SOME SYMMETRIC SEMI-CLASSICAL POLYNOMIAL SETS

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Abstract

We show that if \( v \) is a regular semi-classical form (linear functional), then the symmetric form \( u \) defined by the relation \( x \sigma u = -\lambda v \) where \( \sigma u \) is the even part of \( u \), is also regular and semi-classical form for every complex \( \lambda \) except for a discrete set of numbers depending on \( v \). We give explicitly the recurrence coefficients, integral representation and the structure relation coefficients of the orthogonal polynomials sequence associated with \( u \) and the class of the form \( u \) knowing that of \( v \). We conclude with some illustrative examples.

1 Introduction

In many recent papers, different construction processes of semi-classical orthogonal polynomials (O.P) can be done from well known ones, particularly the classical ones. For instance, we can mention the adjunction of a finite number of Dirac’s masses and their derivatives to semi-classical forms [2, 7-9], the product and the division of a form by a polynomial [1, 3, 6, 10, 13, 15].

The whole idea of the following work is to build a new construction process of semi-classical form, which has not yet been treated in the literature on semi-classical polynomials. The problem we tackle is as follows.

We study the form \( u \), fulfilling \( x \sigma u = -\lambda v \), \( \lambda \neq 0 \), \( (u)_{2n+1} = 0 \), where \( \sigma u \) is the even part of \( u \) and \( v \) is a given semi-classical form.

This paper is organized in sections : The first one is focused on the preliminary results and notations used in the sequel. We will also give the regularity condition and the coefficients of the three-term recurrence relation satisfied by the new family of O.P. In the second, we compute the exact class of the semi-classical form obtained by the above modification and the structure relation of the O.P sequence relatively to the form \( u \) will follow. In the final section, we apply our results to some examples. The regular forms found in the examples are semi-classical of class 2010 Mathematics Subject Classifications. 33C45 ; 42C05

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Let \( \mathcal{P} \) be the vector space of polynomials with coefficients in \( \mathbb{C} \) and let \( \mathcal{P}' \) be its dual. We denote by \( \langle v, f \rangle \) the action of \( v \in \mathcal{P}' \) on \( f \in \mathcal{P} \). In particular, we denote by \( (v)_n := \langle v, x^n \rangle, n \geq 0 \), the moments of \( v \). For any form \( v \) and any polynomial \( h \) let \( Dv = v', hv, \delta_0 \), and \( (x-c)^{-1}v \) be the forms defined by: \( \langle v', f \rangle := -\langle v, f' \rangle, \langle hv, f \rangle := \langle v, hf \rangle, \langle \delta_c, f \rangle := f(c) \), and \( \langle (x-c)^{-1}v, f \rangle := \langle v, \theta_c f \rangle \) where \( \langle \theta_c f \rangle (x) = \frac{f(x) - f(c)}{x - c}, c \in \mathbb{C}, f \in \mathcal{P} \).

Then, it is straightforward to prove that for \( v \in \mathcal{P} \) and \( f \in \mathcal{P}' \), we have
\[
x^{-1}(xv) = v - (v)_0 \delta_0 ,
\]
\[
(fv)' = f'v + f v' .
\]

Let us define the operator \( \sigma : \mathcal{P} \rightarrow \mathcal{P} \) by \( (\sigma f)(x) := f(x^2) \). Then, we define the even part \( \sigma v \) of \( v \) by \( \langle \sigma v, f \rangle := \langle v, \sigma f \rangle \). Therefore, we have [5, 11]
\[
f(x) (\sigma v) = \sigma (f(x^2)v) ,
\]
\[
(\sigma v)_n = (v)_{2n} , \quad n \geq 0 .
\]

The form \( v \) will be called regular if there exists a sequence of polynomials \( \{S_n\}_{n \geq 0} \) of degree \( \deg(S_n) \leq n \) such that \( \langle v, S_n S_m \rangle = r_n \delta_{n,m}, n, m \geq 0, n \neq m \), \( n \geq 0 \). Then \( \deg(S_n) = n, n \geq 0 \), and we can always suppose each \( S_n \) is monic (i.e. \( S_n(x) = x^n + \cdots \)). The sequence \( \{S_n\}_{n \geq 0} \) is said to be orthogonal with respect to \( v \). It is a very well known fact that the sequence \( \{S_n\}_{n \geq 0} \) satisfies the recurrence relation (see, for instance, the monograph by Chihara [5])
\[
S_{n+2}(x) = (x - \xi_{n+1})S_{n+1}(x) - \rho_{n+1}S_n(x) , \quad n \geq 0 ,
\]
\[
S_1(x) = x - \xi_0 , \quad S_0(x) = 1 ,
\]
with \( \{\xi_n, \rho_{n+1}\} \in \mathbb{C} \times \mathbb{C} - \{0\}, n \geq 0 \), by convention we set \( \rho_0 = (v)_0 = 1 \).

In this case, let \( \{S_n^{(1)}\}_{n \geq 0} \) be the associated sequence of first kind for the sequence \( \{S_n\}_{n \geq 0} \) satisfying the three-term recurrence relation
\[
S_{n+2}^{(1)}(x) = (x - \xi_{n+2})S_{n+1}^{(1)}(x) - \rho_{n+2}S_n^{(1)}(x) , \quad n \geq 0 ,
\]
\[
S_{1}^{(1)}(x) = x - \xi_1 , \quad S_0^{(1)}(x) = 1 , \quad \langle S_1^{(1)}(x) \rangle = 0 \).
\]

Also, let \( \{S_n(\mu)\}_{n \geq 0} \) be the co-recursive polynomials for the sequence \( \{S_n\}_{n \geq 0} \) satisfying [5]
\[
S_n(x, \mu) = S_n(x) - \mu S_{n-1}(x) , \quad n \geq 0 .
\]

A form \( v \) is called symmetric if \( (v)_{2n+1} = 0, n \geq 0 \). The conditions \( (v)_{2n+1} = 0, n \geq 0 \) are equivalent to the fact that the corresponding monic orthogonal polynomials sequence (MOPS) \( \{S_n\}_{n \geq 0} \) satisfies the recurrence relation (5) with \( \xi_n = 0, n \geq 0 \) [5].
Proposition 1. [5,11] If the form \( v \) is symmetric, then \( v \) is regular if and only if \( \sigma v \) and \( x \sigma v \) are both regular.

