ON AN INTEGRAL-TYPE OPERATOR FROM $Q_K(p, q)$ SPACES TO $\alpha$-BLOCH SPACES

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Abstract

Let $g \in H(D)$, $n$ be a nonnegative integer and $\varphi$ be an analytic self-map of $D$. We study the boundedness and compactness of the integral operator $C_n^{\varphi, g}$, which is defined by

$$(C_n^{\varphi, g}f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi, \quad z \in D, \quad f \in H(D),$$

from $Q_K(p, q)$ and $Q_K, 0(p, q)$ spaces to $\alpha$-Bloch spaces and little $\alpha$-Bloch spaces.

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let $\alpha > 0$. An $f \in H(\mathbb{D})$ is said to belong to the $\alpha$-Bloch space, denoted by $B^\alpha$, if

$$\|f\|_{B^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$  (1)

Under the above norm, $B^\alpha$ is a Banach space. When $\alpha = 1$, $B^1 = B$ is the classical Bloch space. Let $B^\alpha_0$ denote the subspace of $B^\alpha$ consisting of those $f \in B^\alpha$ for which $(1 - |z|^2)^\alpha |f'(z)| \to 0$ as $|z| \to 1$. This space is called the little $\alpha$-Bloch space.

Let $g(z, a)$ be the Green function with logarithmic singularity at $a$, i.e., $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$ ($\varphi_a$ is a conformal automorphism defined by $\varphi_a(z) = \frac{a - z}{1 - az}$ for $a \in \mathbb{D}$).

Let $p > 0$, $q > -2$, $K : [0, \infty) \to [0, \infty)$ be a nondecreasing continuous function. An $f \in H(\mathbb{D})$ is said to belong to $Q_K(p, q)$ space if (see [9, 29])

$$\|f\| = \left(\sup_{a \in \mathbb{D}} \int_{D} |f'(z)|^p (1 - |z|^2)^q K(g(z, a))dA(z)\right)^{1/p} < \infty.$$  (2)

2010 Mathematics Subject Classifications. Primary 47B35, Secondary 30H05.

Key words and Phrases. $Q_K(p, q)$ space, $\alpha$-Bloch space, integral-type operator.

Received: March 07, 2011
Communicated by Dragana Cvetković Ilić
The author would like to thank the referee for his/her helpful comments.
where \(dA\) is the normalized Lebesgue area measure in \(\mathbb{D}\). For \(p \geq 1\), under the norm \(\|f\|_{\mathcal{Q}_K(p,q)} = |f(0)| + \|f\|\), \(\mathcal{Q}_K(p,q)\) is a Banach space. An \(f \in H(\mathbb{D})\) is said to belong to \(\mathcal{Q}_K,0(p,q)\) space if
\[
\lim_{|a| \to 1} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(g(z,a))dA(z) = 0.
\]
(3)

Throughout the paper we assume that (see [29])
\[
\int_0^1 (1 - r^2)^q K(- \log r) r dr < \infty,
\]
(4)
since otherwise \(\mathcal{Q}_K(p,q)\) consists only of constant functions.

Let \(g \in H(\mathbb{D})\) and \(\varphi\) be an analytic self-map of \(\mathbb{D}\). The composition operator \(C_{\varphi}\) is defined by \(C_{\varphi}(f)(z) = f(\varphi(z)), f \in H(\mathbb{D})\). In [4], Li and Stević defined the generalized composition operator as follows
\[
(C_{\varphi}^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.
\]

The generalized composition operator and its generalizations on various spaces were investigated in [4–7, 13, 14, 19, 21, 22, 24, 28, 30–33, 35, 36]. See, e.g., [1, 11] and the references therein for the study of the composition operator.

Let \(g \in H(\mathbb{D})\), \(n\) be a nonnegative integer and \(\varphi\) be an analytic self-map of \(\mathbb{D}\). In [38], the author defined a new integral-type operator as follows:
\[
(C_{\varphi}^ng f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).
\]

\(C_{\varphi,g}^1\) is the generalized composition operator \(C_{\varphi}^g\). When \(n = 0\), then \(C_{\varphi,g}^0\) is the Volterra composition operator defined by Li in [3], extended by Stević in the \(n\)-dimensional case in [16] and subsequently studied in [15, 17, 18, 20, 23, 25–27].

