On the torsion graph and von Neumann regular rings

P. Malakooti Rad\textsuperscript{a}, Sh. Ghalandarzadeh\textsuperscript{b}, S. Shirinkam\textsuperscript{c}

\textsuperscript{a} Department of Mathematics, K. N. Toosi University of Technology P. O. Box 16315 – 1618, Tehran, Iran.
\textsuperscript{b} Department of Mathematics, Faculty of Science, K. N. Toosi University of Technology P. O. Box 16315 – 1618, Tehran, Iran.
\textsuperscript{c} Faculty of Electronic and Computer and IT, Islamic Azad University, Qazvin Branch, Qazvin, Iran

Abstract. Let $R$ be a commutative ring with identity and $M$ be a unitary $R$-module. A torsion graph of $M$, denoted by $\Gamma(M)$, is a graph whose vertices are the non-zero torsion elements of $M$, and two distinct vertices $x$ and $y$ are adjacent if and only if $[x : M][y : M]M = 0$. In this paper, we investigate the relationship between the diameters of $\Gamma(M)$ and $\Gamma(R)$, and give some properties of minimal prime submodules of a multiplication $R$-module $M$ over a von Neumann regular ring. In particular, we show that for a multiplication $R$-module $M$ over a Bézout ring $R$ the diameter of $\Gamma(M)$ and $\Gamma(R)$ is equal, where $M \neq T(M)$. Also, we prove that, for a faithful multiplication $R$-module $M$ with $|M| \neq 4$, $\Gamma(M)$ is a complete graph if and only if $\Gamma(R)$ is a complete graph.

1. Introduction

In 1999 Anderson and Livingston [1], introduced and studied the zero-divisor graph of a commutative ring with identity whose vertices are nonzero zero-divisors while $x - y$ is an edge whenever $xy = 0$. Since then, the concept of zero-divisor graphs has been studied extensively by many authors including Badawi and Anderson [7], Anderson, Levy and Shapiro [2] and Mulay [17]. This concept has also been introduced and studied for near-rings, semigroups, and non-commutative rings by Cannon, Neuerburg and Redmond [9], DeMeyer, McKenzie and Schneider [10] and Redmond [18]. For recent developments on graphs of commutative rings see Anderson and Badawi [4], and Anderson, Axtell and Stickles [5].

In 2009, the concept of the zero-divisor graph for a ring has been extended to module theory by Ghalandarzadeh and Malakooti Rad [12]. They defined the torsion graph of an $R$-module $M$ whose vertices are the nonzero torsion elements of $M$ such that two distinct vertices $x$ and $y$ are adjacent if and only if $[x : M][y : M]M = 0$. For a multiplication $R$-module $M$, they proved that, $\Gamma(M)$ and $\Gamma(S^{-1}M)$ are isomorphic, where $S = R \setminus \mathbb{Z}(M)$. Also, they showed that, $\Gamma(M)$ is connected and $\text{diam}(\Gamma(M)) \leq 3$ for a faithful $R$-module $M$, see [13].

Let $R$ be a commutative ring with identity and $M$ be a unitary multiplication $R$-module. In this paper, we will investigate the concept of a torsion graph and minimal prime submodules of an $R$-module. Also, we study the relationship among the diameters of $\Gamma(M)$ and $\Gamma(R)$, and minimal prime submodules of a multiplication $R$-module $M$ over a von Neumann regular ring. In particular, we show that for a multiplication $R$-module $M$ over a Bézout ring $R$ the diameter of $\Gamma(M)$ and $\Gamma(R)$ is equal, where $M \neq T(M)$.

2010 Mathematics Subject Classification. Primary 05C99; Secondary 13C12, 13C13

Keywords. Torsion graphs, von Neumann regular rings, multiplication modules

Received: 2 July 2011; Accepted: 12 August 2011
Communicated by Miroslav Ciric

Email addresses: pmalakooti@dena.kntu.ac.ir (P. Malakooti Rad), ghalandarzadeh@kntu.ac.ir (Sh. Ghalandarzadeh), sshirinkam@dena.kntu.ac.ir (S. Shirinkam)
Also, we prove that, if $\Gamma(M)$ is a complete graph, then $\Gamma(R)$ is a complete graph for a multiplication $R$-module $M$ with $|M| \neq 4$. The converse is true if we assume further that $M$ is faithful.

