SEVERAL NEW HARDY-HILBERT’S INEQUALITIES

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Abstract
In this paper, we obtain several extended analogues of Hardy-Hilbert’s inequalities.

1 Introduction
If \( f, g \) are real measurable functions such that
\[
0 < \int_0^\infty f^2(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^2(x)dx < \infty,
\]
then we have the following well known Hilbert’s integral inequality [3],
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}
\]
where the constant factor \( \pi \) is the best possible. Furthermore, we have also the following Hardy-Hilbert’s type inequality [3, Th 341, Th342],
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x, y\}}dxdy < 4 \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2},
\]
\[
\int_0^\infty \int_0^\infty \log \frac{x}{x-y} f(x)g(y)dxdy < \pi^2 \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2},
\]
where the constant factors 4 and \( \pi^2 \) are both the best possible.

There are numerous papers which study the Hilbert’s and Hardy-Hilbert’s type inequalities from different directions [1, 6, 7, 8, 9, 11]. Recently, Li-Wu-He [5] obtained the following inequality: if (1) is satisfied, then we have
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x, y\}}dxdy < c \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2},
\]
where the constant factor \( c \) is the best possible.
where the constant factor $c = 1.7408 \cdots$ is the best possible.

He-Qian-Li [4] have proved the following inequality: if (1) is satisfied, then we have
\[
\int_0^\infty \int_0^\infty \frac{\log x - \log y}{x + y + \min\{x, y\}} f(x)g(y)dx\,dy < c \left( \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{1/2},
\]
(3)
where the constant factor $c = 6.88947 \cdots$ is the best possible.

In this short paper, we will give several extended analogues of Hardy-Hilbert’s inequalities.

2 Main results

Before giving our main results, we need to establish the following

Lemma 1. Let $\gamma, \alpha, \beta$ be three non-negative real numbers. Then we have the following equations
\[
\int_0^\infty \frac{|\log x - \log y|^{\gamma}}{\alpha x + \beta y + \max\{x, y\}} \left( \frac{x}{y} \right)^{1/2} dy = \int_0^\infty \frac{|\log x - \log y|^{\gamma}}{\alpha x + \beta y + \max\{x, y\}} \left( \frac{y}{x} \right)^{1/2} dx
\]
\[= \int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^{2\alpha + (1 + \beta)}} dt + \int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^{2\beta + (1 + \alpha)}} dt =: A,
\]
where $A := A(\gamma, \alpha, \beta) \in [0, \infty]$.

Proof. For any given $y$, let $y = tx$, then it follows that
\[
\int_0^\infty \frac{|\log x - \log y|^{\gamma}}{\alpha x + \beta y + \max\{x, y\}} \left( \frac{x}{y} \right)^{1/2} dy
\]
\[= \int_0^\infty \frac{|\log t|^\gamma}{\alpha + t\beta + \max\{1, t\}} \left( \frac{1}{t} \right)^{1/2} dt
\]
\[= \int_1^1 \frac{|\log t|^\gamma}{t(\beta + 1) + \alpha} \left( \frac{1}{t} \right)^{1/2} dt + \int_1^1 \frac{|\log t|^\gamma}{t\beta + (1 + \alpha)} \left( \frac{1}{t} \right)^{1/2} dt
\]
\[= \int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^{2\alpha + (1 + \beta)}} dt + \int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^{2\beta + (1 + \alpha)}} dt
\]
which implies the desired result. \qed
Theorem 1. If \( f, g \) are real functions such that \( 0 < \int_0^\infty f^2(x)dx < \infty \) and \( 0 < \int_0^\infty g^2(x)dx < \infty \). Then we have

\[
\int_0^\infty \int_0^\infty \frac{\log x - \log y}{ax + \beta y + \max\{x, y\}} f(x)g(y) dxdy < A \left( \int_0^\infty f^2(x)dx \right)^{1/2} \left( \int_0^\infty g^2(y)dy \right)^{1/2},
\]

where \( A \) is defined in Lemma 1 and is the best possible.

