BCC-algebras with pseudo-valuations

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Abstract. The notion of pseudo-valuations (valuations) on a BCC-algebra is introduced by using the Busneag’s model ([1–3]), and a pseudo-metric is induced by a pseudo-valuation on BCC-algebras. Conditions for a real-valued function to be an BCK-pseudo-valuation are provided. The fact that the binary operation in BCC-algebras is uniformly continuous is provided based on the notion of (pseudo) valuation.

1. Introduction

In 1966, Y. Imai and K. Iséki (cf. [8]) defined a class of algebras of type (2,0) called BCK-algebras which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra (cf. [8]). The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [12]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori (cf. [10]) introduced a notion of BCC-algebras, and W. A. Dudek (cf. [4, 5]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [7], W. A. Dudek and X. H. Zhang introduced a new notion of ideals in BCC-algebras and described connections between such ideals and congruences. Busneag [2] defined a pseudo-valuation on a Hilbert algebra, and proved that every pseudo-valuation induces a pseudo metric on a Hilbert algebra. Also, Buşneag [3] provided several theorems on extensions of pseudo-valuations. Buşneag [1] introduced the notions of pseudo-valuations (valuations) on residuated lattices, and proved some theorems of extension for these (using the model of Hilbert algebras ([3])).

In this paper, using the Buşneag’s model, we introduce the notion of (BCK, BCC, strong BCC)-pseudo-valuations (valuations) on BCC-algebras, and we induce a pseudo-metric by using a BCK-pseudo-valuation on BCC-algebras. We provide conditions for a real-valued function on a BCC-algebra $X$ to be a BCK-pseudo-pseudo-valuation on $X$. Based on the notion of (pseudo) valuation, we show that the binary operation $\cdot$ in BCC-algebras is uniformly continuous.

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2. Preliminaries

Recall that a BCC-algebra is an algebra \((X, *, 0)\) of type \((2,0)\) satisfying the following axioms:

\begin{enumerate}
  \item \((x * y) * (z * y)) * (x * z) = 0, \quad (C1)\)
  \item \(0 * x = 0, \quad (C2)\)
  \item \(x * 0 = x, \quad (C3)\)
  \item \(x * y = 0 \text{ and } y * x = 0 \implies x = y \) for every \(x, y, z \in X.\) For any BCC-algebra \(X\), the relation \(\leq\) defined by \(x \leq y\) if and only if \(x * y = 0\) is a partial order on \(X.\) In a BCC-algebra \(X,\) the following holds:
  \begin{enumerate}
    \item \((\forall x \in X) (x * x = 0),\) \(\quad (a1)\)
    \item \((\forall x, y \in X) (x * y \leq x),\) \(\quad (a2)\)
    \item \((\forall x, y, z \in X) (x * y \leq z * y, z * y \leq z * x).\) \(\quad (a3)\)
  \end{enumerate}

A subset \(I\) of a BCC-algebra \(X\) is called a BCK-ideal if it satisfies:

\begin{enumerate}
  \item \(0 \in I, \quad (i)\)
  \item \((\forall x \in X) (\forall y \in I) (x * y \in I \implies x \in I), \quad (ii)\)
\end{enumerate}

A subset \(I\) of a BCC-algebra \(X\) is called a BCC-ideal if it satisfies:

\begin{enumerate}
  \item \(0 \in I, \quad (i)\)
  \item \((\forall x, z \in X) (\forall y \in I) ((x * y) * z \in I \implies x * z \in I). \quad (ii)\)
\end{enumerate}

3. Pseudo-valuations on BCC-algebras

**Definition 3.1.** A real-valued function \(\varphi\) on a BCC-algebra \(X\) is called a weak pseudo-valuation on \(X\) if it satisfies the following condition:

\[
(\forall x, y \in X) (\varphi(x * y) \leq \varphi(x) + \varphi(y)).
\]

**Definition 3.2.** A real-valued function \(\varphi\) on a BCC-algebra \(X\) is called a BCK-pseudo-valuation on \(X\) if it satisfies the following condition:

\[
\begin{align*}
\varphi(0) &= 0, \\
(\forall x, y \in X) (\varphi(x * y) \geq \varphi(x) - \varphi(y)).
\end{align*}
\]