An element $m$ of $M$ is called a torsion element if and only if it has a non-zero annihilator in $R$. Let $T(M)$ be the set of torsion elements of $M$. It is clear that if $R$ is an integral domain, then $T(M)$ is a submodule of $M$, which is called a torsion submodule of $M$. If $T(M) = 0$, then the module $M$ is said to be torsion-free, and it is called a torsion module if $T(M) = M$. Thus, $\Gamma(M)$ is an empty graph if and only if $M$ is a torsion-free $R$-module. An $R$-module $M$ is called a multiplication $R$-module if for every submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$, Barnard [8]. Also, a ring $R$ is called reduced if $\text{Nil}(R) = 0$, and an $R$-module $M$ is called a reduced module if $r^2m = 0$ implies that $rM \cap Rm = 0$, where $r \in R$ and $m \in M$. It is clear that $M$ is a reduced module if $r^2m = 0$ for $r \in R$, $m \in M$ implies that $rm = 0$. Also by the proof of Lemma 3.7, step 1, in Ghalandarzadeh and Malakooti Rad [12], we can check that a multiplication $R$-module $M$ is reduced if and only if $\text{Nil}(M) = 0$. Also, a ring $R$ is a von Neumann regular ring if for each $a \in R$, there exists an element $b \in R$ such that $a = ab \cdot b$. It is clear that every von Neumann regular ring is reduced. A submodule $N$ of an $R$-module $M$ is called a pure submodule of $M$ if $IM \cap N = IN$ for every ideal $I$ of $R$ Ribenboim [19]. Following Kash ([14], p. 105), an $R$-module $M$ is called a von Neumann regular module if and only if every cyclic submodule of $M$ is a direct summand in $M$. If $N$ is a direct summand in $M$, then $N$ is pure but not conversely Matsumara ([16], Example. 2, p. 54) and Ribenboim ([19], Example. 14, p. 100). And so every von Neumann regular module is reduced. A proper submodule $N$ of $M$ is called a prime submodule of $M$, whenever $rM \cap N \neq 0$ implies that $m \in N$ or $r \in [N : M]$, where $r \in R$ and $m \in M$. Also, a prime submodule $N$ of $M$ is called a minimal prime submodule of a submodule $H$ of $M$, if it contains $H$ and there is no smaller prime submodule with this property. A minimal prime submodule of the zero submodule is also known as a minimal prime submodule of the module $M$. Recall that a ring $R$ is called B\'ezout if every finitely generated ideal $I$ of $R$ is principal. We know that every von Neumann regular ring is B\'ezout.

A $G$ is connected if there is a path between any two distinct vertices. The distance $d(x, y)$ between connected vertices $x$ and $y$ is the length of a shortest path from $x$ to $y$ ($d(x, y) = \infty$ if there is no such path). The diameter of $G$ is the diameter of a connected graph, which is the supremum of the distances between vertices. The diameter is $0$ if the graph consists of a single vertex. Also, a complete graph is a simple graph whose vertices are pairwise adjacent; the complete graph with $n$ vertices is denoted by $K_n$.

Throughout, $R$ is a commutative ring with identity and $M$ is a unitary $R$-module. The symbol $\text{Nil}(R)$ will be the ideal consisting of nilpotent elements of $R$. In addition, $\text{Spec}(M)$ and $\text{Min}(M)$ will denote the set of the prime submodules of $M$ and minimal prime submodules of $M$, respectively. And $\text{Nil}(M) := \cap_{\text{Spec}(M)}N$ will denote the nilradical of $M$. We shall often use $[x : M]$ and $[0 : M] = \text{Ann}(M)$ to denote the residual of $Rx$ by $M$ and the annihilator of an $R$-module $M$, respectively. The set $Z(M) := \{r \in R \mid rm = 0 \text{ for some } 0 \neq m \in M\}$ will denote the zero-divisors of $M$. As usual, the rings of integers and integers modulo $n$ will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively.

2. Minimal prime submodules

In this section, we investigate some properties of the class of minimal prime submodules of a multiplication $R$-module $M$. Multiplication $R$- modules have been studied in El-Bast and Smith [11]. In the mentioned paper they have proved the following theorem.

