A common fixed point theorem for cyclic operators on partial metric spaces

Erdal Karapınar\textsuperscript{a}, Nabi Shobkolaei\textsuperscript{b}, Shaban Sedghi\textsuperscript{c}, S. Mansour Vaezpour\textsuperscript{d}

\textsuperscript{a}Department of Mathematics, Atılım University, 06836, İncek, Ankara, Turkey
\textsuperscript{b}Department of Mathematics, Islamic Azad University, Science and Research Branch 14778 93855 Tehran, Iran
\textsuperscript{c}Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
\textsuperscript{d}Department of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Avenue, Tehran 15914, Iran

Abstract. In this paper, we prove a common fixed point theorem for two self-mappings satisfying certain conditions over the class of partial metric spaces. In particular, the main theorem of this manuscript extends some well-known fixed point theorems in the literature on this topic.

1. Introduction

Recently, studies on the existence and uniqueness of fixed points of self-mappings on partial metric spaces have gained momentum (see e.g., [1] - [4],[7], [14]-[? ],[26, 33]). The idea of partial metric space, a generalization of metric space, was introduced by Mathews [25] in 1992. When compared to metric spaces, the innovation of partial metric spaces is that the self distance of a point is not necessarily zero [24]. This feature of partial metrics makes them suitable for many purposes of semantics and domain theory in computer sciences. In particular, partial metric spaces have applications on the Scott-Strachey order-theoretic topological models [32] used in the logics of computer programs.

Mathews [25] proved the analog of Banach contraction mapping principle in the class of partial metric spaces. This remarkable paper of Mathews [25] constructed another important bridge between the domain theory in computer science and fixed point theory in mathematics. Thus, it becomes feasible to transform the tools from Mathematics to Computer Science.

A self-mapping $T$ on a metric space $X$ is called contraction if there exists a constant $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for each $x, y \in X$. Banach contraction mapping principle, which states that a contraction has a fixed point, is one of the most important result in nonlinear analysis. This crucial result has been studied continuously since it was first published (See e.g. [1]-[23],[26]-[30]). As a generalization of this fundamental principle, Kirk-Srinivasan-Veeramani [23] developed the cyclic contraction. A contraction $T : A \cup B \to A \cup B$ on non-empty set $A, B$ is called cyclic if $T(A) \subset B$ and $T(B) \subset A$ hold for closed subsets $A, B$ of a complete metric space $X$. In the last decade, many authors (see e.g.[21, 22, 27–29, 34]) reported some fixed point theorems for cyclic operators.

Rus [29] introduced the following definition which is a further generalization of a cyclic mapping.

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\textbf{Email addresses:} erdalkarapinar@yahoo.com (Erdal Karapınar), nabi_shobe@yahoo.com (Nabi Shobkolaei), sedghi_gb@yahoo.com (Shaban Sedghi), vaez@aut.ac.ir (S. Mansour Vaezpour)
Definition 1.1. Let \( X \) be a nonempty set, \( m \) be a positive integer and \( T : X \to X \) be a mapping. \( X = \cup _{i=1}^{m} A_i \) is said to be a cyclic representation of \( X \) with respect to \( T \) if

(i) \( A_i, i = 1, 2, \ldots, m \) are nonempty sets.

(ii) \( T(A_1) \subset A_2, \ldots, T(A_{m-1}) \subset A_m, T(A_m) \subset A_1. \)

Remark 1.2. For convenience, we denote by \( \mathcal{F} \) the class of functions \( \phi : [0, \infty) \to [0, \infty) \) nondecreasing and continuous satisfying \( \phi(t) > 0 \) for \( t \in (0, \infty) \) and \( \phi(0) = 0. \)

We recall the following definition.

Definition 1.3. (See e.g. [1, 3, 20, 24]) Let \( X \) be a metric space, \( m \) be a positive integer, \( A_1, A_2, \ldots, A_m \) be nonempty subsets of \( X \) and \( X = \cup _{i=1}^{m} A_i. \) An operator \( T : X \to X \) is a cyclic \( (\phi - \psi) \)-contraction if

(i) \( X = \cup _{i=1}^{m} A_i \) is a cyclic representation of \( X \) with respect to \( T, \)

(ii) \( \phi(d(Tx, Ty)) \leq \phi(d(x, y)) - \psi(d(x, y)), \) for any \( x \in A_i, y \in A_{i+1}, i = 1, 2, \ldots, m, \) where \( A_{m+1} = A_1 \) and \( \phi, \psi \in \mathcal{F}. \)

The main result of [22] is the following.