**Example 3.3.** Let \(X := \{0, 1, 2, 3, 4\}\) be a BCC-algebra ([7]), which is not a BCK-algebra, with \(*\)-operation given by Table 1. Let \(\varphi\) be a real-valued function on \(X\) defined by

\[
\varphi = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 3 & 4 & 5
\end{pmatrix}.
\]

It is easy to check that \(\varphi\) is both a weak pseudo-valuation and a BCK-pseudo-valuation on \(X.\)

**Proposition 3.4.** For a weak pseudo-valuation \(\varphi\) on a BCC-algebra \(X,\) we have

\[
(\forall x \in X) (\varphi(x) \geq 0).
\]

**Proof.** For any \(x \in X,\) we have \(\varphi(0) = \varphi(0 * x) \leq \varphi(0) + \varphi(x),\) and so \(\varphi(x) \geq 0.\)
Table 1:  $*$-operation

<table>
<thead>
<tr>
<th>$*$</th>
<th>0</th>
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</table>

Theorem 3.5. Let $S$ be a subalgebra of a BCC-algebra $X$. For any real numbers $t_1$ and $t_2$ with $0 \leq t_1 < t_2$, let $\varphi_S$ be a real-valued function on $X$ defined by

$$\varphi_S(x) = \begin{cases} t_1 & \text{if } x \in S, \\ t_2 & \text{if } x \not\in S \end{cases}$$

for all $x \in X$. Then $\varphi_S$ is a weak pseudo-valuation on $X$.

Proof. Straightforward. $\square$

Corollary 3.6. Let $X$ be a BCC-algebra. For any $a \in X$, let $\varphi_a$ be a real-valued function on $X$ defined by

$$\varphi_a(x) = \begin{cases} t_1 & \text{if } x \in A(a), \\ t_2 & \text{if } x \not\in A(a) \end{cases}$$

for all $x \in X$ where $t_1$ and $t_2$ are real numbers with $t_2 > t_1 \geq 0$ and $A(a)$ is the initial section of $X$ determined by $a$. Then $\varphi_a$ is a weak pseudo-valuation on $X$.

Theorem 3.7. In a BCC-algebra, every BCK-pseudo-valuation is a weak pseudo-valuation.

Proof. Let $\varphi$ be a BCK-pseudo valuation on a BCC-algebra $X$. Using (a2) and (C2), we have $(x \ast y) \ast x = 0 \ast y = 0$ for all $x, y \in X$. Hence

$$0 = \varphi(0) = \varphi((x \ast y) \ast x) \ast y$$

$$\geq \varphi((x \ast y) \ast x) - \varphi(y)$$

$$\geq \varphi(x \ast y) - \varphi(x) - \varphi(y),$$

and so $\varphi(x \ast y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in X$. Therefore $\varphi$ is a weak pseudo-valuation on $X$. $\square$

The following example shows that the converse of Theorem 3.7 is not true.

Example 3.8. Consider the BCC-algebra $X$ which is given in Example 3.3. Let $\theta$ be a real-valued function on $X$ defined by

$$\theta = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 5 \end{pmatrix}.$$ 

It is easy to show that $\theta$ is a weak pseudo-valuation, but not a BCK-pseudo-valuation on $X$ since

$$\theta(3) = 4 \not\leq 3 + 1 = \theta(1) + \theta(2) = \theta(3 \ast 2) + \theta(2).$$
Definition 3.9. A real-valued function $\phi$ on a BCC-algebra $X$ is called a BCC-pseudo-valuation on $X$ if it satisfies (2) and

\[(\forall x, y, z \in X) \ (\phi((x \ast y) \ast z) \geq \phi(x \ast z) - \phi(y)).\]  
\hspace{1cm} (5)

Example 3.10. Consider the set $N_0 = \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the set of natural numbers. Define a binary operation $\ast$ on $N_0$ by

\[(\forall x, y \in N_0) \ (x \ast y := \begin{cases} 0 & \text{if } x \leq y, \\ x - y & \text{if } x > y \end{cases}).\]

Then $(N_0; \ast, 0)$ is a BCK-algebra with the unique small atom $1$, and so it is a BCC-algebra. Define

$$\phi : N_0 \to \mathbb{R}, \ x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ 2x + 1 & \text{otherwise.} \end{cases}$$

It is routine to verify that $\phi$ is a BCC-pseudo-valuation on $N_0$.