**Theorem 2.1.** Let $M$ be a non-zero multiplication $R$-module. Then

1. every proper submodule of $M$ is contained in a maximal submodule of $M$, and
2. $K$ is maximal submodule of $M$ if and only if there exists a maximal ideal $P$ of $R$ such that $K = PM \neq M$.

**Proof.** El-Bast and Smith (Theorem 2.5, [11]).

A consequence of the above theorem is that every non-zero multiplication $R$-module has a maximal submodule, since $0$ is a proper submodule of $M$. Therefore every non-zero multiplication $R$-module has a prime submodule.
Lemma 2.2. Let $M$ be a multiplication $R$-module. Suppose that $S$ is a non empty multiplicatively closed subset of $R$, and $H$ be a proper submodule of $M$ such that $[H : M]$ does not meet $S$. Then there exists a prime submodule $N$ of $M$ which contains $H$ and $[N : M] \cap S = \emptyset$.

Proof. Let $S$ be a non empty multiplicatively closed subset of $R$ and $H$ be a proper submodule of $M$ such that $[H : M]$ does not meet $S$. Set $\Omega := \{[K : M] | K < M, [H : M] \subseteq [K : M], [K : M] \cap S = \emptyset \}$. Since $[H : M] \in \Omega$, we have $\Omega \neq \emptyset$. Of course, the relation of inclusion, $\subseteq$, is a a partial order on $\Omega$. Let $\Delta$ be a non-empty totally ordered subset of $\Omega$ and $G = \bigcup_{[K : M] \in \Delta}[K : M]$. It is clear that $G \in \Omega$; then by Zorn's Lemma $\Omega$ has a maximal element say $[N : M]$. We show that $N = [N : M]M \in \text{Spec}(M)$. Assume $rm \in N$ for some $r \in R$ and $m \in M$, but neither $r \in [N : M]$ nor $m \in N$. Hence $rm \notin N$, and so there is $m_0 \in M$ such that $rm_0 \notin N$. Therefore $N \subseteq H_1 = Rrm_0 + N$, and $N \subseteq H_2 = Rm + N$. Hence $[N : M] \subseteq [H_1 : M]$ and $[N : M] \subseteq [H_2 : M]$. Consequently $[H_1 : M]$ and $[H_2 : M]$ are not elements of $\Omega$. So $[H_1 : M] \cap S \neq \emptyset$ and $[H_2 : M] \cap S \neq \emptyset$. Thus there are two elements $s_1, s_2 \in S$ such that $s_1M \subseteq H_1$ and $s_2M \subseteq H_2$. Hence $s_2s_1M \subseteq s_2H_1 \subseteq s_2(Rrm_0 + N)$, so $s_2s_1M \subseteq Rss_2m_0 + s_2N \subseteq R(rm + N) + N \subseteq N$. Therefore $s_2s_1 \in [N : M] \cap S$, and we have derived the required contradiction. Consequently $N$ is a prime submodule of $M$. \hfill \Box

Lemma 2.3. Let $M$ be an $R$-module with $\text{Spec}(M) \neq \emptyset$, and $H$ be a submodule of $M$. Let $H$ be contained in a prime submodule $N$ of $M$, then $N$ contains a minimal prime submodule of $H$.

Proof. Suppose that $\Omega = \{[K \in \text{Spec}(M), H \subseteq K \subseteq N] \}. Clearly N \in \Omega$, and so $\Omega$ is not empty. If $N'$ and $N''$ belong to $\Omega$, then we shall write $N' \subseteq N''$ if $N'' \subseteq N'$. This gives a partial order on $\Omega$. Now by Zorn’s Lemma $\Omega$ has a maximal element, say $N'$. Since $N'' \in \Omega, N''$ is a prime submodule of $M$. We show that $N''$ is a minimal prime submodule of $H$. Let $H \subseteq N_1 \subseteq N'$. So $N'' \subseteq N_1$, and since $N'$ is a maximal in $\Omega, N'' = N_1$. Consequently $N''$ is minimal with $H \subseteq N'' \subseteq N$. \hfill \Box

Theorem 2.4. Let $M$ be a multiplication $R$-module. Then $\text{Nil}(M) = \bigcap_{N \in \text{Min}(M)} N$.