Here we characterized the boundedness and compactness of the operator \(C_{\varphi,g}^n\) from \(\mathcal{Q}_K(p,q)\) and \(\mathcal{Q}_{K,0}(p,q)\) to \(\alpha\)-Bloch and little \(\alpha\)-Bloch spaces.

Throughout this paper, constants are denoted by \(C\), they are positive and may differ from one occurrence to the other. The notation \(A \asymp B\) means that there is a positive constant \(C\) such that \(B/C \leq A \leq CB\).

## 2 Main results and proofs

In this section we give our main results and proofs. For this purpose, we need some auxiliary results. The following lemma can be proved in a standard way (see, e.g., [10]).

**Lemma 1.** Let \(\alpha, p > 0\), \(q > -2\) and \(K\) be a nonnegative nondecreasing function on \([0,\infty)\). Assume that \(\varphi\) is an analytic self-map of \(\mathbb{D}\) and \(n\) is a nonnegative integer. Then \(C_{\varphi,g}^n : \mathcal{Q}_K(p,q)(\text{or } \mathcal{Q}_{K,0}(p,q)) \to B^\alpha\) is compact if and only if \(C_{\varphi,g}^n : \mathcal{Q}_K(p,q)(\text{or } \mathcal{Q}_{K,0}(p,q)) \to B^\alpha\) is compact.
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\(Q_K(p, q)\) or \(Q_K,0(p, q)\) is bounded and for any bounded sequence \((f_k)_{k \in \mathbb{N}}\) in \(Q_K(p, q)\) or \(Q_K,0(p, q)\) which converges to zero uniformly on compact subsets of \(\mathbb{D}\), we have \(\|C^n_{\varphi,g} f_k\|_{B^{\infty}} \to 0\) as \(k \to \infty\).

The following lemma is essentially proved in [8], hence we omit its proof.

**Lemma 2.** A closed set \(K\) in \(B^{\infty}_0\) is compact if and only if it is bounded and satisfies

\[ \lim_{|z| \to 1^{-}} \sup_{f \in K} (1 - |z|^2)^\alpha |f'(z)| = 0. \]

**Lemma 3.** [29] Let \(p > 0, q > -2\) and \(K\) is a nonnegative nondecreasing function on \([0, \infty)\). For \(f \in Q_K(p, q)\), we have \(f \in B^{\frac{q+2}{p}}\) and

\[ \|f\|_{B^{\frac{q+2}{p}}} \leq \|f\|_{Q_K(p,q)}. \quad (5) \]

**Lemma 4.** [12] Let \(f \in B^{\infty}, 0 < \alpha < \infty\). Then

\[ |f(z)| \leq \begin{cases} C\|f\|_{B^{\infty}}, & 0 < \alpha < 1; \\ C\|f\|_{B^{\infty}} \ln \frac{1}{1-|z|}, & \alpha = 1; \\ C\frac{\|f\|_{B^{\infty}}}{(1-|z|)^{\alpha}}, & \alpha > 1. \end{cases} \]

Now we are in a position to state and prove the main results of this paper.

**Theorem 1.** Let \(\alpha, p > 0, q > -2\) and \(K\) be a nonnegative nondecreasing function on \([0, \infty)\) such that

\[ \int_0^1 K(-\log r)(1 - r)^\min\{-1, q\} \left( \log \frac{1}{1 - r} \right)^{\chi - 1(q)} r dr < \infty, \quad (6) \]

where \(\chi_O(x)\) denote the characteristic function of the set \(O\). Assume that \(\varphi\) is an analytic self-map of \(\mathbb{D}\) and \(n \in \mathbb{N}\). Then the following statements are equivalent.