Let \( v \) be a regular, normalized form (i.e. \( (v)_0 = 1 \)) and \( \{S_n\}_{n \geq 0} \) be its corresponding sequence of monic orthogonal polynomials. For a \( \lambda \in \mathbb{C} - \{0\} \), we can define a new symmetric form \( u \) as follows

\[
x \sigma u = -\lambda v \quad (u)_{2n+1} = 0 , \quad (u)_0 = 1 , \quad n \geq 0.
\]

From (1), we have

\[
\sigma u = -\lambda x^{-1} v + \delta_0.
\]

Proposition 2. The form \( u \) is regular if and only if \( \lambda \neq \lambda_n, n \geq 0 \) where \( \lambda_n = \frac{S_n(0)}{S_{n-1}(0)} \).

Proof. Since \( u \) is a symmetric form then, according to Proposition 1 \( u \) is regular if and only if \( x \sigma u \) and \( \sigma u \) are regular. But \( x \sigma u = -\lambda v \) is regular. So \( u \) is regular if and only if \( \sigma u = -\lambda x^{-1} \sigma v + \delta_0 \) is regular. Or, it was shown in [13] that the form \(-\lambda x^{-1} v + \delta_0 \) is regular if and only if \( \lambda \neq 0 \), and \( S_n(0, \lambda) \neq 0 \), \( n \geq 0 \). Then, we deduce the desired result.

Remark. If \( w \) is the symmetrized form associated with the form \( v \) (i.e. \( (w)_n = (v)_n \) and \( (w)_{2n+1} = 0, n \geq 0 \)), then (8) is equivalent to \( x^2 u = -\lambda v \). Notice that \( w \) is not necessarily a regular form in the problem under study. In [1, 3], the authors have solved it only when \( w \) is regular.

When \( u \) is regular let \( \{Z_n\}_{n \geq 0} \) be its MOPS satisfying the recurrence relation

\[
Z_{n+2}(x) = x Z_{n+1}(x) - \gamma_{n+1} Z_n(x) , \quad n \geq 0 , \\
Z_1(x) = x , \quad Z_0(x) = 1.
\]

Since \( \{Z_n\}_{n \geq 0} \) is symmetric, let us consider its quadratic decomposition [11]:

\[
Z_{2n}(x) = P_n(x^2) , \quad Z_{2n+1}(x) = x R_n(x^2) .
\]

\[
Z_{2n}^{(1)}(x) = R_n \left( x^2, -\gamma_1 \right) , \quad Z_{2n+1}^{(1)}(x) = x P_n^{(1)}(x^2) .
\]

The sequences \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \) are respectively orthogonal with respect to \( \sigma u \) and \( x \sigma u \).

From (8), we have

\[
R_n(x) = S_n(x) , \quad n \geq 0 .
\]

Proposition 3. We may write

\[
\gamma_1 = -\lambda , \quad \gamma_{2n+2} = a_n , \quad \gamma_{2n+3} = \frac{\rho_{n+1}}{a_n} , \quad n \geq 0
\]

where

\[
a_n = -\frac{S_{n+1}(0, \lambda)}{S_n(0, \lambda)} , \quad n \geq 0 .
\]
Proof. Using (8) and the condition \( \langle u, Z_2 \rangle = 0 \), we obtain \( \gamma_1 = -\lambda \).
From (6) and (10) where \( n \rightarrow 2n \) and taking (12)-(13) into account, we get
\[
S_{n+1}(x^2, -\gamma_1) = xZ_{2n+1}^{(1)}(x) - \gamma_{2n+2}S_n(x^2, -\gamma_1)
\]
Substituting \( x \) by 0 in the above equation, we obtain \( \gamma_{2n+2} = a_n \).
From (10), we have
\[
\gamma_{2n+2} \gamma_{2n+3} = \frac{\langle u, Z_{2n+2}^2 \rangle}{\langle u, Z_{2n+1}^2 \rangle} \frac{\langle u, Z_{2n+3}^2 \rangle}{\langle u, Z_{2n+2}^2 \rangle} = \frac{\langle u, Z_{2n+1}^2 \rangle}{\langle u, Z_{2n+1}^2 \rangle} \frac{\langle u, Z_{2n+3}^2 \rangle}{\langle u, Z_{2n+2}^2 \rangle}.
\]
(16)
Using (11), (8) and (5), equation (16) becomes
\[
\gamma_{2n+2} \gamma_{2n+3} = \rho_{n+1},
\]
then, we deduce \( \gamma_{2n+3} = \frac{\rho_{n+1}}{a_n} \).

**Corollary 1.** When the form \( v \) is symmetric, then \( u \) is regular for every \( \lambda \neq 0 \).
Moreover,
\[
\begin{align*}
\gamma_1 &= -\gamma_2 = -\lambda \\
\gamma_{4n+3} &= -\gamma_{4n+4} = -\frac{1}{\lambda} \prod_{k=0}^{n} \frac{\rho_{2k+1}}{\rho_{2k}}, \\
\gamma_{4n+5} &= -\gamma_{4n+6} = \lambda \prod_{k=0}^{n} \frac{\rho_{2k}}{\rho_{2k+1}}, n \geq 0.
\end{align*}
\]
(18)
Proof. Taking into account (5) and (6), with \( \xi_n = 0 \), we get \( S_{n+2}(0) = -\rho_{n+1}S_n(0) \) and \( \langle v, f \rangle = \int_{-\infty}^{+\infty} V(x)f(x)dx \), \( f \in \mathcal{P} \), with \( \langle v \rangle_0 = \int_{-\infty}^{+\infty} V(x)dx = 1 \)
where \( V \) is a locally integrable function with rapid decay and continuous at the origin.
It is obvious that \( f(x) = f^{(2)}(x^2) + xf^{(4)}(x^2), f \in \mathcal{P} \).
Therefore, \( \langle u, f \rangle = \langle u, f^e(x^2) \rangle = \langle \sigma u, f^e(x) \rangle \) since \( u \) is symmetric. Using (8) and taking into account that \( f^e(0) = f(0) \), we obtain

\[
\langle u, f \rangle = f(0) \left\{ 1 + \lambda P \int_{-\infty}^{+\infty} \frac{V(x)}{x} dx \right\} - \lambda P \int_{-\infty}^{+\infty} \frac{V(x)}{x} f^e(x) dx ,
\]