Theorem 1.4. (Theorem 6 of [22]) Let \( (X, d) \) be a complete metric space, \( m \) be a positive integer, \( A_1, A_2, \ldots, A_m \) be nonempty subsets of \( X \) and \( X = \cup _{i=1}^{m} A_i. \) Let \( T : X \to X \) be a cyclic \( (\phi - \psi) \)-contraction with \( \phi, \psi \in \mathcal{F}. \) Then \( T \) has a unique fixed point \( z \in \cap _{i=1}^{m} A_i. \)

In this paper, we proved a common fixed point of two self-mappings \( T, g : X \to X \) on a partial metric space \( X \) under certain conditions.

We start some definitions and results needed in the sequel.

A partial metric on a nonempty set \( X \) is a mapping \( p : X \times X \to [0, \infty) \) such that

(PM1) \( x = y \) if and only if \( p(x, x) = p(x, y) = p(y, y), \)

(PM2) \( p(x, x) \leq p(x, y), \)

(PM3) \( p(x, y) = p(y, x), \)

(PM4) \( p(x, y) \leq p(x, z) + p(z, y) - p(z, z). \)

for all \( x, y, z \in X. \) A pair \( (X, p) \) is said to be partial metric space.

Notice also that if \( p \) is a partial metric on \( X, \) then the functions \( d_p, d_m : X \times X \to \mathbb{R}^+ \) given by

\[
d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y),
\]

\[
p(x, y) - p(x, x), p(x, y) - p(y, y)
\]

are equivalent (usual) metrics on \( X. \) For details see e.g. [?].

Example 1.5. (See e.g. [1, 3, 20, 24]) Consider \( X = [0, \infty) \) with \( p(x, y) = \max(x, y). \) Then \( (X, p) \) is a partial metric space. It is clear that \( p \) is not a (usual) metric. Note that in this case \( d_p(x, y) = |x - y|. \)

Example 1.6. (See e.g. [24]) Let \( X = [a, b] : a, b \in \mathbb{R}, a \leq b \) and define \( p([a, b], [c, d]) = \max(b, d) - \min(a, c). \) Then \( (X, p) \) is a partial metric spaces.

Lemma 1.7. (See e.g. [14, 15]) Let \( (X, p) \) be a PMS. Then

(A) If \( p(x, y) = 0 \) then \( x = y, \)

(B) If \( x \neq y, \) then \( p(x, y) > 0. \)
Example 1.8. (See e.g.[24]) Let \((X, d)\) and \((X, p)\) be a metric space and a partial metric space, respectively. Mappings \(p_i : X \times X \rightarrow [0, \infty) (i \in \{1, 2, 3\})\) defined by

\[
\begin{align*}
p_1(x, y) &= d(x, y) + p(x, y) \\
p_2(x, y) &= d(x, y) + \max\{\omega(x), \omega(y)\} \\
p_3(x, y) &= d(x, y) + a
\end{align*}
\]

induce partial metrics on \(X\), where \(\omega : X \rightarrow [0, \infty)\) is an arbitrary function and \(a \geq 0\).

We notice also that each partial metric \(p\) on \(X\) generates a \(T_0\) topology \(\tau_p\) on \(X\) which has a family of open \(p\)-balls

\[B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}\]

as a base where \(B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}\) for all \(x \in X\) and \(\varepsilon > 0\).

Definition 1.9. (See e.g. [24]) Let \((X, p)\) be a partial metric space.