Proof. Clearly $\text{Nil}(M) \subseteq \bigcap_{N \in \text{Min}(M)} N$. To establish the reverse inclusion, let $x \notin \text{Nil}(M)$. We show that there is a minimal prime submodule which does not contain $x$. Since $x \notin \text{Nil}(M)$, there is a prime submodule $N$ of $M$ such that $x \notin N$. If for all $0 \neq \alpha \in [x : M]$ there exists $n \in \mathbb{N}$ such that $\alpha^nx = 0$, then $x \in N$; which is a contradiction. Thus there exists non-zero element $\alpha \in [x : M]$ such that $\alpha^nx \neq 0$ for all $n \in \mathbb{N}$. Let $S = \{\alpha^n | n \geq 0\}$. It is clear that $S$ is a multiplicatively closed subset of $R$, and $0 \notin S$. A simple check yields that $S \cap [0 : M] = \emptyset$. By Lemma 2.2, there exists a prime submodule $N$ of $M$ such that $0 \subseteq N$ and $[N : M] \cap S = \emptyset$. Therefore by Lemma 2.3, there exists a minimal prime submodule $N''$ of $M$ such that $0 \subseteq N'' \subseteq N$. Since $x \notin N$, we have $x \notin N''$. Consequently $\text{Nil}(M) = \bigcap_{N \in \text{Min}(M)} N$. \hfill \Box

Lemma 2.5. Let $R$ be a von Neumann regular ring. Then every $R$-module is reduced.

Proof. Let $R$ be a von Neumann regular ring. So any finitely generated ideal is generated by an idempotent, and therefore any $R$-module is reduced. \hfill \Box

Proposition 2.6. Let $R$ be a von Neumann regular ring, and $M$ be a multiplication $R$-module. Suppose that $\Gamma(M)$ be a connected graph, and $\Gamma(M) \neq K_1$. Then $T(M) = \bigcup_{N \in \text{Min}(M)} N$. 

Proof. Let $N$ be a prime submodule of $M$ such that $N \not\subseteq T(M)$. It will be sufficient to show that $N \not\subseteq \text{Min}(M)$. Since $N \not\subseteq T(M)$, we may suppose that there exists an element $x \in N$ such that $x \not\in T(M)$. Since $M$ is a multiplication module, we may assume $x = \sum_{i=1}^{n} a_i m_i$. Since $R$ is a von Neumann regular ring, we have $\sum_{i=1}^{n} R a_i = Re$ for some non-zero idempotent element $e$ of $R$. Therefore there exists $m \in M$ such that $x = em$. Now put $\Omega = \{e^p \beta l : 0 < 1$ and $\beta \in R \setminus [N : M]\}$. Since $x = em \notin T(M)$, we have $R \setminus [N : M] \subset \Omega$, and $0 \notin \Omega$.

Now a simple check shows that $\Omega$ and $R \setminus [N : M]$ are multiplicatively closed subsets of $R$. Let $\Delta = |S|S$ is a multiplicatively closed subset of $R$. $R \setminus [N : M]$ is not a maximal element of $\Delta$. Since $R \setminus [N : M] \subset \Omega$. Thus $[N : M]$ is not a minimal prime ideal of $R$, and so there exists a prime ideal $h_1$ of $R$ such that $h_1 M \subset N$.

Therefore $h_1 M \not\subseteq M$ and by El-Bast and Smith (Corollary 2.11, [11]), $h_1 M$ is a prime submodule of $M$.

Now let $x \in T(M)^*$ but, $x \notin \bigcup_{N \in \text{Min}(M)} N$. Therefore $x \notin N$ for all minimal prime submodules $N$ of $M$. Since $\Gamma(M)$ is connected and $\Gamma(M) \not\subseteq K_1$, there is $y \in T(M)^*$ such that $x \neq y$ and $[x : M]y : M(M) = 0$ and so $\text{Ann}(x) \neq \text{Ann}(M)$. There is a non-zero element $r \in \text{Ann}(x)$ such that $r \notin \text{Ann}(M)$. Thus $Rx \neq 0$ in $M$ for all minimal prime submodules $N$ of $M$. Since $x \notin N$, then $R \subseteq \bigcap_{N \in \text{Min}(M)} N$. Now by Theorem 2.4, $R \subseteq \text{Nil}(M)$ and since $R$ is a von Neumann regular ring, by Lemma 2.5, $M$ is a reduced module and $\text{Nil}(M) = 0$. Hence $r \in \text{Ann}(M)$, which is a contradiction. Therefore, $x \notin \bigcup_{N \in \text{Min}(M)} N$. □

The next result give some properties and characterizations of multiplication von Neumann regular modules as a generalization of von Neumann regular rings.