(21)

where

\[
P \int_{-\infty}^{+\infty} \frac{V(x)}{x} f(x) dx = \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{-\epsilon} \frac{V(x)}{x} f(x) dx + \int_{\epsilon}^{+\infty} \frac{V(x)}{x} f(x) dx \right\} .
\]

It is easy to see that

\[
P \int_{-\infty}^{+\infty} \frac{V(x)}{x} f(x) dx = \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{+\infty} \frac{V(x)}{x} f^e(x) dx - \int_{\epsilon}^{+\infty} \frac{V(-x)}{x} f^e(-x) dx \right\} .
\]

Using the fact that \( f^e(x) = \frac{f(\sqrt{\epsilon}) + f(-\sqrt{\epsilon})}{2} \) and \( f^e(-x) = \frac{f(i\sqrt{\epsilon}) + f(-i\sqrt{\epsilon})}{2} \) for \( x \geq 0 \) and making the change of variables \( t = \sqrt{\epsilon} \), we get

\[
P \int_{-\infty}^{+\infty} \frac{V(x)}{x} f(x) dx = - \lim_{\epsilon \to 0} \int_{\sqrt{\epsilon}}^{+\infty} \frac{V(-t^2)}{t} (f(it) + f(-it)) dt + \lim_{\epsilon \to 0} \int_{\sqrt{\epsilon}}^{+\infty} \frac{V(t^2)}{t} (f(t) + f(-t)) dt .
\]

Inserting the last equation into (21), we get after a change variables in the obtained equation

\[
\langle u, f \rangle = f(0) \left\{ 1 + \lambda P \int_{-\infty}^{+\infty} \frac{V(x)}{x} dx \right\} + \lambda P \int_{-\infty}^{+\infty} \frac{V(-x^2)}{|x|} f(ix) dx - \lambda P \int_{-\infty}^{+\infty} \frac{V(x^2)}{|x|} f(x) dx .
\]

(22)

Remark. When \( v \) is symmetric, (22) becomes

\[
\langle u, f \rangle = f(0) - \lambda P \int_{-\infty}^{+\infty} \frac{V(x^2)}{|x|} (f(x) - f(ix)) dx .
\]

(23)

Our aim is to give examples of semi-classical forms (8) through data of semi-classical form \( v \).

2 The semi-classical case

Let us recall that a form \( v \) is called semi-classical when it is regular and there exist two polynomials \( \Phi \) and \( \Psi \) such that:

\[
(\Phi v)' + \Psi v = 0 , \quad \deg(\Psi) \geq 1 , \quad \Phi \text{ monic}.
\]

(24)
The class of the semi-classical form \( v \) is \( s = \max(\deg \Psi - 1, \deg \Phi - 2) \) if and only if the following condition is satisfied
\[
\prod_{c} \left( |\Phi'(c) + \Psi(c)| + |\langle u, \theta_{c}\Psi + \theta_{c}^{2}\Phi' \rangle| \right) > 0 ,
\] where \( c \) goes over the roots set of \( \Phi \) [12].

The corresponding orthogonal sequence \( \{S_n\}_{n \geq 0} \) is also called semi-classical of class \( s \).

We can state characterizations of semi-classical orthogonal sequences. \( \{S_n\}_{n \geq 0} \) is semi-classical of class \( s \) if and only if one of the following statements holds:

(a) The formal Stieltjes function of \( v \), namely
\[
S(v)(z) = -\sum_{n \geq 0} \frac{(v)_{n}}{z^{n+1}}
\] satisfies a linear non-homogeneous first order differential equation [4,12]
\[
\Phi(z)S'(v)(z) = C_{0}(z)S(v)(z) + D_{0}(z),
\] where
\[
C_{0}(x) = -\Phi'(x) - \Psi(x).
\] and
\[
D_{0}(z) = -(v\theta_{0}\Phi)'(x) - (v\theta_{0}\Psi)(x).
\] with \((v\theta_{0}f)(x) = \left\langle v, \frac{f(x) - f(\zeta)}{x - \zeta} \right\rangle \), \( f \in \mathcal{P} \). \( \Phi \) and \( \Psi \) are the same polynomials as in (24).

(b) \( \{S_n\}_{n \geq 0} \) fulfills the following structure recurrence relation (written in a compact form):
\[
\Phi(x)S_{n+1}'(x) = \frac{C_{n+1}(x) - C_{0}(x)}{2} S_{n+1}(x) - \rho_{n+1}D_{n+1}(x)S_{n}(x) , n \geq 0
\] where
\[
\begin{cases}
C_{n+1}(x) = -C_{n}(x) + 2(x - \beta_{n})D_{n}(x) , & n \geq 0 , \\
\rho_{n+1}D_{n+1}(x) = -\Phi(x) + \rho_{n}D_{n-1}(x) - (x - \xi_{n})C_{n}(x) + (x - \xi_{n})^{2}D_{n}(x) , & n \geq 0 ,
\end{cases}
\] \( \Phi, \Psi, C_{0} \) and \( D_{0} \) are the same polynomials introduced in (a); \( \xi_{n}, \rho_{n} \) are the coefficients of the three term recurrence relation (5). Notice that \( D_{-1}(x) = 0, \deg C_{n} \leq s + 1 \) and \( \deg D_{n} \leq s, n \geq 0 \) [12].