Putting $z = 0$ in (5) and using (C3), we get $\phi(x \ast y) \geq \phi(x) - \phi(y)$ for all $x, y \in X$. Thus we know that every BCC-pseudo-valuation is a BCK-pseudo-valuation. We will state this as a theorem.

Theorem 3.11. In a BCC-algebra, every BCC-pseudo-valuation is a BCK-pseudo-valuation.

The converse of Theorem 3.11 is not true as seen in the following example.

Example 3.12. Consider the BCC-algebra $X$ which is given in Example 3.3. Let $\phi$ be as in Example 3.3. Then $\phi$ is a BCK-pseudo-valuation, but not a BCC-pseudo-valuation on $X$ since

$$\phi((4 \ast 1) \ast 2) = \phi(1) = 1 \nleq 4 = \phi(4 \ast 2) - \phi(1).$$

Theorem 3.13. In a BCK-algebra, every BCK-pseudo-valuation is a BCC-pseudo-valuation.

Proof. Let $\phi$ be a BCK-pseudo-valuation on a BCK-algebra $X$ and let $x, y, z \in X$. Then

$$\phi(x \ast z) \leq \phi((x \ast z) \ast y) + \phi(y) = \phi((x \ast y) \ast z) + \phi(y)$$

and so $\phi$ is a BCC-pseudo-valuation on $X$. \hfill \Box

Lemma 3.14. Let $\phi$ be a BCC-pseudo-valuation on a BCC-algebra $X$. If $x \leq y$ then $\phi(x) \leq \phi(y)$ for all $x, y \in X$.

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x \ast y = 0$, and so

$$\phi(x) = \phi(x \ast 0) \leq \phi((x \ast y) \ast 0) + \phi(y)$$

$$= \phi(x \ast y) + \phi(y) = \phi(0) + \phi(y) = \phi(y).$$

This completes the proof. \hfill \Box

Lemma 3.15. Every BCC-pseudo-valuation on a BCC-algebra $X$ is a weak pseudo-valuation on $X$.

Proof. It is clear. \hfill \Box

Corollary 3.16. Every BCC-pseudo-valuation on a BCC-algebra $X$ satisfies the following assertions: for all $x, y, z \in X$,

(a) $\phi(x \ast y) \leq \phi(x)$,
(b) $\phi(x \ast (y \ast z)) \leq \phi(x) + \phi(y) + \phi(z)$,
(c) $\phi((x \ast y) \ast (z \ast y)) \leq \phi(x \ast z)$,

The following example shows that the converse of Lemma 3.15 is not true.

Example 3.17. Consider the BCC-algebra \( X \) which is given in Example 3.3. Let \( \varphi \) be a real-valued function on \( X \) defined by

\[
\varphi = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
1 & 1 & 2 & 3 & 3
\end{pmatrix}.
\]

It is easy to check that \( \varphi \) is a weak pseudo-valuation, but not a BCK-pseudo-valuation since \( \varphi(0) \neq 0 \). Also it is not a BCC-pseudo-valuation since

\[
\varphi((4 \ast 1) \ast 2) \not\geq \varphi(4 \ast 2) - \varphi(1).
\]

Proposition 3.18. Every BCC-pseudo-valuation on a BCC-algebra \( X \) satisfies the following implication:

\[
(\forall x, y, z, a \in X) ((x \ast y) \ast z \leq a \Rightarrow \varphi(x \ast z) \leq \varphi(y) + \varphi(a)). \tag{6}
\]

Proof. Let \( x, y, z, a \in X \) be such that \( (x \ast y) \ast z \leq a \). It follows from Lemma 3.14 that \( \varphi((x \ast y) \ast z) \leq \varphi(a) \) so from (5) that

\[
\varphi(x \ast z) \leq \varphi((x \ast y) \ast z) + \varphi(y) \leq \varphi(a) + \varphi(y).
\]

This completes the proof. \( \square \)

We provide a condition for a real-valued function \( \varphi \) on a BCC-algebra \( X \) to be a BCC-pseudo-valuation on \( X \).