Proof. By Cauchy-Schwarz inequality and Lemma 1, we get

\[
\int_0^\infty \int_0^\infty \frac{\log x - \log y}{ax + \beta y + \max\{x, y\}} f(x)g(y) dxdy \\
\leq A \left( \int_0^\infty f^2(x)dx \right)^{1/2} \left( \int_0^\infty g^2(y)dy \right)^{1/2},
\]

(5)

If the equality in (5) holds, then there exist two constant \( c \) and \( d \), not both zero (without loss of generality, suppose that \( c \neq 0 \)) and

\[
c \frac{\log x - \log y}{ax + \beta y + \max\{x, y\}} \left( \frac{x}{y} \right)^{1/2} f^2(x) = d \frac{\log x - \log y}{ax + \beta y + \max\{x, y\}} \left( \frac{y}{x} \right)^{1/2} g^2(y), \quad \text{a.e.}
\]

in \((0, \infty) \times (0, \infty)\). That is to say, we have

\[
 ex f^2(x) = dy g^2(y) = \text{constant}, \quad \text{a.e.}
\]

in \((0, \infty) \times (0, \infty)\). Thus

\[
\int_0^\infty f^2(x)dx = \infty,
\]

which contradicts the assumption \( 0 < \int_0^\infty f^2(x)dx < \infty \). Hence, the inequality (5) takes the form of strict inequality.

Assume that the constant \( A \) in the inequality (4) is not the best possible, then there exists a positive number \( K \) with \( K < A \) and \( a > 0 \), such that

\[
\int_a^\infty \int_0^\infty \frac{\log x - \log y}{ax + \beta y + \max\{x, y\}} f(x)g(y) dxdy < K \left( \int_a^\infty f^2(x)dx \right)^{1/2} \left( \int_a^\infty g^2(y)dy \right)^{1/2}.
\]

(6)
Suppose then by (5), we have

\[ \text{The contradiction implies that the constant } \]
\[ \text{which yields } \]
\[ \text{Letting } y = tx, \text{ we get } \]
\[ \int_a^\infty \int_0^\infty \frac{\log x - \log y}{\alpha x + \beta y + \max\{x, y\}} f(x)g_t(y) dy dx \]
\[ = \int_b^\infty \int_0^\infty \frac{\log x - \log y}{\alpha x + \beta y + \max\{x, y\}} x^{-\frac{\epsilon+1}{t}} y^{-\frac{\gamma+1}{t}} dy dx \]
\[ = \int_b^\infty \int_{b/x}^\infty \frac{\log t}{\alpha + t/\beta + \max\{1, t\}} t^{-\frac{\epsilon+1}{t}} dt dx \]

(7)

Letting \( b \to 0^+ \), by (6) and Fatou’s lemma, we have

\[ \int_a^\infty \int_0^\infty \frac{\log t}{\alpha + t/\beta + \max\{1, t\}} t^{-\frac{\epsilon+1}{t}} dt dx \]
\[ = \frac{1}{\varepsilon \alpha^2} \int_0^\infty \frac{\log t}{\alpha + t/\beta + \max\{1, t\}} t^{-\frac{\epsilon+1}{t}} dt \leq K \frac{1}{\varepsilon \alpha^2}, \]

which yields

\[ \lim_{\varepsilon \to 0^+} \int_0^\infty \frac{\log t}{\alpha + t/\beta + \max\{1, t\}} t^{-\frac{\epsilon+1}{t}} dt = A \leq K. \]

The contradiction implies that the constant \( A \) is the best possible.

Theorem 2. Suppose \( f \geq 0 \) and \( 0 < \int_0^\infty f^2(x) dx < \infty \). Then

\[ \int_0^\infty \left( \int_0^\infty \frac{\log x - \log y}{\alpha x + \beta y + \max\{x, y\}} f(x) dx \right)^2 dy < A^2 \int_0^\infty f^2(x) dx \]  

(8)

Proof. Let

\[ g(y) = \int_0^\infty \frac{\log x - \log y}{\alpha x + \beta y + \max\{x, y\}} f(x) dx, \]

then by (5), we have

\[ 0 < \int_0^\infty g^2(y) dy \]
\[ = \int_0^\infty \left( \int_0^\infty \frac{\log x - \log y}{\alpha x + \beta y + \max\{x, y\}} f(x) dx \right)^2 dy \]
\[ = \int_0^\infty \int_0^\infty \frac{\log x - \log y}{\alpha x + \beta y + \max\{x, y\}} f(x)g(y) dx dy \]
\[ \leq A \left( \int_0^\infty f^2(x) dx \right)^{1/2} \left( \int_0^\infty g^2(y) dy \right)^{1/2}, \]  

(9)
which yields
\[ 0 < \int_0^\infty g^2(y)dy \leq A^2 \int_0^\infty f^2(x)dx < \infty. \] (10)

By (4), both (9) and (10) take the form of strict inequality, so we have the inequality (8). On the other hand, suppose that (8) is valid. Again, you use the Cauchy-Schwarz inequality, we get
\[
\int_0^\infty \int_0^\infty \frac{|\log x - \log y|\gamma}{\alpha x + \beta y + \max\{x, y\}} f(x)g(y)dxdy
\]
\[
= \int_0^\infty \left( \int_0^\infty \frac{|\log x - \log y|\gamma}{\alpha x + \beta y + \max\{x, y\}} f(x)dx \right) g(y)dy
\]
\[
< A \left( \int_0^\infty f^2(x)dx \right)^{1/2} \left( \int_0^\infty g^2(y)dy \right)^{1/2},
\]
which is the inequality (4).

\[ \square \]

Remark 1. If we take \( \gamma = \alpha = \beta = 1 \), then the inequality (3) can be induced by the inequality (4).