(i) A sequence \(\{x_n\}\) in \(X\) converges to \(x \in X\) whenever \(\lim_{n \to \infty} p(x_n, x) = p(x, x)\).
(ii) A sequence \(\{x_n\}\) in \(X\) is called Cauchy whenever \(\lim_{n, m \to \infty} p(x_n, x_m)\) exists (and finite),
(iii) \((X, p)\) is said to be complete if every Cauchy sequence \(\{x_n\}\) in \(X\) converges, with respect to \(\tau_p\), to a point \(x \in X\), that is, \(\lim_{n, m \to \infty} p(x_n, x_m) = p(x, x)\).

We define \(L(x_n) = \{x | x_n \to x\}\) where \(\{x_n\}\) is a sequence in a partial metric space \((X, p)\). The example below shows that a convergent sequence \(\{x_n\}\) in a partial metric space may not be a Cauchy. In particular, it shows that the limit of a convergent sequence is not unique.

Example 1.10. (See e.g.[24]) Let \(X = [0, \infty)\) and \(p(x, y) = \max\{x, y\}\). Let

\[x_n = \begin{cases} 0, & n = 2k, \\ 1, & n = 2k + 1. \end{cases}\]

Then clearly it is convergent sequence and for every \(x \geq 1\) we have \(\lim_{n \to \infty} p(x_n, x) = p(x, x)\), therefore \(L(x_n) = [1, \infty)\). But \(\lim_{n, m \to \infty} p(x_n, x_m)\) does not exist.

We state a lemma that shows the limit of a convergent sequence \(\{x_n\}\) in a partial metric space is unique.

Lemma 1.11. (See e.g.[24]) Let \(\{x_n\}\) be a convergent sequence in partial metric space \(X\) such that \(x_n \to x\) and \(x_n \to y\). If

\[\lim_{n \to \infty} p(x_n, x_n) = p(x, x) = p(y, y),\]

then \(x = y\).

Lemma 1.12. (See e.g.[24]) Let \(\{x_n\}\) and \(\{y_n\}\) be two sequences in partial metric space \(X\) such that

\[\lim_{n \to \infty} p(x_n, x) = \lim_{n \to \infty} p(x_n, x_n) = p(x, x),\]

and

\[\lim_{n \to \infty} p(y_n, y) = \lim_{n \to \infty} p(y_n, y_n) = p(y, y),\]

then \(\lim_{n \to \infty} p(x_n, y_n) = p(x, y)\). In particular, \(\lim_{n \to \infty} p(x_n, z) = p(x, z)\) for every \(z \in X\).
Lemma 1.13. (See e.g. [24],[26]) Let \((X, p)\) be a partial metric space.

(a) \(\{x_n\}\) is a Cauchy sequence in \((X, p)\) if and only if it is a Cauchy sequence in the metric space \((X, d_p)\).

(b) A partial metric space \((X, p)\) is complete if and only if the metric space \((X, d_p)\) is complete. Furthermore, \(\lim_{n \to \infty} d_p(x_n, x) = 0\) if and only if

\[ p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n,m \to \infty} p(x_n, x_m). \]

Lemma 1.14. (See e.g. [4]) If \(\{x_n\}\) is a convergent sequence in \((X, d_p)\), then it is a convergent sequence in the partial metric space \((X, p)\).

In this paper, we prove a common fixed point theorem on the class of the partial metric spaces as a generalization of Theorem 1.4 and the main theorem of [31].

2. Main Result

We start this section with the following definition for two self-mappings \(T, g : X \to X\).

Definition 2.1. Let \(X\) be a nonempty set, \(m\) be a positive integer and \(T, g : X \to X\) be two mappings. \(X = \bigcup_{i=1}^{m} A_i\) is said to be a cyclic representation of \(X\) with respect to \((T - g)\) if

(i) \(A_i, i = 1, 2, \ldots, m\) are nonempty sets.

(ii) \(T(A_1) \subseteq g(A_2), \ldots, T(A_{m-1}) \subseteq g(A_m), T(A_m) \subseteq g(A_1)\).