(i) \(C^n_{\varphi,g} : Q_K(p, q) \to B^{\infty}\) is bounded;
(ii) \(C^n_{\varphi,g} : Q_K,0(p, q) \to B^{\infty}\) is bounded;
(iii)

\[ M_1 := \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\alpha |g(z)|}{(1 - |\varphi(z)|^2)^{\frac{q+2}{p}+n}} < \infty. \quad (7) \]

**Proof.** (iii) \(\Rightarrow\) (i). Suppose that (7) holds. First it is easy to see that \((C^n_{\varphi,g} f)(0) = 0\) and \((C^n_{\varphi,g} f)'(z) = f^{(n)}(\varphi(z))g(z)\) for every \(f \in H(\mathbb{D})\). For any
z ∈ D and \( f ∈ Q_K(p, q) \), by Lemma 3 we have
\[
(1 - |z|^2)^α |(C^n_{φ, g}f)'(z)| = (1 - |z|^2)^α |f^{(n)}(φ(z))g(z)|
\leq \frac{(1 - |z|^2)^α |g(z)|}{(1 - |φ(z)|^2)^{\frac{2p}{p} + n}} \|f\|_{B^α}^{\frac{1}{2}}
\leq \frac{(1 - |z|^2)^α |g(z)|}{(1 - |φ(z)|^2)^{\frac{2p}{p} + n}} \|f\|_{Q_K(p, q)},
\]
where we have used the following well-known characterization for \( α \)-Bloch functions (see, e.g., [34])
\[
\sup_{z ∈ D}(1 - |z|^2)^α |f'(z)| = |f'(0)| + \cdots + |f^{(n-1)}(0)| + \sup_{z ∈ D}(1 - |z|^2)^α + 1 |f^{(n)}(z)|.
\]
Taking the supremum in (8) for \( z ∈ D \), then employing (7) we obtain that \( C^n_{φ, g} : Q_K(p, q) → B^α \) is bounded.

(i) ⇒ (ii). It is clear.

(ii) ⇒ (iii). Suppose that \( C^n_{φ, g} : Q_{K, 0}(p, q) → B^α \) is bounded, i.e. there exists a constant \( C \) such that \( \|C^n_{φ, g}f\|_{B^α} ≤ C\|f\|_{Q_K(p, q)} \) for all \( f ∈ Q_{K, 0}(p, q) \). Taking the function \( f(z) ≡ z^n \), which belongs to \( Q_{K, 0}(p, q) \), we get
\[
\sup_{z ∈ D}(1 - |z|^2)^α |g(z)| < \infty.
\]
For \( w ∈ D \), let \( f_w(z) = \frac{1 - |w|^2}{(1 - zw)^{\frac{2p}{p} + n}} \). Using the condition (6), we see that \( f_w ∈ Q_{K, 0}(p, q) \), for each \( w ∈ D \) (see [2]), moreover there is a positive constant \( C \) such that \( \sup_{w ∈ D} \|f_w\|_{Q_K(p, q)} ≤ C \) and
\[
|f_w^{(n)}(w)| = \prod_{j=0}^{n-1} \left( \frac{q + 2}{p} + j \right) \frac{|w|^n}{(1 - |w|^2)^{\frac{2p}{p} + n}}.
\]
Hence,
\[
∞ > C\|C^n_{φ, g}\|_{Q_{K, 0}(p, q) → B^α} ≥ \|C^n_{φ, g}f_φ(λ)\|_{B^α}
\geq \prod_{j=0}^{n-1} \left( \frac{q + 2}{p} + j \right) (1 - |λ|^2)^α |g(λ)| |φ(λ)|^n
\leq (1 - |φ(λ)|^2)^{\frac{2p}{p} + n} \|f_φ(λ)\|^α
\]
for each \( λ ∈ D \).

From (10), we have
\[
\sup_{|φ(λ)| > \frac{1}{2}} \frac{(1 - |λ|^2)^α |g(λ)|}{(1 - |φ(λ)|^2)^{\frac{2p}{p} + n}} \leq 2^n \sup_{|φ(λ)| > \frac{1}{2}} \frac{(1 - |λ|^2)^α |g(λ)| |φ(λ)|^n}{(1 - |φ(λ)|^2)^{\frac{2p}{p} + n}} \leq C\|C^n_{φ, g}\|_{Q_{K, 0}(p, q) → B^α} < ∞.
\]
Inequality (9) gives
\[
\sup_{|\varphi(\lambda)| \leq \frac{1}{2}} \frac{(1 - |\lambda|^2)^\alpha |g(\lambda)|}{(1 - |\varphi(\lambda)|^2)^\frac{2 + \frac{2}{p} - n}{2} + n} \leq \frac{4^{\frac{2 + \frac{2}{p} - n}{2} + n}}{3} \sup_{|\varphi(\lambda)| \leq \frac{1}{2}} (1 - |\lambda|^2)^\alpha |g(\lambda)| < \infty, \quad (12)
\]
where we used the assumption \((g + 2 - p)/p + n > 0\). Therefore, (7) follows from (11) and (12). This completes the proof of Theorem 1. \(\square\)