3 Several special inequalities

In this section, by choosing different \( \gamma, \alpha, \beta \), we establish several special inequalities. In what follows, assume that (1) is satisfied.

(1) If \( \gamma = 0, \alpha = \beta = 1 \), then
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x, y\}}dxdy
\]
\[
< A \left( \int_0^\infty f^2(x)dx \right)^{1/2} \left( \int_0^\infty g^2(y)dy \right)^{1/2},
\] (11)

where
\[ A = 4 \int_0^1 \frac{1}{t^2 + 2}dt = 2\sqrt{2} \arctan(\sqrt{2}/2). \]

(2) If \( \alpha = \beta = 0, \gamma = 1 \), then
\[
\int_0^\infty \int_0^\infty \frac{|\log x - \log y|}{\max\{x, y\}} f(x)g(y)dxdy
\]
\[
< A \left( \int_0^\infty f^2(x)dx \right)^{1/2} \left( \int_0^\infty g^2(y)dy \right)^{1/2},
\] (12)

where
\[ A = -8 \int_0^1 \log tdt = 7.99988 \cdots . \]
If $\alpha = 0$, $\beta = \gamma = 1$, then
\[
\int_0^\infty \int_0^\infty |\log x - \log y| f(x)g(y) dx dy < A \left( \int_0^\infty f^2(x) dx \right)^{1/2} \left( \int_0^\infty g^2(y) dy \right)^{1/2},
\]
where
\[
A = -2 \int_1^1 \log t dt - 4 \int_1^1 \frac{\log t}{1 + t^2} dt = 5.66377 \cdots.
\]

If $\gamma = 2$, $\alpha = \beta = 1$, then
\[
\int_0^\infty \int_0^\infty \frac{|\log x - \log y|^2}{x + y + \max\{x, y\}} f(x)g(y) dx dy < A \left( \int_0^\infty f^2(x) dx \right)^{1/2} \left( \int_0^\infty g^2(y) dy \right)^{1/2},
\]
where
\[
A = 16 \int_0^1 \frac{\log t^2}{t^2 + 2} dt = 15.72916 \cdots.
\]

If $\gamma = \alpha = 0$, $\beta = 1$, then
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{y + \max\{x, y\}} dx dy < A \left( \int_0^\infty f^2(x) dx \right)^{1/2} \left( \int_0^\infty g^2(y) dy \right)^{1/2},
\]
where
\[
A = 1 + 2 \int_0^1 \frac{1}{t^2 + 1} dt = 1 + \pi/2.
\]

If $\gamma = \beta = \alpha = 0$, then
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x, y\}} dx dy < A \left( \int_0^\infty f^2(x) dx \right)^{1/2} \left( \int_0^\infty g^2(y) dy \right)^{1/2},
\]
where
\[
A = 4.
\]

4 Further discussions

In this section, we give two new Hardy-Hilbert’s inequalities. Before our works, the following result need be mentioned.
Lemma 2. [2] Let \( f \) be a nonnegative integrable function. Define
\[
F(x) = \int_0^x f(t) dt.
\]
Then
\[
\int_0^\infty \left( \frac{F(x)}{x} \right)^p dx < \left( \frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx, \quad p > 1.
\]

Theorem 3. Let \( f, g \geq 0 \),
\[
F(x) = \int_0^x f(t) dt, \quad G(x) = \int_0^x g(t) dt.
\]
Furthermore assume that \( 0 < \int_0^\infty f^2(x) dx < \infty \) and \( 0 < \int_0^\infty g^2(x) dx < \infty \) and let \( A \in (0, \infty) \), then we have
\[
\int_0^\infty \int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x,y\}} \frac{F(x) G(y)}{x} \frac{dy}{y} dx dy < 4A \left( \int_0^\infty f^2(x) dx \right)^{1/2} \left( \int_0^\infty g^2(y) dy \right)^{1/2}.
\]