**Proposition 2.7.** Let $M$ be a multiplication $R$-module.

1. If $R$ be a von Neumann regular ring, then $M$ is a von Neumann regular module.

2. If $R$ be a von Neumann regular ring, then $S^{-1}M$ is a von Neumann regular module, and $\text{Nil}(S^{-1}M) = 0$, where $S = R \setminus Z(M)$.

Proof. (1) Let $0 \neq x = \sum_{i=1}^{n} a_i m_i \in M$, where $a_i \in [x : M], m_i \in M$. Since $R$ is a von Neumann regular ring, we have $\sum_{i=1}^{n} R a_i = Re$ for some non-zero idempotent element $e$ of $R$; therefore there exists $m \in M$ such that $x = em$ and $e \in [x : M]$. So $1 = e + 1 - e$, thus

$$M = eM + (1 - e)M \subseteq Rx + M(1 - e).$$

Now, let $y \in Rx \cap M(1 - e)$. Hence $y = r_1 x = (1 - e)m$ for some $r_1 \in R$ and $m \in M$; so $y = ey = r_1 em_1 = e(1 - e)m = 0$. Therefore $M = Rx \oplus M(1 - e)$ and $M$ is a von Neumann regular module.

(2) We show that $SM = M$ for all $s \in S$, where $S = R \setminus Z(M)$. Since $R$ is a von Neumann regular ring, for any $s \in S$ there exists $t \in S$ such that $s + t = u$ is a regular element of $R$ and $st = 0$. So $u$ is a unit of $R$; hence $uM = M$. Since $st = 0$ and $s \notin Z(M), tM = 0$. Therefore $M = sM$ for all $s \in S$. Thus $S^{-1}M = M$. By (1), $S^{-1}M$ is a von Neumann regular module. □

3. The diameter of torsion graphs

In this section we establish some basic and important results on the diameter of torsion graphs over a multiplication module. Moreover, we investigate the relationship between the diameter of $\Gamma(M)$ and $\Gamma(R)$.

**Theorem 3.1.** Let $M$ be a multiplication $R$-module with $[M] \neq 4$. If $\Gamma(M)$ is a complete graph, then $\Gamma(R)$ is a complete graph. The converse is true if we assume further that $M$ is faithful.

Proof. Let $\Gamma(M)$ be a complete graph. By Ghalandarzadeh and Malakooti (Theorem 2.11, [13]), $\text{Nil}(M) = T(M)$. Also by Theorem 2.4, $\text{Nil}(M) = \bigcap_{N \in \text{Min}(M)} N$, so $T(M) \neq M$. Hence there exists $m \in M$ such that $\text{Ann}(m) = 0$. Suppose that $a, \beta \in \text{vertices of } \Gamma(R)$. One can easily check that $am, \beta m \in T(M)^*$. Therefore $[am : M][\beta m : M] = 0$, so $a\beta = 0$. Consequently $\Gamma(R)$ is a complete graph.

Now, let $\Gamma(R)$ be a complete graph, and $m, n \in T(M)^*$. So $\text{Ann}(m) = 0$ and $\text{Ann}(n) = 0$. Suppose that $0 \neq a \in [m : M]$ and $0 \neq \beta \in [n : M]$. Since $M$ is a faithful, $R$-module then $a$ and $\beta$ are the vertices $\Gamma(R)$. Therefore $a\beta = 0$, and so $[m : M][n : M]M = 0$. Hence $\Gamma(M)$ is a complete graph. □
The following example shows that the multiplication condition in the above theorem is not superfluous.

**Example 3.2.** Let \( R = \mathbb{Z} \) and \( M = \mathbb{Z} \oplus \mathbb{Z}_6 \). So by El-Bast and Smith (Corollary 2.3, [11]), \( M \) is not a multiplication \( R \)-module. Also \( \Gamma(M) \) is a complete graph, but \( V(\Gamma(R)) = \emptyset \).