**Theorem 2.** Let \(\alpha, p > 0, q > -2\) and \(K\) be a nonnegative nondecreasing function on \([0, \infty)\) such that (6) holds. Assume that \(\varphi\) is an analytic self-map of \(\mathbb{D}\) and \(n \in \mathbb{N}\). Then the following statements are equivalent.

(i) \(C_{\varphi, g}^n : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha\) is compact;

(ii) \(C_{\varphi, g}^n : \mathcal{Q}_{K, 0}(p, q) \rightarrow \mathcal{B}^\alpha\) is compact;

(iii) \(C_{\varphi, g}^n : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha\) is bounded and

\[
\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\alpha |g(z)|}{(1 - |\varphi(z)|^2)^\frac{2 + \frac{2}{p} - n}{2} + n} = 0. \quad (13)
\]

**Proof.** (iii) \(\Rightarrow\) (i). Suppose that \(C_{\varphi, g}^n : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha\) is bounded and (13) holds. Let \((f_k)_{k \in \mathbb{N}}\) be a sequence in \(\mathcal{Q}_K(p, q)\) such that \(\sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{Q}_K(p, q)} \leq C\) and \(f_k\) converges to 0 uniformly on compact subsets of \(\mathbb{D}\) as \(k \rightarrow \infty\). By the assumption, for any \(\varepsilon > 0\), there exists a \(\delta \in (0, 1)\) such that

\[
\frac{(1 - |z|^2)^\alpha |g(z)|}{(1 - |\varphi(z)|^2)^\frac{2 + \frac{2}{p} - n}{2} + n} < \varepsilon \quad (14)
\]

when \(\delta < |\varphi(z)| < 1\). Since \(C_{\varphi, g}^n : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha\) is bounded, then from the proof of Theorem 1 we have

\[
M_2 := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |g(z)| < \infty. \quad (15)
\]

Let \(\Omega = \{z \in \mathbb{D} : |\varphi(z)| \leq \delta\}\). Then, we have
\[
\|C_{\varphi, g}^n f_k\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |(C_{\varphi, g}^n f_k)(z)| \\
\leq \sup_{\Omega} (1 - |z|^2)^\alpha |g(z)||f_k^{(n)}(\varphi(z))| + \sup_{\Omega} (1 - |z|^2)^\alpha |g(z)||f_k^{(n)}(\varphi(z))| \\
\leq \sup_{\Omega} (1 - |z|^2)^\alpha |g(z)||f_k^{(n)}(\varphi(z))| + C \sup_{\Omega} \frac{(1 - |z|^2)^\alpha |g(z)|}{(1 - |\varphi(z)|^2)^\frac{2 + \frac{2}{p} - n}{2} + n} \|f_k\|_{\mathcal{Q}_K(p, q)} \\
\leq M_2 \sup_{|w| \leq \delta} |f_k^{(n)}(w)| + C\varepsilon \|f_k\|_{\mathcal{Q}_K(p, q)}. \quad (16)
\]
\( k \to \infty \) in (16) and using the fact that \( \epsilon \) is an arbitrary positive number, we obtain 
\[
\lim_{k \to \infty} \| C^n_{\varphi, g} f_k \|_{B^\alpha} = 0.
\]
Applying Lemma 1, the result follows.

\((i) \Rightarrow (ii)\). This implication is obvious.