Theorem 3.19. Let \( \varphi \) be a real-valued function on a BCC-algebra \( X \). If \( \varphi \) satisfies conditions (2) and (6), then \( \varphi \) is a BCC-pseudo-valuation on \( X \).

Proof. Assume that \( \varphi \) satisfies conditions (2) and (6). We note that \( (x \ast y) \ast z \leq (x \ast y) \ast z \) for all \( x, y, z \in X \), and so \( \varphi(x \ast z) \leq \varphi((x \ast y) \ast z) + \varphi(y) \). Therefore \( \varphi \) is a BCC-pseudo-valuation on \( X \). \( \square \)

Definition 3.20. A real-valued function \( \varphi \) on a BCC-algebra \( X \) is called a strong BCC-pseudo-valuation on \( X \) if it satisfies (2) and

\[
(\forall x, y, z \in X) (\varphi((x \ast y) \ast z) \geq \varphi(x) - \varphi(y)). \tag{7}
\]

Lemma 3.21. Every strong BCC-pseudo-valuation \( \varphi \) on a BCC-algebra \( X \) is order preserving.

Proof. Let \( x, y \in X \) be such that \( x \leq y \). Then \( x \ast y = 0 \), and so

\[
\varphi(x) \leq \varphi((x \ast y) \ast 0) + \varphi(y) = \varphi(0 \ast 0) + \varphi(y) = \varphi(0) + \varphi(y) = \varphi(y)
\]

by (7), (2) and (a1). Hence \( \varphi \) is order preserving. \( \square \)

Theorem 3.22. Every strong BCC-pseudo-valuation \( \varphi \) on a BCC-algebra \( X \) is a BCC-pseudo-valuation on \( X \).

Proof. By (a2) and Lemma 3.21, we have \( \varphi((x \ast y) \ast z) \leq \varphi(x) \) for all \( x, z \in X \). It follows from (7) that

\[
\varphi((x \ast y) \ast z) \geq \varphi(x) - \varphi(y) \geq \varphi(x \ast z) - \varphi(y). \tag{8}
\]

Hence \( \varphi \) is a BCC-pseudo-valuation on \( X \). \( \square \)

The following example shows that the converse of Theorem 3.22 may not be true.
Example 3.23. Let \( X := \{0, 1, 2, 3, 4, 5\} \) be a BCC-algebra ([7]), which is not a BCK-algebra, with \( \ast \)-operation given by Table 2. Let \( \varphi \) be a real-valued function on \( X \) defined by

\[
\varphi = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 1 & 1 & 1 & 7
\end{pmatrix}.
\]

It is easy to check that \( \varphi \) is a BCC-pseudo-valuation on \( X \), but not a strong BCC-pseudo-valuation on \( X \), since \( \varphi((1 \ast 0) \ast 1) = 0 \neq 1 - 0 = \varphi(1) - \varphi(0) \).

Definition 3.24. ([6]) A non-zero element \( a \) of a BCC-algebra \( X \) is called an atom of \( X \) if for any \( x \in X \), \( x \leq a \) implies \( x = 0 \) or \( x = a \).

Lemma 3.25. ([6]) Let \( a \) and \( b \) be atoms of a BCC-algebra \( X \). If \( a \neq b \), then \( a \ast b = a \).

We provide a condition for a BCC-pseudo-valuation to be a strong BCC-pseudo-valuation.

Theorem 3.26. In a BCC-algebra containing only atoms, every BCC-pseudo-valuation is a strong BCC-pseudo-valuation.

Proof. Let \( X \) be a BCC-algebra containing only atoms and let \( \varphi \) be a BCC-pseudo-valuation on \( X \). Using Lemma 3.25 and (5), we have

\[
\varphi(x) = \varphi(x \ast z) \leq \varphi((x \ast y) \ast z) + \varphi(y)
\]

for all \( x, y, z \in X \). Hence \( \varphi \) is a strong BCC-pseudo-valuation on \( X \).