Definition 2.2. Let \((X, p)\) be a partial metric space, \(m\) be a positive integer, \(A_1, A_2, \ldots, A_m\) be nonempty subsets of \(X\) and \(X = \bigcup_{i=1}^{m} A_i\). Two operators \(T, g : X \to X\) are cyclic \((\phi - \psi)\)-contraction if

(i) \(X = \bigcup_{i=1}^{m} A_i\) is a cyclic representation of \(X\) with respect to \((T - g)\),

(ii) \(\phi(p(Tx, Ty)) \leq \phi(p(gx, gy)) - \psi(p(gx, gy)), \) for any \(x \in A_i, y \in A_{i+1}, i = 1, 2, \ldots, m,\) where \(A_{m+1} = A_1\) and \(\phi, \psi \in \mathcal{F}\).

Our main result is the following.

Theorem 2.3. Let \((X, p)\) be a complete partial metric space, \(m\) be a positive integer, \(A_1, A_2, \ldots, A_m\) be nonempty subsets of \(X\) and \(X = \bigcup_{i=1}^{m} A_i\). Let \(T, g : X \to X\) be two cyclic \((\phi - \psi)\)-contraction such that \(g(A_i)\) closed subsets of \(X\).

(i) If \(g\) is one to one then there exists \(z \in \bigcap_{i=1}^{m} A_i\) such that \(gz = Tz\).

(ii) If the pair \((T, g)\) are weakly compatible, then \(T\) and \(g\) has a unique common fixed point \(z \in \bigcap_{i=1}^{m} A_i\).

Proof. Let \(x_1\) be an arbitrary point in \(A_1\). By cyclic representation of \(X\) with respect to pair \((T, g)\), we choose a point \(x_2\) in \(A_2\) such that \(Tx_1 = gx_2\). For this point \(x_2\) there exists a point \(x_3\) in \(A_3\) such that \(Tx_2 = gx_3\), and so on. Continuing in this manner we can define a sequence \(\{x_n\}\) as follows

\[ Tx_n = gx_{n+1}, \]

for \(n = 1, 2, \ldots\). We prove that \(\{x_n\}\) is a Cauchy sequence. If there exists \(n_0 \in \mathbb{N}\) such that \(gx_{n_0+1} = gx_{n_0}\), then, since \(gx_{n_0+1} = Tx_{n_0} = gx_{n_0}\), the part of existence of the coincidence point of \(T\) and \(g\) is proved. Suppose that \(gx_{n_0+1} \neq gx_{n_0}\) for any \(n = 1, 2, \ldots\). Then, since \(X = \bigcup_{i=1}^{m} A_i\), for any \(n > 0\) there exists \(i_n \in \{1, 2, \cdots, m\}\) such that \(x_{n-1} \in A_{i_n}\) and \(x_n \in A_{i_{n+1}}\). Since \((T, g)\) are cyclic \((\phi - \psi)\)-contraction, we have

\[ \phi(p(gx_n, gx_{n+1})) = \phi(p(Tx_{n-1}, Tx_n)) \leq \phi(p(gx_{n-1}, gx_n)) - \psi(p(gx_{n-1}, gx_n)) \leq \phi(p(gx_{n-1}, gx_n)) \]

(3)
From (3) and taking into account that \( \phi \) is nondecreasing we obtain
\[
p(gx_n, gx_{n+1}) \leq p(gx_{n-1}, gx_n) \quad \text{for any } n = 2, 3, \ldots
\]
Thus \( \{p(gx_n, gx_{n+1})\} \) is a nondecreasing sequence of nonnegative real numbers. Consequently, there exists \( \gamma \geq 0 \) such that \( \lim_{n \to \infty} p(gx_n, gx_{n+1}) = \gamma \). Taking \( n \to \infty \) in (3) and using the continuity of \( \phi \) and \( \psi \), we have
\[
\phi(\gamma) \leq \phi(\gamma) - \psi(\gamma) \leq \phi(\gamma),
\]
and therefore, \( \psi(\gamma) = 0 \). Since \( \psi \in \mathcal{F}, \gamma = 0 \), that is,
\[
\lim_{n \to \infty} p(gx_n, gx_{n+1}) = 0.
\]
Since \( p(gx_n, gx_n) \leq p(gx_n, gx_{n+1}) \) and \( p(gx_{n+1}, gx_{n+1}) \leq p(gx_n, gx_{n+1}) \), hence
\[
\lim_{n \to \infty} p(gx_n, gx_n) = \lim_{n \to \infty} p(gx_{n+1}, gx_{n+1}) = \lim_{n \to \infty} p(gx_n, gx_{n+1}) = 0. \tag{4}
\]
Since
\[
d_p(gx_n, gx_{n+1}) = 2p(gx_n, gx_{n+1}) - p(gx_n, gx_n) - p(gx_{n+1}, gx_{n+1}).
\]
This shows that \( \lim_{n \to \infty} d_p(gx_n, gx_{n+1}) = 0 \).