**Corollary 3.3.** Let \( M \) be a faithful multiplication \( R \)-module with \( |M| \neq 4 \). If \( \Gamma(R) \) is a complete graph, then \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) or \( \text{Nil}(M) = \text{Nil}(R)M = Z(R)M = T(M) \).

**Proof.** Let \( \Gamma(M) \) be a faithful multiplication \( R \)-module. By Theorem 3.1, \( \Gamma(M) \) is a complete graph, and by Ghalandarzadeh and Malakooti (Theorem 2.11, [13]), \( \text{Nil}(M) = T(M) \). Let \( R \neq \mathbb{Z}_2 \times \mathbb{Z}_2 \), by Anderson and Livingston (Theorem 2.8, [1]), \( \text{Nil}(R) = Z(R) \). Hence \( Z(R) \) is an ideal of \( R \) and \( T(M) = Z(R)M \). Therefore, we have that \( \text{Nil}(M) = \text{Nil}(R)M = Z(R)M = T(M) \). \( \square \)

**Corollary 3.4.** Let \( M \) be a faithful multiplication \( R \)-module with \( |M| \neq 4 \). If \( \Gamma(R) \) is a complete graph, then \( |\text{Min}(M)| = 1 \).

**Proof.** Let \( M \) be a faithful multiplication \( R \)-module. By Theorem 3.1, \( \Gamma(M) \) is a complete graph. Thus \( T(M) \) is a submodule of \( M \). We show that \( \bigcup_{N \in \text{Min}(M)} N \subseteq T(M) \). Suppose that \( N \) be a prime submodule of \( M \), such that \( N \notin T(M) \). It will be sufficient to show that \( N \notin \text{Min}(M) \). Since \( N \notin T(M) \) there exists an element \( x \in N \) such that \( x \notin T(M) \). So there are \( \alpha \in [x : M] \) and \( m \in M \) such that \( \alpha m \notin T(M) \). Now by putting \( \Omega = \{ \alpha^i \beta \geq 0 \text{ and } \beta \in R \setminus [N : M] \} \), and similar to the proof of Proposition 2.6, one can check that \( \bigcup_{N \in \text{Min}(M)} N \subseteq T(M) \). By Ghalandarzadeh and Malakooti (Theorem 2.11, [13]) and Theorem 2.4, we have

\[
\bigcup_{N \in \text{Min}(M)} N \subseteq T(M) = \text{Nil}(M) = \bigcap_{N \in \text{Min}(M)} N,
\]

which completes the proof. \( \square \)

**Theorem 3.5.** Let \( R \) be a Bézout ring and \( M \) be a multiplication \( R \)-module such that \( |M| \neq 4 \) and \( M \neq T(M) \); then \( \text{diam} \Gamma(M) = \text{diam} \Gamma(R) \).

**Proof.** Let \( R \) be a Bézout ring and \( M \) be a multiplication \( R \)-module. By Theorem 3.1, \( \text{diam} \Gamma(M) = 1 \) if and only if \( \text{diam} \Gamma(R) = 1 \). Suppose that \( \text{diam} \Gamma(R) = 2 \) and \( x, y \in T(M) \) such that \( d(x, y) \neq 1 \). Let \( x = \sum_{i=1}^{m} \alpha_i m_i \) and \( y = \sum_{j=1}^{n} \beta_j m_j \), where \( \alpha_i \notin [x : M] \), \( 0 \neq \beta_j \notin [y : M] \). Since \( R \) is a Bézout ring, \( \sum_{i=1}^{m} R \alpha_i = Ra \) and \( \sum_{j=1}^{n} R \beta_j = Rb \), for some \( \alpha, \beta \in R \). Hence there exist \( m, n \in M \) such that \( x = am, y = bm \). Thus \( \alpha, \beta \in Z(R) \). If \( d(\alpha, \beta) = 1 \), then \( d(x, y) = 1 \), and so we have a contradiction. Thus \( d(\alpha, \beta) = 2 \), so there exists \( y \in Z(R) \) such that \( \alpha - y = -\beta \) is a path of length 2. Since \( M \neq T(M) \), then there is \( n \in M \) such that \( yn \notin T(M) \). Therefore \( \text{diam} \Gamma(M) = 2 \).