Proposition 3.27. For any BCK-pseudo-valuation \( \varphi \) on a BCC-algebra \( X \), we have the following assertions:

(a) \( \varphi \) is order preserving,

(b) \( (\forall x, y \in X)(\varphi(x \ast y) + \varphi(y \ast x) \geq 0) \),

(c) \( (\forall x, y, z \in X)(\varphi(x \ast y) \leq \varphi(x \ast z) + \varphi(z \ast y)) \).

Proof. (a) Let \( x, y \in X \) be such that \( x \leq y \). Then \( x \ast y = 0 \), and so \( \varphi(x) \leq \varphi(x \ast y) + \varphi(y) = \varphi(0) + \varphi(y) = \varphi(y) \).

(b) Let \( x, y \in X \). Using (3), we have \( \varphi(x \ast y) \geq \varphi(x) - \varphi(y) \) and \( \varphi(y \ast x) \geq \varphi(y) - \varphi(x) \). It follows that \( \varphi(x \ast y) + \varphi(y \ast x) \geq 0 \).

(c) Let \( x, y, z \in X \). Since \( \varphi \) is order preserving, it follows from (C1) and (3) that

\[
\varphi(x \ast z) \geq \varphi((x \ast y) \ast (z \ast y)) \geq \varphi(x \ast y) - \varphi(z \ast y).
\]

Hence (c) is valid.
Corollary 3.28. Every BCC-pseudo-valuation \( \phi \) on a BCC-algebra \( X \) satisfies conditions (a), (b) and (c) in Proposition 3.27.

Theorem 3.29. If a real-valued function \( \phi \) on a BCC-algebra \( X \) satisfies the condition (2) and

\[
(\forall x, y, z \in X)(\phi((x \ast y) \ast y) \ast z) \geq \phi(x \ast y) - \phi(z)
\]

then \( \phi \) is a BCK-pseudo-valuation on \( X \).

Proof. Taking \( y = 0 \) in (9) and using (C3), we have

\[
\phi(x \ast z) = \phi(((x \ast 0) \ast 0) \ast z) \geq \phi(x \ast 0) - \phi(z) = \phi(x) - \phi(z).
\]

Hence \( \phi \) is a BCK-pseudo-valuation on \( X \). \( \square \)

Corollary 3.30. Let \( \phi \) be a real-valued function on a BCK-algebra \( X \). If \( \phi \) satisfies conditions (2) and (9), then \( \phi \) is a BCC-pseudo-valuation on \( X \).

By a pseudo-metric space we mean an ordered pair \((M, d)\), where \( M \) is a non-empty set and \( d : M \times M \to \mathbb{R} \) is a positive function satisfying the following properties: \( d(x, x) = 0, d(x, y) = d(y, x) \) and \( d(x, z) \leq d(x, y) + d(y, z) \) for every \( x, y, z \in M \). If in the pseudo-metric space \((M, d)\) the implication \( d(x, y) = 0 \Rightarrow x = y \) holds, then \((M, d)\) is called a metric space. For a real-valued function \( \phi \) on a BCC-algebra \( X \), define a mapping \( d_{\phi} : X \times X \to \mathbb{R} \) by \( d_{\phi}(x, y) = \phi(x \ast y) + \phi(y \ast x) \) for all \((x, y) \in X \times X \).

Theorem 3.31. If a real-valued function \( \phi \) on a BCC-algebra \( X \) is a BCK-pseudo-valuation on \( X \), then \((X, d_{\phi})\) is a pseudo-metric space.

We say \( d_{\phi} \) is the pseudo-metric induced by a BCK-pseudo-valuation \( \phi \) on a BCC-algebra \( X \).