In the sequel, we prove that \( \{gx_n\} \) is a Cauchy sequence in the metric space \((X, d_p)\).

First, we prove the following claim.

Claim: For every \( \epsilon > 0 \) there exists \( n \in \mathbb{N} \) such that if \( b, q \geq n \) with \( b - q \equiv 1(m) \) then \( d_p(x_b, x_q) < \epsilon \).

In fact, suppose the contrary case. This means that there exists \( \epsilon > 0 \) such that for any \( n \in \mathbb{N} \) we can find \( b_n > q_n \geq n \) with \( b_n - q_n \equiv 1(m) \) satisfying
\[
d_p(gx_{b_n}, gx_{q_n}) \geq \epsilon. \tag{5}
\]
Now, we take \( n > 2m \). Then, corresponding to \( q_n \geq n \) use can choose \( b_n \) in such a way that it is the smallest integer with \( b_n > q_n \) satisfying \( b_n - q_n \equiv 1(m) \) and \( d_p(gx_{b_n}, gx_{q_n}) \geq \epsilon \). Therefore, \( d_p(gx_{b_n}, gx_{q_n}) \leq \epsilon \).

Using the triangular inequality
\[
\epsilon \leq d_p(gx_{b_n}, gx_{q_n}) \leq d_p(gx_{b_n}, gx_{b_{n-1}}) + \sum_{i=1}^{m} d_p(gx_{b_{n-i}}, gx_{b_{n-i+1}}) < \epsilon + \sum_{i=1}^{m} d_p(gx_{b_{n-i}}, gx_{b_{n-i+1}}).
\]
Letting \( n \to \infty \) in the last inequality and taking into account that
\[
\lim_{n \to \infty} d_p(gx_n, gx_{n+1}) = 0,
\]
we obtain
\[
\lim_{n \to \infty} d_p(gx_{b_n}, gx_{q_n}) = \epsilon \implies \lim_{n \to \infty} p(gx_{b_n}, gx_{q_n}) = \frac{\epsilon}{2}. \tag{6}
\]
Again, by the triangular inequality
\[
\epsilon \leq d_p(gx_{b_n}, gx_{q_n}) \leq d_p(gx_{b_n}, gx_{b_{n-1}}) + d_p(gx_{b_{n-1}}, gx_{b_n}) \leq d_p(gx_{b_n}, gx_{b_{n-1}}) + d_p(gx_{b_{n-1}}, gx_{b_{n-1}}) + d_p(gx_{b_{n-1}}, gx_{b_n}) + d_p(gx_{b_{n-1}}, gx_{b_{n-1}}) = 2d_p(gx_{b_n}, gx_{b_{n-1}}) + d_p(gx_{b_{n-1}}, gx_{b_n}) + 2d_p(gx_{b_{n-1}}, gx_{b_{n-1}})
\]
Letting \( n \to \infty \) in (6) and taking into account that \( \lim_{n \to \infty} d_p(gx_n, gx_{n+1}) = 0 \) and (6), we get
\[
\lim_{n \to \infty} d_p(gx_{b_n}, gx_{q_n}) = \epsilon.
\]
Hence
\[
\lim_{n \to \infty} p(gx_{q_{n+1}}, gx_{b_{n+1}}) = \frac{\epsilon}{2}.
\] (8)

Since \(gx_q\) and \(gx_b\) lie in different adjacently labeled sets \(A_i\) and \(A_{i+1}\) for certain \(1 \leq i \leq m\), using the fact that \(T\) and \(g\) are cyclic \((\phi - \psi)\)-contraction, we have
\[
\phi(p(gx_{q_{n+1}}, gx_{b_{n+1}})) = \phi(p(Tx_q, Tx_b)) \\
\leq \phi(p(gx_{q_{n}}, gx_{b_{n}})) - \psi(p(gx_{q_{n}}, gx_{b_{n}})) \\
\leq \phi(p(gx_{q_{n}}, gx_{b_{n}})).
\]