Proof. By Hölder’s inequality, Lemma 1 and Lemma 2, we have
\[
\int_0^\infty \int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x,y\}} \frac{F(x) G(y)}{x} \frac{dy}{y} dx dy \leq \left\{ \int_0^\infty \left( \int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x,y\}} \left( \frac{x}{y} \right)^{1/2} dy \right) \left( \frac{F(x)}{x} \right)^2 dx \right\}^{1/2} \times \left\{ \int_0^\infty \left( \int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \max\{x,y\}} \left( \frac{y}{x} \right)^{1/2} dx \right) \left( \frac{G(y)}{y} \right)^2 dy \right\}^{1/2} < 4A \left( \int_0^\infty f^2(x) dx \right)^{1/2} \left( \int_0^\infty g^2(y) dy \right)^{1/2}.
\]
The proof of the theorem can be completed. \( \square \)

Theorem 4. Let \( f, g \geq 0 \),
\[
F(x) = \int_0^x f(t) dt, \quad G(x) = \int_0^x g(t) dt.
\]
Furthermore assume that \( p, q > 1, \alpha, \beta, s, t, \mu, \nu > 0 \), such that
\[
\frac{1}{p} + \frac{1}{q} = 1, \quad sp > \beta q + 1, \quad tq > \alpha p + 1
\]
and
\[
(\beta + \mu - s)p + 1 = 0, \quad (\alpha + \nu - t)q + 1 = 0.
\]
Then we have
\[
\int_0^\infty \int_0^\infty \frac{x^\alpha y^\beta F^\mu(x)G^\nu(y)}{(x+y)^{s+t}} \, dx \, dy < \kappa \left( \left( \frac{pq}{pq-1} \right)^{\mu} \left( \int_0^\infty f^\mu(x) \, dx \right)^{1/p} \left( \int_0^\infty g^\nu(y) \, dy \right)^{1/q} \right),
\]
where
\[
\kappa = B^{1/p}(\beta p + 1, sp - (\beta p + 1))B^{1/q}(\alpha p + 1, tq - (\alpha p + 1))
\]
and \( B(\cdot, \cdot) \) denotes the Beta function.

**Proof.** By Hölder’s inequality, it is easy to see
\[
\int_0^\infty \int_0^\infty \frac{x^\alpha y^\beta F^\mu(x)G^\nu(y)}{(x+y)^{s+t}} \, dx \, dy
\]
\[
= \int_0^\infty \int_0^\infty \frac{x^\alpha F^\mu(x) y^\beta G^\nu(y)}{(x+y)^s (x+y)^t} \, dx \, dy
\]
\[
\leq \left( \int_0^\infty \int_0^\infty \frac{y^{\beta p} F^\mu(x) \, dx \, dy}{(x+y)^{sp}} \right)^{1/p} \left( \int_0^\infty \int_0^\infty \frac{x^\alpha G^\nu(y) \, dx \, dy}{(x+y)^{tq}} \right)^{1/q}
\]
\[
= P^{1/p}Q^{1/q}.
\]

Next by Lemma 2, we obtain
\[
P = \int_0^\infty \left( \frac{F(x)}{x} \right)^{\mu} \, dx \int_0^\infty \frac{y^{\beta p} x^{\mu}}{(x+y)^{sp}} \, dy
\]
\[
= \int_0^\infty \frac{F(x)}{x} \, dx \int_0^\infty \frac{\frac{y}{x}^{\beta p} x^{-1}}{(1 + \frac{y}{x})^{sp}} \, dy
\]
\[
= \int_0^\infty \frac{F(x)}{x} \, dx \int_0^\infty \frac{u^{\beta p}}{(1 + u)^{sp}} \, du
\]
\[
< B(\beta p + 1, sp - (\beta p + 1)) \left( \frac{pq}{pq-1} \right)^{\mu} \int_0^\infty f^\mu(x) \, dx.
\]

Similarly, it can be shown that
\[
Q = \int_0^\infty \left( \frac{G(y)}{y} \right)^{\nu} \, dy \int_0^\infty \frac{x^{\alpha q} y^{\nu}}{(x+y)^{tq}} \, dx
\]
\[
< B(\alpha q + 1, tq - (\alpha q + 1)) \left( \frac{q^v}{q^v-1} \right)^{\nu} \int_0^\infty g^\nu(y) \, dy
\]
which implies the desired result. \( \square \)
Several new Hardy-Hilbert’s inequalities

Remark 2. Let $\mu = \nu = 1$, $\alpha = 1/p$, $\beta = 1/q$, $s = t = 2$, then we have the following special inequality (see [10]),

$$\int_0^\infty \int_0^\infty \frac{x^{1/p}y^{1/q}F(x)G(y)}{(x+y)^4} \, dx \, dy \leq B^{1/p}(p,p)B^{1/q}(q,q) \frac{p}{p-1} \frac{q}{q-1} \left( \int_0^\infty f^p(x) \, dx \right)^{1/p} \left( \int_0^\infty g^q(y) \, dy \right)^{1/q}.$$ 

References


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