(c) Each polynomial of \( \{S_n\}_{n \geq 0} \) satisfies a second order differential equation of Laguerre-Perron type, i.e.
\[
\Phi D_{n+1}S_{n+1}' + \{C_{0}D_{n+1} - W(\Phi, D_{n+1})\} S_{n+1}' + \left\{W \left( \frac{C_{n+1} - C_{0}}{2}, D_{n+1} \right) - D_{n+1} \sum_{k=0}^{n} D_{k} \right\} S_{n+1} = 0 , n \geq 0,
\]
where $W(f, g) = fg' - f'g$. $\Phi$, $D_n$, $C_n$, $n \geq 0$ are the same parameters introduced in the previous characterizations [4,14].

**Remark.** The structure relation gives information about the multiplicity of the zeros of orthogonal polynomials.

In the sequel the form $v$ will be supposed semi-classical of class $s$ satisfying (24) – (25).

**Proposition 4.** If $v$ is a semi-classical form and satisfies (24), then for every $\lambda \in \mathbb{C} - \{0\}$ such that $S_n(0, \lambda) \neq 0, n \geq 0$, the form $u$ defined by (8) is regular and semi-classical. It satisfies

$$
\left(\tilde{\Phi}u\right)' + \tilde{\Psi}u = 0 \quad (33)
$$

with

$$
\tilde{\Phi}(x) = x\Phi(x^2), \quad \tilde{\Psi}(x) = 2x^2\Psi(x^2). \quad (34)
$$

and $u$ is of class $\tilde{s}$ with $\tilde{s} \leq 2s + 3$.

**Proof.** Assume that $v$ fulfills (24). To prove that $u$ satisfies (33)-(34), we will show that the forms $(\tilde{\Phi}u)'$ and $-\tilde{\Psi}u$ coincide on the basis $\{x^n\}_{n \geq 0}$ of $P$.

Taking into account (34) and using the operator $\sigma$, we obtain

$$
\left<(\tilde{\Phi}u)', x^{2n}\right> = -2n\left<\Phi(x^2)u, x^{2n}\right> = -2n\left<\Phi(x)\sigma u, x^n\right>, n \geq 1.
$$

By virtue of (8) and (24), we deduce

$$
\left<(\tilde{\Phi}u)', x^{2n}\right> = -2\lambda \left<\Phi(x)v', x^n\right> = 2\lambda \left<\Psi(x)v, x^n\right>.
$$

Now, using (8) again and the definition of the operator $\sigma$, we get

$$
\left<(\tilde{\Phi}u)', x^{2n}\right> = -\left<\tilde{\Psi}u, x^{2n}\right>.
$$

Since $u$ is symmetric, it is clear that $\left<(\tilde{\Phi}u)', x^{2n+1}\right> = -\left<\tilde{\Psi}u, x^{2n+1}\right> = 0$.

Thus, (33)-(34) is proved.

Finally, we have $s = \max\{\deg \Psi - 1, \deg \Phi - 2\}$, then $\deg(\tilde{\Phi}) \leq 2s + 5$ and $\deg(\tilde{\Psi}) = \tilde{p} \leq 2s + 4$. Thus $\tilde{s} \leq 2s + 3$.

**Proposition 5.** The class of $u$ depends only on the zero $x = 0$.

For the proof, we use the following lemma:

**Lemma 1.** For $c \in \mathbb{C}$ such that $c^2$ be a root of $\Phi$, we have

$$
\left<u, \theta_c^2 \tilde{\Psi} + \theta_c^2 \tilde{\Phi}\right> = -2c\lambda \left<v, \theta_c \Psi + \theta_c^2 \Phi\right> + 2c\left(\Phi'(c^2) + \Psi(c^2)\right) \quad (35)
$$

and

$$
\tilde{\Psi}(c) + \tilde{\Phi}'(c) = 2c^2\left(\Phi'(c^2) + \Psi(c^2)\right). \quad (36)
$$
Proof. Using the definition of the operator $\theta_c$, it is easy to prove that, for two polynomials $f$ and $g$, we have

$$\theta_c(fg)(x) = g(x)(\theta_c f)(x) + f(c)(\theta_c g)(x), \quad (37)$$

$$\theta_c(f(\xi^2))(x) = (x + c)(\theta_c f)(x^2). \quad (38)$$

Let $c \in \mathbb{C}$ such that $c^2$ be a root of $\Phi$.

Using successively (37) and (38), we obtain

$$\left(\theta_c \tilde{\Phi}\right)(x) = x(\theta_c \Phi(\xi^2))(x) = x(x + c)(\theta_{c^2} \Phi)(x^2), \quad \text{since } \Phi(c^2) = 0. \quad \text{Then,}$$

$$\left(\theta_c^2 \tilde{\Phi}\right)(x) = x(x + c)^2 (\theta_{x^2} \Phi)(x^2) + (x + 2c)\Phi'(c^2), \quad (39)$$

because $\theta_c(\xi(\xi + c))(x) = x + 2c$, $\theta_c((\theta_{x^2} \Phi)(\xi^2))(x) = (x + c)\left(\theta_{x^2} \Phi\right)(x^2)$

and $(\theta_{x^2} \Phi)(c^2) = \Phi'(c^2)$.

Using the same procedure, we prove that

$$\theta_c \tilde{\Psi}(x) = x^2(x + c)(\theta_{x^2} \Psi)(x^2) + (x + c)\Psi(c^2). \quad (40)$$

Therefore, with (39)-(40) and the fact $u$ is symmetric, we obtain

$$\left<u, \theta_c \tilde{\Psi} + \theta_c^2 \tilde{\Phi}\right> = \left<x^2 u, 2\theta_{x^2} \Psi + \theta_{x^2}^2 \Phi\right> + 2c\left(\Phi'(c^2) + \Psi(c^2)\right). \quad (41)$$

Now applying the operator $\sigma$ for (41) and using (8), we get (35). Finally, from (34), we easily get (36).

Proof of Proposition 5. Let $c$ be a root of $\tilde{\Phi}$ such that $c \neq 0$.

If $\Phi'(c^2) + \Psi(c^2) \neq 0$ then $\tilde{\Phi}'(c) + \tilde{\Psi}(c) \neq 0$, from (36).

If $\Phi'(c^2) + \Psi(c^2) = 0$, using (35), we have $\left<u, \theta_c \Psi + \theta_c^2 \Phi\right> \neq 0$, since $\nu$ is semiconvex and so satisfies (25).