Taking into account (6) and (8) and the continuity of \(\phi\) and \(\psi\), letting \(n \to \infty\) in the last inequality, we obtain
\[
\phi\left(\frac{\epsilon}{2}\right) \leq \phi\left(\frac{\epsilon}{2}\right) - \psi\left(\frac{\epsilon}{2}\right) \leq \phi\left(\frac{\epsilon}{2}\right)
\]
and consequently, \(\psi\left(\frac{\epsilon}{2}\right) = 0\). Since \(\psi \in \mathcal{F}\), then \(\epsilon = 0\) which is contradiction. Therefore, our claim is proved.

In the sequel, we will prove that \([gx_n]\) is a Cauchy sequence in metric space \((X, d_p)\). Fix \(\epsilon > 0\). By the claim, we find \(n_0 \in \mathbb{N}\) such that if \(b, q \geq n_0\) with \(b - q \equiv 1(m)\)
\[
d_p(gx_b, gx_q) \leq \frac{\epsilon}{2}.
\] (9)

Since \(\lim_{n \to \infty} d_p(gx_n, gx_{n+1}) = 0\) we also find \(n_1 \in \mathbb{N}\) such that
\[
d_p(gx_n, gx_{n+1}) \leq \frac{\epsilon}{2m}
\] (10)
for any \(n \geq n_1\).

Suppose that \(r, s \geq \max\{n_0, n_1\}\) and \(s > r\). Then there exists \(k \in \{1, 2, \cdots, m\}\) such that \(s - r \equiv k(m)\). Therefore, \(s - r + j \equiv 1(m)\) for \(j = m - k + 1\). So, we have
\[
d_p(gx_r, gx_s) \leq d_p(gx_r, gx_{s+j}) + d_p(gx_{s+j}, gx_{s+j-1}) + \cdots + d_p(gx_{s+1}, gx_s).
\]

By (9) and (10) and from the last inequality, we get
\[
d_p(gx_r, gx_s) \leq \frac{\epsilon}{2} + \frac{j \epsilon}{2m} \leq \frac{\epsilon}{2} + \frac{m \epsilon}{2m} = \epsilon.
\]
This proves that \([gx_n]\) is a Cauchy sequence in metric space \((X, d_p)\). Since \((X, p)\) is complete then from Lemma 1.13, the sequence \([gx_n]\) converges in the metric space \((X, d_p)\), say \(\lim_{n \to \infty} p(gx_n, x) = 0\) for some \(x \in X\).

Therefore, by Lemma 1.13 we have
\[
p(x, x) = \lim_{n \to \infty} p(gx_n, x) = \lim_{n \to \infty} p(gx_n, gx_m).
\]

That is, there exists \(x \in X\) such that \(\lim_{n \to \infty} gx_n = x\) in partial metric \((X, p)\). Since \(g(A_i)\) are closed subsets of \(X\), we have \(x \in g(A_i)\) for every \(i \in \{1, 2, \cdots, m\}\). That is, \(x \in \cap_{i=1}^m g(A_i)\). Hence, there exists \(z_i \in A_i\) such that \(gz_i = x\). Since \(g\) is one to one we have
\[
g(z_1) = g(z_2) = \cdots = g(z_m) = x \implies z_1 = z_2 = \cdots = z_m = z.
\]

Therefore, \(g(z) = x\) for \(z \in \cap_{i=1}^m A_i\). In fact, \(\lim_{n \to \infty} gx_n = g\). On the other hand since the sequence \([gx_n]\) has infinite terms in each \(A_i\) for \(i \in \{1, 2, \cdots, m\}\), we take a subsequence \([gx_n]\) of \([gx_n]\) with \(gx_n \in g(A_{i-1})\) where \(x_m \in A_{i-1}\). Using the contractive condition, we can obtain
\[
\phi(p(gx_{m+1}, Tz)) = \phi(p(Tx_m, Tz)) \\
\leq \phi(p(gx_m, gz)) - \psi(p(gx_m, gz)) \\
\leq \phi(p(gx_m, gz)).
\]
Since \( gx_m \to g \) and \( \phi \) and \( \psi \) belong to \( F \), letting \( k \to \infty \) in the last inequality, we have

\[
\phi(p(gz, Tz)) \leq \phi(p(gz, gz)) - \psi(p(gz, gz)) \leq \phi(p(gz, gz)).
\]