Proof. Obviously, \( d_{\phi}(x, y) \geq 0, d_{\phi}(x, x) = 0 \) and \( d_{\phi}(x, y) = d_{\phi}(y, x) \) for all \( x, y \in X \). Let \( x, y, z \in X \). Using Proposition 3.27(c), we have

\[
d_{\phi}(x, y) + d_{\phi}(y, z) = [\phi(x \ast y) + \phi(y \ast x)] + [\phi(y \ast z) + \phi(z \ast y)]
\]

\[
= [\phi(x \ast y) + \phi(y \ast z)] + [\phi(z \ast y) + \phi(y \ast x)]
\]

\[
\geq \phi(x \ast z) + \phi(z \ast x) = d_{\phi}(x, z).
\]

Therefore \((X, d_{\phi})\) is a pseudo-metric space. \( \square \)

The following example illustrates Theorem 3.31.

Example 3.32. Consider the BCC-pseudo-valuation \( \phi \) on \( \mathbb{N}_0 \) which is described in Example 3.10. Using Theorem 3.11, we know that \( \phi \) is a BCK-pseudo-valuation on \( \mathbb{N}_0 \). The pseudo-metric \( d_{\phi} \) induced by \( \phi \) is given as follows:

\[
d_{\phi}(x, y) = \begin{cases} 
0 & \text{if } x = y, \\
2y + 1 & \text{if } x = 0 \text{ and } y \neq 0, \\
2x + 1 & \text{if } x \neq 0 \text{ and } y = 0, \\
2(y \ast x) + 1 & \text{if } \begin{cases} 
x \ast y = 0 \\
y \ast x = 0
\end{cases} \text{ for } 0 \neq x \neq y \\
2(x \ast y) + 1 & \text{if } \begin{cases} 
x \ast y \neq 0 \\
y \ast x = 0 \text{ or } y \neq 0
\end{cases} \text{ for } 0 \neq x \neq y, \\
2(x \ast y) + 2(y \ast x) + 2 & \text{if } \begin{cases} 
x \ast y = 0 \\
y \ast x = 0 \text{ or } y \neq 0
\end{cases} \text{ for } 0 \neq x \neq y,
\end{cases}
\]

and \((\mathbb{N}_0, d_{\phi})\) is a pseudo-metric space.
Proposition 3.33. Let $\varphi$ be a BCK-pseudo-valuation on a BCC-algebra $X$. Then every pseudo-metric $d_\varphi$ induced by $\varphi$ satisfies the following inequalities:

(a) $d_\varphi(x, y) \geq \max\{d_\varphi(x \ast a, y \ast a), d_\varphi(a \ast x, a \ast y)\}$,

(b) $d_\varphi(x \ast y, a \ast b) \leq d_\varphi(x \ast y, a \ast y) + d_\varphi(a \ast y, a \ast b)$

for all $x, y, a, b \in X$.

Proof. (a) Let $x, y, a \in X$. Since

$$(y \ast a) \ast (x \ast a) \ast (y \ast x) = 0 \text{ and } ((x \ast a) \ast (y \ast a)) \ast (x \ast y) = 0,$$

it follows from Proposition 3.27(a) that $\varphi(y \ast x) \geq \varphi((y \ast a) \ast (x \ast a))$ and $\varphi(x \ast y) \geq \varphi((x \ast a) \ast (y \ast a))$ so that

$$d_\varphi(x, y) = \varphi(x \ast y) + \varphi(y \ast x)$$
$$\geq \varphi((x \ast a) \ast (y \ast a)) + \varphi((y \ast a) \ast (x \ast a))$$
$$= d_\varphi(x \ast a, y \ast a).$$

Similarly, we have $d_\varphi(x, y) \geq d_\varphi(a \ast x, a \ast y)$. Hence (a) is valid.

(b) Using Proposition 3.27(c), we have

$$\varphi((x \ast y) \ast (a \ast b)) \leq \varphi((x \ast y) \ast (a \ast y)) + \varphi((a \ast y) \ast (a \ast b)),$$

$$\varphi((a \ast b) \ast (x \ast y)) \leq \varphi((a \ast b) \ast (a \ast y)) + \varphi((a \ast y) \ast (x \ast y))$$