In any case, we cannot simplify by $x - c$. \hfill \Box

Proposition 6. Under the conditions of proposition 4, for the class of $u$, we have the four different cases

1) $\tilde{s} = 2s + 3$ if $\Phi(0) \neq 0$.

2) $\tilde{s} = 2s + 2$ if $\Phi(0) = 0$ and $X_1 = -2\lambda \left<v, \theta_0 \Psi + \theta_0^2 \Phi\right> + 2\left(\Phi'(0) + \Psi(0)\right) \neq 0$.

3) $\tilde{s} = 2s + 1$ if $\Phi(0) = 0$, $X_1 = 0$ and $X_2 = 3\Phi'(0) + 2\Psi(0) \neq 0$.

4) $\tilde{s} = 2s$ if $\Phi(0) = 0$, $X_1 = 0$ and $X_2 = 0$.

Proof. 1) From (34), we have $\tilde{\Phi}'(0) + \tilde{\Psi}(0) = \Phi(0)$

and $\left<u, \theta_0 \tilde{\Psi} + \theta_0^2 \tilde{\Phi}\right> = \left<u, 2x \Psi(x^2) + x(\theta_0 \Phi)(x^2)\right> = 0$, since $u$ is symmetric. Therefore, if $\Phi(0) \neq 0$ it is not possible to simplify (33)-(34), which means that the class of $u$ is $\tilde{s} = 2s + 3$. \

2) If $\Phi(0) = 0$, then it is possible to simplify by $x$. Then, $u$ fulfills (33) with
\[ \tilde{\Phi}(x) = \Phi(x^2), \quad \tilde{\Psi}(x) = x((\theta_0\Phi)(x^2) + 2\Psi(x^2)). \] (42)

Here, we have $\tilde{\Phi}'(0) + \tilde{\Psi}(0) = 0$ and $\langle u, \theta_0\Psi + \theta_0^2\tilde{\Phi} \rangle = \langle u, 2\Psi(x^2) + 2(\theta_0\Phi)(x^2) \rangle$.

Applying the operator $\sigma$ for the second equation and using (9), we obtain
\[ \langle u, \theta_0\Psi + \theta_0^2\tilde{\Phi} \rangle = -2\lambda\langle \nu, \theta_0\Psi + \theta_0^2\Phi \rangle + 2(\Phi'(0) + \Psi(0)) = X_1. \]

Therefore, if $X_1 \neq 0$ it is not possible to simplify, which means that the class of $u$ is $\tilde{s} = 2s + 2$.

3) If $\Phi(0) = 0$ and $X_1 = 0$, then it is possible to simplify (33)-(34) by $x^2$. Then, $u$ fulfills (33) with
\[ \tilde{\Phi}(x) = x(\theta_0\Phi)(x^2), \quad \tilde{\Psi}(x) = 2((\theta_0\Phi)(x^2) + \Psi(x^2)). \] (43)

Here, we have $\tilde{\Phi}'(0) + \tilde{\Psi}(0) = 3\Phi'(0) + 2\Psi(0) = X_2$ and
\[ \langle u, \theta_0\Psi + \theta_0^2\tilde{\Phi} \rangle = \langle u, x(2(\theta_0\Psi)(x^2) + (\theta_0^2\Phi)(x^2)) \rangle = 0, \] since $u$ is symmetric.

Therefore, if $X_2 \neq 0$ it is not possible to simplify, which means that the class of $u$ is $\tilde{s} = 2s + 1$.

4) If $\Phi(0) = 0$, $X_1 = 0$ and $X_2 = 0$, then it is possible to simplify (33)-(34) by $x^3$. Then, $u$ fulfills (33) with
\[ \tilde{\Phi}(x) = (\theta_0\Phi)(x^2), \quad \tilde{\Psi}(x) = x(3(\theta_0^2\Phi)(x^2) + 2(\theta_0\Psi)(x^2)). \] (44)

Under these conditions $x = 0$ can’t be a root of $(\theta_0\Phi)(x^2)$. Assuming the contrary, that $(\theta_0\Phi)(0) = \Phi'(0) = 0$, then from the conditions $\Phi(0) = 0$, $X_1 = 0$ and $X_2 = 0$ we obtain $\langle \nu, \theta_0\Psi + \theta_0^2\Phi \rangle = 0$ and $\Phi'(0) + \Psi(0) = 0$ which is a contradiction with (25). Then it is not possible to simplify, which means that the class of $u$ is $\tilde{s} = 2s$.

\[ \Box \]

**Proposition 7.** If $\nu$ is a semi-classical form and satisfies (27), then for every $\lambda \in \mathbb{C} - \{0\}$ such that $S_n(0, \lambda) \neq 0, n \geq 0$, the form $u$ defined by (8) is regular and semi-classical. It satisfies
\[ \tilde{\Phi}(z)S'(u)(z) = \tilde{C}_0(z)S(u)(z) + \tilde{D}_0(z), \] (45)

where
\[ \begin{cases} \tilde{\Phi}(z) = z\Phi(z^2), \\ \tilde{C}_0(z) = -\Phi(z^2) + 2z^2C_0(z^2), \\ \tilde{D}_0(z) = -2z\lambda D_0(z^2) + 2zC_0(z^2). \end{cases} \] (46)

**Proof.** From (26), we have
\[ S'(\nu)(z^2) = -\sum_{n \geq 0} \frac{(\nu)_n}{z^{2n+2}}. \]

Using (8), we get
\[ -\lambda S'(\nu)(z^2) = zS(u)(z) + 1, \] (47)
Deriving (47), we obtain
\[ -2z\lambda S'(v)(z^2) = zS'(u)(z) + S(u)(z). \tag{48} \]

Make a change of variable \( z \rightarrow z^2 \) in (27) and multiply by \(-2\lambda z\), we obtain (45)-(46) by taking into account (47)-(48).

We are going to establish the expression of structure relation coefficients \( \tilde{C}_n \) and \( \tilde{D}_n \), \( n \geq 0 \) of \( \{Z_n\}_{n \geq 0} \) in terms of those of the sequence \( \{S_n\}_{n \geq 0} \).