\((ii) \Rightarrow (iii)\). Suppose that \( C^n_{\varphi, g} : Q_{K, 0}(p, q) \to B^\alpha \) is compact. Then it is clear
that \( C^n_{\varphi, g} : Q_{K, 0}(p, q) \to B^\alpha \) is bounded and from Theorem 1 we see that \( C^n_{\varphi, g} : Q_K(p, q) \to B^\alpha \) is bounded. Let \((z_k)_{k \in \mathbb{N}}\) be a sequence in \( \mathbb{D} \) such that \( |\varphi(z_k)| \to 1 \) as \( k \to \infty \) (if such a sequence does not exist then condition (13) is vacuously satisfied). Let \( f_k(z) = \frac{1 - |\varphi(z_k)|^2}{(1 - |\varphi(z_k)|^2)^{\alpha/2}} \). Then, \( f_k \in Q_{K, 0}(p, q) \), \( \sup_{k \in \mathbb{N}} \| f_k \|_{Q_K(p, q)} < \infty \) and \( f_k \) converges to 0 uniformly on compact subsets of \( \mathbb{D} \) as \( k \to \infty \). Since
\( C^n_{\varphi, g} : Q_{K, 0}(p, q) \to B^\alpha \) is compact, by Lemma 1 we have
\[
\lim_{k \to \infty} \| C^n_{\varphi, g} f_k \|_{B^\alpha} = 0.
\]

On the other hand, from (10) we have
\[
\| C^n_{\varphi, g} f_k \|_{B^\alpha} \geq \prod_{j=0}^{n-1} \left( g + 2 - \frac{j}{p} \right) \left( 1 - |z_k|^2 \right)^\alpha |g(z_k)| |\varphi(z_k)|^n
\]
which together with (17) implies that
\[
\lim_{|\varphi(z_k)| \to 1} \frac{(1 - |z_k|^2)^\alpha |g(z_k)| |\varphi(z_k)|^n}{(1 - |\varphi(z_k)|^2)^{\alpha/2 - \frac{\alpha}{p} + n}} = \lim_{k \to \infty} \frac{(1 - |z_k|^2)^\alpha |g(z_k)| |\varphi(z_k)|^n}{(1 - |\varphi(z_k)|^2)^{\alpha/2 - \frac{\alpha}{p} + n}} = 0,
\]
from which (13) easily follows. \( \square \)

**Theorem 3.** Let \( \alpha, p > 0, q > -2 \) and \( K \) be a nonnegative nondecreasing function on \([0, \infty)\) such that (6) holds. Assume that \( \varphi \) is an analytic self-map of \( \mathbb{D} \) and \( \alpha, n \in \mathbb{N} \).

Then \( C^n_{\varphi, g} : Q_{K, 0}(p, q) \to B^\alpha_0 \) is bounded if and only if \( C^n_{\varphi, g} : Q_{K, 0}(p, q) \to B^\alpha_0 \) is bounded and
\[
\lim_{|z| \to 1} (1 - |z|^2)^\alpha |g(z)| = 0.
\]

*Proof.* Suppose that \( C^n_{\varphi, g} : Q_{K, 0}(p, q) \to B^\alpha_0 \) is bounded. It is obvious that \( C^n_{\varphi, g} : Q_{K, 0}(p, q) \to B^\alpha_0 \) is bounded. Taking the function \( f(z) = z^n \), and employing the boundedness of \( C^n_{\varphi, g} : Q_{K, 0}(p, q) \to B^\alpha_0 \) we see that (19) holds.

Conversely, assume that \( C^n_{\varphi, g} : Q_{K, 0}(p, q) \to B^\alpha_0 \) is bounded and (19) holds. Then, for each polynomial \( p(z) \), we have that
\[
(1 - |z|^2)^\alpha |(C^n_{\varphi, g} p)'(z)| \leq (1 - |z|^2)^\alpha |g(z)||p^{(n)}||(z)|,
\]
from which it follows that \( C^n_{\varphi, 0}p \in B^\alpha_0 \). Since the set of all polynomials is dense in \( Q_{K, 0}(p, q) \) (see [2]), we have that for every \( f \in Q_{K, 0}(p, q) \) there is a sequence of polynomials \( (p_k)_{k \in \mathbb{N}} \) such that \( \| f - p_k \|_{Q_K(p, q)} \to 0 \), as \( k \to \infty \). Hence
\[
\| C^n_{\varphi, g} f - C^n_{\varphi, g} p_k \|_{B^\alpha} \leq \| C^n_{\varphi, g} \|_{Q_{K, 0}(p, q) \to B^\alpha} \| f - p_k \|_{Q_K(p, q)} \to 0
\]
as $k \to \infty$. Since $B^0_\alpha$ is closed subset of $B^\alpha$, we obtain $C^{\alpha}_{\varphi,g}(Q_{K,0}(p,q)) \subset B^0_\alpha$. Therefore $C^{\alpha}_{\varphi,g} : Q_{K,0}(p,q) \to B^0_\alpha$ is bounded. □