for all $x, y, a, b \in X$. Hence

$$d_\varphi(x \ast y, a \ast b) = \varphi((x \ast y) \ast (a \ast b)) + \varphi((a \ast b) \ast (x \ast y))$$
$$\leq \varphi((x \ast y) \ast (a \ast y)) + \varphi((a \ast y) \ast (a \ast b))$$
$$+ \varphi((a \ast b) \ast (a \ast y)) + \varphi((a \ast y) \ast (x \ast y))$$
$$+ \varphi((a \ast b) \ast (a \ast y)) + \varphi((a \ast y) \ast (a \ast b))$$
$$= d_\varphi(x \ast y, a \ast y) + d_\varphi(a \ast y, a \ast b)$$

for all $x, y, a, b \in X.$

Theorem 3.34. For a real-valued function $\varphi$ on a BCC-algebra $X$, if $d_\varphi$ is a pseudo-metric on $X$, then $(X \times X, d_\varphi^*)$ is a pseudo-metric space, where

$$d_\varphi^*((x, y), (a, b)) = \max\{d_\varphi(x, a), d_\varphi(y, b)\}$$

for all $(x, y), (a, b) \in X \times X$.

Proof. Suppose $d_\varphi$ is a pseudo-metric on $X$. For any $(x, y), (a, b) \in X \times X$, we have

$$d_\varphi^*((x, y), (x, y)) = \max\{d_\varphi(x, x), d_\varphi(y, y)\} = 0$$

and

$$d_\varphi^*((x, y), (a, b)) = \max\{d_\varphi(x, a), d_\varphi(y, b)\}$$
$$= \max\{d_\varphi(a, x), d_\varphi(b, y)\}$$
$$= d_\varphi^*((a, b), (x, y)).$$
Now let \((x, y), (a, b), (u, v) \in X \times X\). Then
\[
d'_{\varphi}(x, y, (u, v)) + d'_{\varphi}((u, v), (a, b))
= \max[d_{\varphi}(x, u), d_{\varphi}(y, v)] + \max[d_{\varphi}(u, a), d_{\varphi}(v, b)]
\geq \max[d_{\varphi}(x, u) + d_{\varphi}(u, a), d_{\varphi}(y, v) + d_{\varphi}(v, b)]
\geq \max[d_{\varphi}(x, a), d_{\varphi}(y, b)]
= d'_{\varphi}((x, y), (a, b)).
\]
Therefore \((X \times X, d'_{\varphi})\) is a pseudo-metric space. □

**Corollary 3.35.** If \(\varphi : X \to \mathbb{R}\) is a BCK-pseudo-valuation on a BCC-algebra \(X\), then \((X \times X, d'_{\varphi})\) is a pseudo-metric space.

A BCK/BCC-pseudo-valuation \(\varphi\) on a BCC-algebra \(X\) satisfying the following condition:
\[
(\forall x \in X) \ (x \neq 0 \Rightarrow \varphi(x) \neq 0)
\]
(11)
is called a BCK/BCC-valuation on \(X\).

**Theorem 3.36.** If \(\varphi : X \to \mathbb{R}\) is a BCK-valuation on a BCC-algebra \(X\), then \((X, d_{\varphi})\) is a metric space.

**Proof.** Suppose \(\varphi\) is a BCK-valuation on a BCC-algebra \(X\). Then \((X, d_{\varphi})\) is a pseudo-metric space by Theorem 3.31. Let \(x, y \in X\) be such that \(d_{\varphi}(x, y) = 0\). Then \(0 = d_{\varphi}(x, y) = \varphi(x \ast y) + \varphi(y \ast x)\), and so \(\varphi(x \ast y) = 0\) and \(\varphi(y \ast x) = 0\). It follows from (11) that \(x \ast y = 0\) and \(y \ast x = 0\) so from (C4) that \(x = y\). Therefore \((X, d_{\varphi})\) is a metric space. □

**Theorem 3.37.** If \(\varphi : X \to \mathbb{R}\) is a BCK-valuation on a BCC-algebra \(X\), then \((X \times X, d'_{\varphi})\) is a metric space.