**Proposition 8.** The sequence \( \{Z_n\}_{n \geq 0} \) fulfills
\[ \tilde{\Phi}(x)Z_{n+1}'(x) = \frac{\tilde{C}_{n+1}(x) - \tilde{C}_0(x)}{2} Z_{n+1}(x) - \gamma_{n+1} \tilde{D}_{n+1}(x) Z_n(x), \quad n \geq 0 \tag{49} \]

with
\[
\begin{align*}
\tilde{C}_{2n+1}(x) &= 2x^2C_n(x^2) + \Phi(x^2) + 4\gamma_{2n+1}x^2D_n(x^2), \quad n \geq 0, \\
\tilde{D}_{2n+1}(x) &= 2x^3D_n(x^2), \quad n \geq 0, \\
\tilde{C}_{2n+2}(x) &= 2x^2C_n(x^2) - \Phi(x^2) + 4\gamma_{2n+2}x^2D_n(x^2), \quad n \geq 0, \\
\tilde{D}_{2n+2}(x) &= x(C_{n+1}(x^2) - C_n(x^2)) + 2x(\gamma_{2n+3}D_n(x^2) - 2\gamma_{2n+1}D_n(x^2)) + 2x^3D_n(x^2), \quad n \geq 0. 
\end{align*}
\tag{50} \tag{51} \tag{52} \tag{53} \tag{54} \tag{55}
\]

\( \tilde{C}_0(x) \) and \( \tilde{D}_0(x) \) are given by (46) and \( \gamma_{n+1} \) by (14)-(15).

**Proof.** Change \( x \rightarrow x^2 \) in (29) and multiply by \( 2x^3 \) we obtain by taking (11) and (13) into account,
\[ x\Phi(x^2)Z_{2n+3}'(x) = (x^2\left(C_{n+1}(x^2) - C_0(x^2)\right) + \Phi(x^2)) Z_{2n+3}(x) - 2x^2D_{n+1}(x^2)Z_{2n+1}(x). \]

Using (16) and (10) where \( n \rightarrow 2n \), the last equation becomes
\[ \tilde{\Phi}(x)Z_{2n+3}'(x) = (x^2\left(C_{n+1}(x^2) - C_0(x^2)\right) + \Phi(x^2)) Z_{2n+3}(x) - 2\gamma_{2n+3}x^2D_{n+1}(x^2)Z_{2n+2}(x). \]

From (49) and the above equation, we have
\[
\begin{align*}
\left\{\frac{\tilde{C}_{2n+3}(x) - \tilde{C}_0(x)}{2} - (x^2\left(C_{n+1}(x^2) - C_0(x^2)\right) + \Phi(x^2) + 2x^2\gamma_{2n+3}D_{n+1}(x^2))\right\} 	imes Z_{2n+3}(x) = \gamma_{2n+3} \left\{ \tilde{D}_{2n+3}(x) - 2x^2D_{n+1}(x^2) \right\} Z_{2n+2}(x).
\end{align*}
\]

\( Z_{2n+3} \) and \( Z_{2n+2} \) have no common roots, then \( Z_{2n+3} \) divides \( \tilde{D}_{2n+3}(x) - 2x^2D_{n+1}(x^2) \), which is a polynomial of degree at most equal to \( 2s + 3 \). Then we have necessarily
\[ \tilde{D}_{2n+3}(x) = 2x^2D_{n+1}(x^2) \text{ for } n > s, \text{ and also } \frac{\tilde{C}_{2n+3}(x) - \tilde{C}_0(x)}{2} = x^2\left(C_{n+1}(x^2) - C_0(x^2)\right) + \Phi(x^2) + 2x^2\gamma_{2n+3}D_{n+1}(x^2), \text{ for } n > s. \]

Then, by (45), we get (49) for \( n > s \).

By virtue of the recurrence relation (30) and (46), we can easily prove by induction.
that the system (50) is valid for $0 \leq n \leq s$. Hence (50) is valid for $n \geq 0$.

After a derivation of (10) where $n \rightarrow 2n + 1$ multiplying by $x\Phi(x^2)$ and using (49),
we obtain

$$\begin{align*}
x^2\Phi(x^2)Z_{2n+2}'(x) &= \frac{C_{2n+3}(x) - C_0(x)}{2}Z_{2n+3}(x) - \gamma_{2n+3}D_{2n+3}(x)Z_{2n+2}(x) - x\Phi(x^2)Z_{2n+2}(x) + \\
&\gamma_{2n+2} \left\{ \frac{\tilde{C}_{2n+1}(x) - \tilde{C}_0(x)}{2}Z_{2n+1}(x) - \gamma_{2n+1}\tilde{D}_{2n+1}(x)Z_{2n}(x) \right\}.
\end{align*}$$

Applying the recurrence relation (10), we get

$$\begin{align*}
x^2\Phi(x^2)Z_{2n+2}'(x) &= \left\{ x\tilde{C}_{2n+3}(x) - \tilde{C}_0(x) - \gamma_{2n+3}\tilde{D}_{2n+3}(x) - x\Phi(x^2) + \gamma_{2n+2}\tilde{D}_{2n+1}(x) \right\} \times \\
&\times Z_{2n+2}(x) - \gamma_{2n+2} \left\{ \frac{\tilde{C}_{2n+3}(x) - \tilde{C}_{2n+1}(x)}{2} + x\tilde{D}_{2n+1}(x) \right\} Z_{2n+1}.
\end{align*}$$

Now, using (49) and taking into account the fact that $Z_{2n+2}(x)$ and $Z_{2n+1}(x)$ are coprime, we get from the last equation after simplification by $x$ (51) for $n > s$.

Finally, by virtue of the recurrence relation (30) and (50) with $n = 0$, we can easily prove by induction that the system (51) is valid for $0 \leq n \leq s$. Hence (51) is also proved for $n \geq 0$.