**Theorem 4.** Let $\alpha, p > 0$, $q > -2$ and $K$ be a nonnegative nondecreasing function on $[0, \infty)$ such that (6) holds. Assume that $\varphi$ is an analytic self-map of $D$ and $n \in \mathbb{N}$. Then the following statements are equivalent.

(i) $C^{\alpha}_{\varphi,g} : Q_{K,0}(p,q) \to B^0_\alpha$ is compact;

(ii) $C^{\alpha}_{\varphi,g} : Q_{K,0}(p,q) \to B^0_\alpha$ is compact;

(iii) $\lim\limits_{|z| \to 1} \frac{(1 - |z|^2)^\alpha |g(z)|}{(1 - |\varphi(z)|^2)^{\frac{2q - p}{p} + n}} = 0$. (20)

**Proof.** (iii) ⇒ (i). Assume that (20) holds. Let $f \in Q_{K,0}(p,q)$. By the proof of Theorem 1 we have

$$\|(C^{\alpha}_{\varphi,g} f)'(z)\| \leq C \frac{(1 - |z|^2)^\alpha |g(z)|}{(1 - |\varphi(z)|^2)^{\frac{2q - p}{p} + n}} \|f\|_{Q_{K,0}(p,q)}. \quad (21)$$

Taking the supremum in (21) over all $f \in Q_{K,0}(p,q)$ such that $\|f\|_{Q_{K,0}(p,q)} \leq 1$, then letting $|z| \to 1$, we get

$$\lim\limits_{|z| \to 1} \sup_{\|f\|_{Q_{K,0}(p,q)} \leq 1} (1 - |z|^2)^\alpha (C^{\alpha}_{\varphi,g} f)'(z) = 0.$$ (22)

From which by Lemma 2 we see that $C^{\alpha}_{\varphi,g} : Q_{K,0}(p,q) \to B^0_\alpha$ is compact.

(i) ⇒ (ii). This implication is obvious.

(ii) ⇒ (iii). Suppose that $C^{\alpha}_{\varphi,g} : Q_{K,0}(p,q) \to B^0_\alpha$ is compact. Then $C^{\alpha}_{\varphi,g} : Q_{K,0}(p,q) \to B^0_\alpha$ is bounded and by Theorem 3 we get

$$\lim\limits_{|z| \to 1} (1 - |z|^2)^\alpha |g(z)| = 0.$$ (22)

If $\|\varphi\|_\infty < 1$, from (22), we obtain that

$$\lim\limits_{|z| \to 1} \frac{(1 - |z|^2)^\alpha |g(z)|}{(1 - |\varphi(z)|^2)^{\frac{2q - p}{p} + n}} \leq \frac{1}{(1 - \|\varphi\|_\infty^2)^{\frac{2q - p}{p} + n}} \lim\limits_{|z| \to 1} (1 - |z|^2)^\alpha |g(z)| = 0,$$

from which the result follows in this case.

Assume that $\|\varphi\|_\infty = 1$. Let $(\varphi(z_k))_{k \in \mathbb{N}}$ be a sequence such that $\lim_{k \to \infty} |\varphi(z_k)| = 1$. From the compactness of $C^{\alpha}_{\varphi,g} : Q_{K,0}(p,q) \to B^0_\alpha$ we see that the operator $C^{\alpha}_{\varphi,g} : Q_{K,0}(p,q) \to B^\alpha$ is compact. □From Theorem 2 we get

$$\lim\limits_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^\alpha |g(z)|}{(1 - |\varphi(z)|^2)^{\frac{2q - p}{p} + n}} = 0 \quad (23)$$

From (23), we have that for every $\varepsilon > 0$, there exists an $r \in (0, 1)$ such that

$$\frac{(1 - |z|^2)^\alpha |g(z)|}{(1 - |\varphi(z)|^2)^{\frac{2q - p}{p} + n}} < \varepsilon \quad (24)$$
when \( r < |\varphi(z)| < 1 \). From (22), there exists a \( \sigma \in (0,1) \) such that
\[
(1 - |z|^2)^\alpha |g(z)| \leq \varepsilon (1 - r^2)^{\frac{q + 2}{p} + n}
\] (25)
when \( \sigma < |z| < 1 \).