Suppose that \( \text{diam} \Gamma(M) = 2 \) and \( \alpha, \beta \in Z(R) \) such that \( d(\alpha, \beta) \neq 1 \). So \( \alpha \beta \neq 0 \); if \( M \neq T(M) \), then there is \( n \in M \) such that \( \alpha n \neq 0 \). Hence \( \text{diam} \Gamma(M) \leq 3 \). Therefore \( \text{diam} \Gamma(M) \leq 3 \). If \( \text{diam} \Gamma(M) = 3 \), then \( \text{diam} \Gamma(R) \geq 3 \), and by Anderson and Livingston, (Theorem 2.3, [1]), \( \text{diam} \Gamma(R) \leq 3 \). Therefore \( \text{diam} \Gamma(R) = 3 \). Consequently \( \text{diam} \Gamma(M) = \text{diam} \Gamma(R) \). \( \square \)
Lemma 3.6. Let $M$ be a reduced multiplication $R$-module and $H$ be a finitely generated submodule of $M$. Then $Ann(H)M \neq 0$ if and only if $H \subseteq N$ for some $N \in \text{Min}(M)$.

Proof. Let $Ann(H)M \neq 0$, so $Ann(H)M \not\subseteq \text{Nil}(M) = \bigcap_{N \in \text{Min}(M)} N$. Thus there exists $N_0 \in \text{Min}(M)$ such that $Ann(H)M \not\subseteq N_0$. Assume that $r \in R$ and $m \in M$ and $rm \in Ann(H)M$, but $rm \not\in N_0$. Therefore $rm[H : M] = 0 \subseteq N_0$. Since $rm \not\in N_0$, we have $H \subseteq N_0$.

To establish the reverse, let $N = PM \in \text{Min}(M)$, where $P = [N : M]$, and $H \subseteq N$. Since $M$ is a reduced $R$-module, $M_P$ will be a reduced $R_P$-module. We show that $M_P$ has exactly one maximal submodule. Let $M_P$ has two maximal submodules $S^{-1}H_1$ and $S^{-1}H_2$; so there exist two ideals $S^{-1}h_1$ and $S^{-1}h_2$ of $\text{Max}(S^{-1}R)$, such that $S^{-1}H_1 = S^{-1}h_1S^{-1}M$ and $S^{-1}H_2 = S^{-1}h_2S^{-1}M$. Since $R_P$ is a local ring, $S^{-1}H_1 = S^{-1}H_2$. We claim that $S^{-1}N$ is a proper submodule of $S^{-1}M$, and so by Theorem 2.1, $S^{-1}P$ is $S^{-1}N$ is the unique maximal submodule of $M_P$. Also, if $S^{-1}H_0$ is a prime submodule of $M_P$, then by Theorem 2.1, $S^{-1}H_0 \subseteq S^{-1}N$. By a routine argument $H_0 \subseteq N$, so $H_0 = N$; hence $S^{-1}H_0 = S^{-1}N$. Therefore by Theorem 2.4, $\text{Nil}(M_P) = S^{-1}N$. Since $M_P$ is reduced, $\text{Nil}(M_P) = 0$. Thus $S^{-1}N = 0$. On the other hand, $H \subseteq N$; hence $S^{-1}H = 0$. Suppose that $H = \sum_{i=1}^{n} R_{j_i}$ and $\frac{1}{i} = 0$ for all $1 \leq i \leq n$. Hence there exists $s_i \in R \setminus P$ such that $s_ih_i = 0$. Let $s = s_1s_2 \cdots s_n$, thus $sH = 0$. If $sM = 0$ then $s \in [N : M] = P$, which is a contradiction. So there is an element $m \in M$ such that $0 \neq sm \in Ann(H)M$.

Theorem 2.6 in [15] characterizes the diameter of $\Gamma(R)$ in terms of the ideals of $R$. Our results obtained in Theorems 3.7 and 3.8 specifies the diameter of $\Gamma(R)$ in terms of minimal prime submodules of a multiplication module $M$ over a von Neumann regular ring.

Theorem 3.7. Let $R$ be a von Neumann regular ring and $M$ be a multiplication $R$-module. If $M$ has more than two minimal prime submodules and $T(M^* \cap Ann(Rm + Rn)) = 0$. Hence $M$ is faithful. First, suppose that $[m : M][n : M]M \neq 0$, so $d(m, n) \neq 1$. If $d(m, n) = 2$, then there exists a vertex $x \in T(M^*)$ such that $m - x - n$ is a path. Thus $[m : M][x : M]M = 0 = [x : M][n : M]M$.

