} S_{u_n,x} 0 \right] \left[ x \right] + \frac{1}{\beta^2 \eta} A_{T,S}^{(2)} u_n
\]

\[
= \frac{1}{\alpha^2} \left[ A_{T,S}^{(2)} A_{u_n,x} 0 \right] \left[ x \right] + \frac{1}{\beta^2 \eta} A_{T,S}^{(2)} u_n
\]

\[
= \frac{1}{\alpha^2} \left[ A_{T,S}^{(2)} A_{u_n,x} 0 \right] \left[ x \right] + \frac{1}{\beta^2 \eta} A_{T,S}^{(2)} u_n
\]

\[
= \frac{1}{\alpha^2} \left[ \|x\|^2 p A_{T,S}^{(2)} \right] \left[ x \right] + \frac{1}{\beta^2 \eta} A_{T,S}^{(2)} u_n
\]

\[
= \frac{1}{\alpha^2} \left[ \|x\|^2 p A_{T,S}^{(2)} \right] \left[ x \right] + \frac{1}{\beta^2 \eta} A_{T,S}^{(2)} u_n
\]

\[
= \left( \|x\|^2 p \frac{\|A_{T,S}^{(2)} \|^2}{\eta} + \frac{1}{\beta^2} \right) \dot{v}_n
\]

So

\[
\|F'(A, b)|_{(E_n, f_n)}\|_P \rightarrow \|A_{T,S}^{(2)}\|_P \|x\|^2 p \left( \frac{\|x\|^2 p}{\alpha^2} + \frac{1}{\beta^2} \right) (n \rightarrow \infty).
\]

Knowing \( \alpha^2 \|E_n\|_P^2 + \beta^2 \|f_n\|_P^2 \leq 1, \) we get

\[
\|F'(A, b)|_{(E_n, f_n)}\| \rightarrow \|A_{T,S}^{(2)}\|_P \|x\|^2 p \left( \frac{\|x\|^2 p}{\alpha^2} + \frac{1}{\beta^2} \right), (n \rightarrow \infty)
\]

and we complete the proof. \( \square \)

### 3 Concluding remarks

In this paper, we consider the absolute condition number of a operator between Hilbert spaces, which is related with the outer generalized inverse of a given operator. In [5, 12] our Theorem 2.1 is proved for complex matrices. In [10] the author proved Theorem 2.1 considering the weighted Drazin inverse in Hilbert spaces. It is of interest to extend our results to the outer inverse of a operator between Banach spaces.

**Acknowledgement.** I am grateful to Professor Dragan Djordjević for helpful comments and suggestions concerning the paper.
References


Estimation of a condition number related to $A_{T,S}^{(2)}$


Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, Višegradska 33,18000 Niš, Serbia

E-mail: sknme@ptt.rs