Therefore, when \( \sigma < |z| < 1 \) and \( r < |\varphi(z)| < 1 \), we have
\[
(1 - |z|^2)^\alpha |g(z)| \leq \varepsilon (1 - r^2)^{\frac{q + 2}{p} + n}
\]
(26)

On the other hand, if \( \sigma < |z| < 1 \) and \( |\varphi(z)| \leq r \), we obtain
\[
(1 - |z|^2)^\alpha |g(z)| \leq \varepsilon (1 - r^2)^{\frac{q + 2}{p} + n}
\]
(27)

From (26) and (27) we get (20), as desired. The proof is completed. \( \Box \)

Next, we consider the case \( n = 0 \).

Theorem 5. Let \( \alpha, p > 0, q > -2 \) such that \( q + 2 \geq p \). Let \( K \) be a nonnegative nondecreasing function on \([0,\infty)\) such that (6) holds. Assume that \( \varphi \) is an analytic self-map of \( \mathbb{D} \). Then the following statements are equivalent.

(i) \( C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha \) is bounded;
(ii) \( C^0_{\varphi,g} : \mathcal{Q}_K,0(p,q) \to \mathcal{B}^\alpha \) is bounded;
(iii)
\[
\begin{align*}
&\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |g(z)| \ln \frac{e}{1 - |\varphi(z)|^2} < \infty, \quad q + 2 = p; \\
&\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\alpha |g(z)|}{(1 - |\varphi(z)|^2)^{\frac{q + 2}{p}}} < \infty, \quad q + 2 > p.
\end{align*}
\]

Proof. (ii) \( \Rightarrow \) (iii). Assume that \( C^0_{\varphi,g} : \mathcal{Q}_K,0(p,q) \to \mathcal{B}^\alpha \) is bounded. For \( w \in \mathbb{D} \), let
\[
f_w(z) = \begin{cases} 
\ln \frac{e}{1 - |z|^2}, & q + 2 = p; \\
\ln \frac{1 - |z|^2}{(1 - |w|^2)^{\frac{q + 2}{p}}}, & q + 2 > p.
\end{cases}
\]
Then \( f_w \in \mathcal{Q}_K,0(p,q) \) (see [2]). The other proof is similar to the proof of Theorem 1 and hence we omit it.

(i) \( \Rightarrow \) (ii) is obvious.

(iii) \( \Rightarrow \) (i). Using Lemma 4, similar to the proof of Theorem 1, the implication follows. We omit the details of the proofs.

Let \((z_k)_{k \in \mathbb{N}}\) be a sequence in \( \mathbb{D} \) such that \( |\varphi(z_k)| \to 1 \) as \( k \to \infty \). Taking the test function
\[
f_{\varphi(z_k)}(z) = \begin{cases} 
\ln \frac{e}{1 - |z|^2}, & q + 2 = p; \\
\ln \frac{1 - |z|^2}{(1 - |\varphi(z_k)|)^{\frac{q + 2}{p}}}, & q + 2 > p,
\end{cases}
\]
similar to the proof of Theorem 2, we obtain the following result.

**Theorem 6.** Let \( \alpha, p > 0, q > -2 \) such that \( q + 2 \geq p \). Let \( K \) be a nonnegative nondecreasing function on \([0, \infty)\) such that (6) holds. Assume that \( \varphi \) is an analytic self-map of \( D \). Then the following statements are equivalent.