Using (32), Proposition 8. and simplifying, we get the following result:

**Corollary 2.** Each polynomial of $\{Z_n\}_{n\geq0}$ satisfies a second order differential equation of Laguerre-type, (or holonomic second order differential equation)

$$J(x,n)Z_{2n+1}''(x) + K(x,n)Z_{2n+1}'(x) + L(x,n)Z_{2n+1}(x) = 0, \quad n \geq 1,$$

with

$$\begin{align*}
J(x,2n+1) &= \Phi(x^2) \{ x(C_{n+1}(x^2) - C_n(x^2)) + 2x(\gamma_{2n+3}D_{2n+1}(x^2) - \gamma_{2n+1}D_n(x^2)) + 2x^3D_n(x^2) \} \\
K(x,2n+1) &= 2x^2(\Phi(x^2) + C_0(x^2)) \left\{ C_{n+1}(x^2) - C_n(x^2) + 2(\gamma_{2n+3}D_{2n+1}(x^2) - \gamma_{2n+1}D_n(x^2)) + \\
&+ 2x^2D_n(x^2) + \Phi(x^2) \right\} \{ C'_{n+1}(x^2) - C'_n(x^2) + 2x^2(C_{n+1}(x^2) - C_n(x^2)) + 2x^2(\gamma_{2n+3}D'_{2n+1}(x^2) - \\
&- \gamma_{2n+1}D'_n(x^2)) + 2x^2D'_n(x^2) + 2x^2(C'_{n+1}(x^2) - C'_n(x^2)) + 4x^2(\gamma_{2n+3}D''_{2n+1}(x^2) - \\
&- \gamma_{2n+1}D''_n(x^2)) + 4x^2D''_n(x^2) \} - x \{ C'_{n+1}(x^2) - C'_n(x^2) + 2\gamma_{2n+3}D'_{2n+1}(x^2) - \\
&- 2\gamma_{2n+1}D'_n(x^2) + 2x^2D'_n(x^2) \} \{ 2x^2C'_{n+1}(x^2) + 4\gamma_{2n+3}D'_{2n+1}(x^2) - 2x^2C'_n(x^2) + \\
&+ C'_{n+1}(x^2) + C_{n+1}(x^2) + 2\gamma_{2n+3}D_n(x^2) - 2\lambda D_0(x^2) + 4x^2\sum_{k=0}^n D_k(x^2) + 2x^2D_n(x^2) \}.
\end{align*}$$

and

$$\begin{align*}
J(x,2n) &= 2x^3\Phi(x^2)D_n(x^2), \\
K(x,2n) &= 2x^2D_n(x^2)(2x^2\Phi'(x^2) + 2x^2C_n(x^2) - 3\Phi(x^2)) - 4x^4\Phi(x^2)D'_n(x^2), \\
L(x,2n) &= 2x^2D_n(x^2) \left\{ 3\Phi(x^2) - 2x^2C_0(x^2) - 4x^4\Phi(x^2) + 2x^4C'_n(x^2) + \\
&+ 2\lambda x^2D_0(x^2) - 4x^4\sum_{k=0}^n D_k(x^2) \right\} + 4x^3D'_n(x^2)(2x^2C_n(x^2) - x^2C_0(x^2) + \Phi(x^2)).
\end{align*}$$

3. Illustrative examples

(1) We study the problem (8), with $v = \mathcal{L}(\alpha)$ where $\mathcal{L}(\alpha)$ is the Laguerre form. In this case, the form $v$ is not symmetric. This form is classical (semi-classical of class
\( s = 0 \). We have [12]

\[
\xi_n = 2n + \alpha + 1, \quad \rho_{n+1} = (n+1)(n+\alpha+1), \quad n \geq 0, \tag{52}
\]

the regularity condition is \( \alpha \neq -n, \ n \geq 1 \)

\[
\Phi(x) = x, \quad \Psi(x) = x - \alpha - 1, \tag{53}
\]

\[
C_n(x) = -x + (2n + \alpha), \quad D_n(x) = -1, \quad n \geq 0. \tag{54}
\]

Using (5) and (52), we get

\[
S_n(0) = (-1)^n \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)}, \quad n \geq 0. \tag{55}
\]

From (6) and (52), we obtain by induction for \( n \geq 0 \)

\[
S_n(\lambda) = \left\{
\begin{array}{ll}
(-1)^{n+1} \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)}, & \alpha \neq 0, \\
(-1)^n \frac{\Gamma(n + 1)}{\Gamma(n + \alpha + 1)}, & \alpha = 0.
\end{array}
\right. \tag{56}
\]

By virtue of (7) and (55)-(56), we deduce

\[
S_n(0, \lambda) = (-1)^n \frac{\Gamma(n + \alpha + 1) d_{\alpha,n}}{\alpha \Gamma(\alpha + 1)}, \quad n \geq 0 \tag{57}
\]

where

\[
d_{\alpha,n} = \left\{
\begin{array}{ll}
\frac{(\alpha + \lambda) - \frac{\lambda \Gamma(n + 1)}{\Gamma(n + \alpha + 1)} + \Gamma(n + \alpha + 1)}{\Gamma(n + \alpha + 1)}, & \alpha \neq 0, \ n \geq 0, \\
1 + \lambda \sum_{k=0}^{n-1} \frac{1}{k + 1}, & \alpha = 0, \ n \geq 0.
\end{array}
\right. \tag{58}
\]

Then, \( u \) is regular for every \( \lambda \neq 0 \) such that

\[
\lambda \neq \left\{
\begin{array}{ll}
-\alpha + \frac{\lambda \Gamma(n + 1)}{\Gamma(n + \alpha + 1)}, & \alpha \neq 0, \ n \geq 0, \\
- \sum_{k=0}^{n-1} \frac{1}{k + 1} \right)^{-1}, & \alpha = 0, \ n \geq 1.
\right. \tag{59}
\]

(15) and (57) give

\[
a_n = \frac{(n + \alpha + 1) d_{\alpha,n+1}}{d_{\alpha,n}}, \quad n \geq 0. \tag{60}
\]