Accordingly $[x : M][m : M]^2 \cap Ann(Rm + Rn) = 0$, and so $[x : M] \not\subseteq Ann(Rm + Rn) = 0$. Which is a contradiction. We shall now assume that $d(m, n) \neq 2$. By Ghalandarzadeh and Malakooti (Theorem 2.6, [13]), $\Gamma(M)$ is connected with diam$(\Gamma(M)) \leq 3$; therefore $d(m, n) = 3$. Next, assume $[m : M][n : M]M = 0$, then by Proposition 2.6, $m, n \in \bigcup_{N \in \text{Min}(M)} N$. Since Ann$(Rm + Rn)M = 0$, by Lemma 3.6, $m$ and $n$ belong to two distinct minimal prime submodules. Suppose that $P, N$ and $Q$ are distinct minimal prime submodules of $M$ such that $m \in P \setminus (Q \cup N)$ and $n \in (Q \cap N) \setminus P$. Hence $[m : M][n : M] \not\subseteq N$; thus $am \not\subseteq N$ for some $a \in [m : M]$ and $m \in M$. Let $x \in (Q \cap P) \setminus N$. A simple check yields that $a^2x \neq 0$. On the other hand, since $[m : M][n : M]M = 0$, we have $a(x + ax) = a^2x$. Therefore $0 \neq a^2x \in [m : M][n + ax : M]M$. Also, by a routine argument, we have $Rm + Rn = Rm + R(n + ax)$ . So $Ann(Rm + R(n + ax)) = 0$. Similar to the above argument, we have $d(m, (n + ax)) = 3$. Consequently $diam(\Gamma(M)) = 3$.

Theorem 3.8. Let $R$ be a von Neumann regular ring and $M$ be a multiplication $R$-module. If $T(M)$ is not a submodule of $M$, then diam$(\Gamma(M)) \leq 2$ if and only if $M$ has exactly two minimal prime submodules.

Proof. Suppose that diam$(\Gamma(M)) \leq 2$, and $T(M)$ is not a submodule of $M$, so there exist $m, n \in T(M^*)$ with Ann$(Rm + Rn) = 0$. So $M$ is faithful and by Ghalandarzadeh and Malakooti (Theorem 2.6, [13]), $\Gamma(M)$ is connected. Since $\Gamma(M)$ is a connected graph and $T(M)$ is a submodule of $M$, by Proposition 2.6 and Lemma 3.6, there are at least two distinct minimal prime submodules $P$ and $Q$ of $M$ such that $m \in P \setminus Q$ and $n \in Q \setminus P$. On the other hand, by Theorem 3.7, $M$ can not have more than two minimal prime submodules; therefore $M$ has exactly two minimal prime submodules. Conversely, suppose that $P$ and $Q$ be only two minimal prime submodules of $M$. By Proposition 2.6, $T(M) = P \cup Q$. Assume that $m, n \in T(M^*)$ such that $m \in P \setminus Q$ and $n \in Q \setminus P$. Thus $[m : M][n : M]M \subseteq P \setminus Q = \text{Nil}(M) = 0$, by Lemma 2.5. So $d(m, n) = 1$. Also if $m, n \in P$, then $Rm + Rn \subseteq P$. By Lemma 3.6, Ann$(Rm + Rn)M = 0$; therefore there is $0 \neq \alpha \in R$ such that $am = an = 0$. On the other hand, there exists a non-zero element $x$ of $M$ such that $ax \neq 0$ and so $m - ax - n$ is a path, hence $d(m, n) = 2$, thus diam$(\Gamma(M)) \leq 2$. Moreover, if $m, n \in Q$, then similarly $diam(\Gamma(M)) \leq 2$. \qed
As an immediate consequence from Theorem 3.5 and Theorem 3.8, we obtain the following result.

**Corollary 3.9.** Let $R$ be a von Neumann regular ring and let $M$ be a multiplication $R$-module. If $T(M)$ is not a submodule of $M$, then $M$ has exactly two minimal prime submodules if and only if $R$ has exactly two minimal prime ideals.

**References**