1. \( C_{\varphi, \alpha}^0 : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha \) is compact;
2. \( C_{\varphi, \alpha}^0 : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha \) is compact;
3. \( C_{\varphi, \alpha}^0 : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha \) is bounded and

\[
\begin{align*}
\lim_{|z| \to 1} (1 - |z|^2)\alpha |\varphi(z)|^{\frac{\alpha}{1 - |\varphi(z)|^2}} &= 0, \quad q + 2 = p; \\
\lim_{|z| \to 1} (1 - |z|^2)\alpha |\varphi(z)|^{\frac{\alpha}{1 - |\varphi(z)|^2}} &= 0, \quad q + 2 > p.
\end{align*}
\]

Similar to the proofs of Theorems 3 and 4, we obtain Theorems 7 and 8 respectively. We omit the proofs.

**Theorem 7.** Let \( \alpha, p > 0, q > -2 \) such that \( q + 2 \geq p \). Let \( K \) be a nonnegative nondecreasing function on \([0, \infty)\) such that (6) holds. Assume that \( \varphi \) is an analytic self-map of \( D \). Then \( C_{\varphi, \alpha}^0 : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha \) is bounded if and only if \( C_{\varphi, \alpha}^0 : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha \) is bounded and

\[
\lim_{|z| \to 1} (1 - |z|^2)\alpha |\varphi(z)| = 0.
\]

**Theorem 8.** Let \( \alpha, p > 0, q > -2 \) such that \( q + 2 \geq p \). Let \( K \) be a nonnegative nondecreasing function on \([0, \infty)\) such that (6) holds. Assume that \( \varphi \) is an analytic self-map of \( D \). Then the following statements are equivalent.

1. \( C_{\varphi, \alpha}^0 : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha \) is compact;
2. \( C_{\varphi, \alpha}^0 : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha \) is compact;
3. \( C_{\varphi, \alpha}^0 : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha \) is bounded and

\[
\begin{align*}
\lim_{|z| \to 1} (1 - |z|^2)\alpha |\varphi(z)|^{\frac{\alpha}{1 - |\varphi(z)|^2}} &= 0, \quad q + 2 = p; \\
\lim_{|z| \to 1} (1 - |z|^2)\alpha |\varphi(z)|^{\frac{\alpha}{1 - |\varphi(z)|^2}} &= 0, \quad q + 2 > p.
\end{align*}
\]

The proof of the following two theorems are similar to the proofs of Theorems 12-14 of [37]. We omit the details.

**Theorem 9.** Let \( \alpha, p > 0, q > -2 \) such that \( q + 2 < p \). Let \( K \) be a nonnegative nondecreasing function on \([0, \infty)\) such that (6) holds. Assume that \( \varphi \) is an analytic self-map of \( D \). Then the following statements are equivalent.

1. \( C_{\varphi, \alpha}^0 : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha \) is bounded;
2. \( C_{\varphi, \alpha}^0 : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha \) is compact;
3. \( C_{\varphi, \alpha}^0 : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha \) is bounded;
4. \( C_{\varphi, \alpha}^0 : \mathcal{Q}_K(p, q) \rightarrow \mathcal{B}^\alpha \) is compact;
\( (v) \quad \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |g(z)| < \infty. \)

**Theorem 10.** Let \( \alpha, p > 0, q > -2 \) such that \( q + 2 < p \). Let \( K \) be a nonnegative nondecreasing function on \([0, \infty)\) such that (6) holds. Assume that \( \varphi \) is an analytic self-map of \( \mathbb{D} \). Then the following statements are equivalent.

(i) \( C_{\varphi, g}^0 : Q_K(p, q) \to B_0^\alpha \) is bounded;

(ii) \( C_{\varphi, g}^0 : Q_{K,0}(p, q) \to B_0^\alpha \) is bounded;

(iii) \( C_{\varphi, g}^0 : Q_K(p, q) \to B_0^\alpha \) is compact;

(iv) \( C_{\varphi, g}^0 : Q_{K,0}(p, q) \to B_0^\alpha \) is compact;

(v) \( \lim_{|z| \to 1} (1 - |z|^2)^\alpha |g(z)| = 0. \)

**References**


S. Stević, Integral-type operators from a mixed norm space to a Bloch-type space on the unit ball, *Siberian Math. J.* 50 (6) (2009), 1098-1105.