Then, with (14), we get

\[
\left\{
\begin{array}{ll}
\gamma_1 = -\lambda, \\
\gamma_{2n+3} = \frac{d_{\alpha,n}}{d_{\alpha,n+1}}, & n \geq 0, \\
\gamma_{2n+2} = \frac{(n + \alpha + 1) d_{\alpha,n+1}}{d_{\alpha,n}}, & n \geq 0.
\end{array}
\right. \tag{61}
\]
Taking into account that the form \( v \) is semi-classical and by virtue of Proposition 4., the form \( u \) is also semi-classical. It satisfies (33) and (45) with
\[
\Phi(x) = x^2, \quad \Psi(x) = 2x^3 - (2\alpha + 1)x,
\]
\[
\tilde{C}_0(x) = -2x^3 + (2\alpha - 1)x, \quad \tilde{D}_0(x) = -2x^2 + 2(\alpha + \lambda).
\] (62)
From (53), we have
\[
\Phi(0) = 0, X_1 = -2(\alpha + \lambda) \text{ and } X_2 = 1 - 2\alpha \text{ (we take } \lambda = -\alpha \text{ in calculation of } X_2).
\]
Now, it is enough to use Proposition 6. in order to obtain the following results:
1. If \( \lambda \neq -\alpha \) and verifies (59), then the class of \( u \) is \( \tilde{s} = 2 \).
2. If \( \lambda = -\alpha \) and \( 2\alpha \neq 1 \), then the class of \( u \) is \( \tilde{s} = 1 \).
3. If \( \lambda = -\alpha \) and \( 2\alpha = 1 \), then the class of \( u \) is \( \tilde{s} = 0 \).

Now, we are going to give the elements of the structure relation of the sequence \( \{Z_n\}_{n \geq 0} \).
Using (53), (54) and Proposition 8., we obtain after simplifying by \( x \)
\[
\begin{align*}
\tilde{C}_0(x) &= -2x^3 + (2\alpha - 1)x, \quad C_1(x) = -2x^3 + (2\alpha + 4\lambda + 1)x, \\
\tilde{C}_{2n+2}(x) &= -2x^3 - X_n, \quad \tilde{C}_{2n+3}(x) = -2x^3 + X_{n+1}, \\
\tilde{D}_0(x) &= -2x^2 + 2(\alpha + \lambda), \quad \tilde{D}_{2n+1}(x) = -2x^2, \\
\tilde{D}_{2n+2}(x) &= -2x^2 - \frac{2(\alpha^2 + \delta_{0,\alpha})(\alpha + \lambda)\Gamma(\alpha + 1)\Gamma(n + 1)}{\Gamma(n + \alpha + 2)\alpha_n\alpha_{n+1}}, n \geq 0.
\end{align*}
\] (63)
The form \( v \) has the following integral representation[5]
\[
\langle v, f \rangle = \frac{1}{\Gamma(\alpha + 1)} \int_{0}^{+\infty} x^\alpha e^{-x} f(x) dx, \quad \Re(\alpha) > -1, \quad f \in \mathcal{P}.
\] (64)
Then, using (22), we obtain the following integral representation of \( u \)
\[
\langle u, f \rangle = \left(1 + \frac{\lambda}{\alpha}\right)f(0) - \frac{\lambda}{\Gamma(\alpha + 1)} \int_{-\infty}^{+\infty} |x|^{2\alpha - 1} e^{-x^2} f(x) dx, \quad \Re(\alpha) > 0.
\] (65)

(2) Let us describe the case \( v := \mathcal{H} \) where \( \mathcal{H} \) denotes the Hermite form. In this case, the form \( v \) is symmetric. This form is classical (semi-classical of class \( s = 0 \)). Here [12]
\[
\xi_n = 0, \quad \rho_{n+1} = \frac{1}{2}(n + 1), \quad n \geq 0, \\
\Phi(x) = 1, \quad \Psi(x) = 2x,
\] (66)
\[
C_n(x) = -2x, \quad D_n(x) = -2, \quad n \geq 0.
\] (68)
In accordance with Corollary 1. and (66), $u$ is regular for every $\lambda \neq 0$ and we have

$$
\begin{align*}
\gamma_1 &= -\gamma_2 = \lambda \\
\gamma_{4n+3} &= -\gamma_{4n+4} = -\frac{1}{2^{2n+1} \Gamma(n+1) \Gamma(n+2)} \Gamma(2n+2), n \geq 0 \\
\gamma_{4n+5} &= -\gamma_{4n+6} = \lambda \frac{1}{2^{2n+1} \Gamma(n+1) \Gamma(n+2)} \Gamma(2n+2), n \geq 0
\end{align*}
$$

(69)

By virtue of Proposition 6. and Proposition 7., the form $u$ is semi-classical of class $\tilde{s} = 3$ for any $\lambda \neq 0$ and fulfils (33) and (45) with

$$
\tilde{\Phi}(x) = x, \quad \tilde{\Psi}(x) = 4x^3, \quad \tilde{C}_0(x) = -4x^4 - 1, \quad \tilde{D}_0(x) = -4x^3 + 4\lambda x.
$$

(70)

According to Proposition 8., (67) and (68), we have, for $n \geq 0$

$$
\begin{align*}
\tilde{C}_0(x) &= -4x^4 - 1 \\
\tilde{C}_{2n+1}(x) &= -4x^4 - 8\gamma_{2n+1}x^2 + 1 \\
\tilde{C}_{2n+2}(x) &= -4x^4 - 8\gamma_{2n+2}x^2 - 1 \\
\tilde{D}_0(x) &= -4x^3 + 4\lambda x \\
\tilde{D}_{2n+1}(x) &= -4x^3 \\
\tilde{D}_{2n+2}(x) &= -4x^3 + 4(\gamma_{2n+1} - \gamma_{2n+3})x.
\end{align*}
$$

(71)

The form $v$ has the following integral representation[5]

$$
\langle v, f \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} f(x) dx, \quad f \in \mathcal{P}.
$$

(72)

Therefore, for $\lambda \neq 0$ and $f \in \mathcal{P}$, (23) becomes

$$
\langle u, f \rangle = f(0) - \frac{\lambda}{\sqrt{\pi}} P \int_{-\infty}^{+\infty} \frac{e^{-x^4}}{|x|} (f(x) - f(ix)) dx.
$$

(73)

References


Symmetric semi-classical polynomials


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