**Proof.** Note from Corollary 3.35 that \((X \times X, d'_{\varphi})\) is a pseudo-metric space. Let \((x, y), (a, b) \in X \times X\) be such that \(d'_{\varphi}((x, y), (a, b)) = 0\). Then
\[
0 = d'_{\varphi}((x, y), (a, b)) = \max[d_{\varphi}(x, a), d_{\varphi}(y, b)],
\]
and so \(d_{\varphi}(x, a) = 0 = d_{\varphi}(y, b)\) since \(d_{\varphi}(x, y) \geq 0\) for all \((x, y) \in X \times X\). Hence
\[
0 = d_{\varphi}(x, a) = \varphi(x \ast a) + \varphi(a \ast x)
\]
and
\[
0 = d_{\varphi}(y, b) = \varphi(y \ast b) + \varphi(b \ast y).
\]
It follows that \(\varphi(x \ast a) = 0 = \varphi(a \ast x)\) and \(\varphi(y \ast b) = 0 = \varphi(b \ast y)\) so that \(x \ast a = 0 = a \ast x\) and \(y \ast b = 0 = b \ast y\). Using (C4), we have \(a = x\) and \(b = y\), and so \((x, y) = (a, b)\). Therefore \((X \times X, d'_{\varphi})\) is a metric space. □

**Theorem 3.38.** If \(\varphi : X \to \mathbb{R}\) is a BCK-valuation on a BCC-algebra \(X\), then the operation \(\ast\) in the BCC-algebra \(X\) is uniformly continuous.

**Proof.** For any \(\varepsilon > 0\), if \(d'_{\varphi}((x, y), (a, b)) < \frac{\varepsilon}{2}\), then \(d_{\varphi}(x, a) < \frac{\varepsilon}{2}\) and \(d_{\varphi}(y, b) < \frac{\varepsilon}{2}\). Using Proposition 3.33, we have
\[
d_{\varphi}(x \ast y, a \ast b) \leq d_{\varphi}((x, y), (a \ast y, a \ast b))
\leq d_{\varphi}(x, a) + d_{\varphi}(y, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Therefore the operation \(\ast : X \times X \to X\) is uniformly continuous. □
Table 3: *-operation

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
</tbody>
</table>

The following example illustrates Theorem 3.38.

**Example 3.39.** Let $X = \{0, a, b, c\}$ be a set with the *-operation given by Table 3. Then $(X, *, 0)$ is a proper BCC-algebra. Let $\varphi$ be a real-valued function on $X$ defined by

$$\varphi = \begin{pmatrix} 0 & a & b & c \\ 0 & 3 & 4 & 5 \end{pmatrix}.$$ 

Then $\varphi$ is a BCK-valuation on $X$ and $(X, d_{\varphi})$ is a metric space where

$$d_{\varphi} = \begin{pmatrix} (0,0) & (0,a) & (0,b) & (a,a) & (a,b) & (a,c) & (b,b) & (b,c) & (c,c) \\ 0 & 3 & 4 & 5 & 0 & 3 & 4 & 0 & 4 \end{pmatrix}.$$ 

Also, $(X \times X, d'_{\varphi})$ is a metric space where $d'_{\varphi}$ is obtained by (10), for example, 

$$d'_{\varphi}((0, b), (a, c)) = \max\{d_{\varphi}(0, a), d_{\varphi}(b, c)\} = \max\{3, 4\} = 4,$$

$$d'_{\varphi}((a, b), (c, a)) = \max\{d_{\varphi}(a, c), d_{\varphi}(b, a)\} = \max\{4, 3\} = 4,$$

$$d'_{\varphi}((c, a), (b, 0)) = \max\{d_{\varphi}(c, 0), d_{\varphi}(a, 0)\} = \max\{5, 3\} = 5,$$

$$d'_{\varphi}((a, c), (b, 0)) = \max\{d_{\varphi}(a, b), d_{\varphi}(c, 0)\} = \max\{3, 5\} = 5,$$

$$d'_{\varphi}((a, c), (b, c)) = \max\{d_{\varphi}(a, b), d_{\varphi}(c, c)\} = \max\{3, 0\} = 3,$$

and so on. Now, it is routine to verify that the operation * in the BCC-algebra $X$

$$*: X \times X \to X, (x, y) \mapsto x * y$$

is uniformly continuous.

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**References**