Moreover, we obtain \( p(gz, Tz) = p(gz, gz) \), because \( \phi \) is nondecreasing and \( p(gz, gz) \leq p(gz, Tz) \). Hence, if \( p(gz, gz) \neq 0 \) then by the last inequality we have,

\[
\phi(p(gz, gz)) = \phi(p(gz, Tz)) \\
\leq \phi(p(gz, gz)) - \psi(p(gz, gz)) \\
< \phi(p(gz, gz)),
\]

which is contradiction. Since \( \phi \in F \), then, \( p(Tz, Tz) = p(gz, gz) = p(gz, Tz) = 0 \), it follows that, \( Tz = gz = x \).

ii) Since \( g \) and \( T \) are two weakly compatible mappings, we have \( TTz = Tgz = gTz = gz \). That is \( Tx = gx \).

Next, we prove that \( Tx = x \). Since \( Tz \in X \) hence there exists some \( i \) such that \( Tz \in A_i \). By \( z \in \cap_{i=1}^m A_i \) we have \( z \in A_{i-1} \), by using the contractive condition we obtain

\[
\phi(p(Tz, TTz)) \leq \phi(p(Tz, gTz)) - \psi(p(gz, gTz)) \\
\leq \phi(p(gz, gTz)) = \phi(p(Tz, TTz)),
\]

from the last inequality we have

\[
\psi(p(Tz, TTz)) = 0.
\]

Since \( \psi \in F \), \( p(Tz, TTz) = 0 \) and, consequently, \( x = Tz = TTz = Tx = gx \).

Finally, in order to prove the uniqueness of a fixed point, we have \( y, z \in X \) with \( y \) and \( z \) common fixed points of \( T \) and \( g \). The cyclic character of \( T - g \) and the fact that \( y, z \in X \) are common fixed points of \( T \) and \( g \), imply that \( y, z \in \cap_{i=1}^m A_i \). If \( p(y, z) \neq 0 \) then by using the contractive condition we obtain

\[
\phi(p(y, z)) = \phi(p(Ty, Tz)) \leq \phi(p(gy, gz)) - \psi(p(gy, gz)) \\
< \phi(p(gy, gz)) = \phi(p(y, z)),
\]

which is a contradiction. Since \( \phi \in F \), \( p(y, z) = 0 \) and, consequently, \( y = z \). This finishes the proof. □

**Corollary 2.4.** Let \( (X, p) \) be a complete partial metric space, \( m \) be a positive integer, \( A_1, A_2, \ldots, A_m \) be nonempty closed subsets of \( X \) and \( X = \bigcup_{i=1}^m A_i \). Let \( T : X \to X \) be a cyclic weak \( (\phi - \psi) \)-contraction. Then \( T \) has a unique fixed point \( z \in \cap_{i=1}^m A_i \).

**Proof.** Take \( g(x) = x \) in Theorem 2.3. □

**Corollary 2.5.** Let \( (X, p) \) be a complete partial metric space, \( m \) be a positive integer, \( A_1, A_2, \ldots, A_m \) be nonempty closed subsets of \( X \). Suppose that \( T : X \to X \) is a self-mapping and \( X = \bigcup_{i=1}^m A_i \) is a cyclic representation of \( X \) with respect to \( T \). Further, \( T \) satisfies \( d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)) \), for any \( x \in A_i, y \in A_{i+1}, i = 1, 2, \ldots, m \), where \( A_{m+1} = A_1 \) and \( \psi \in F \). Then \( T \) has a unique fixed point \( z \in \cap_{i=1}^m A_i \).

**Proof.** Take \( \phi(t) = t \) in Corollary 2.4. □

**Example 2.6.** Let \( X = [0, 1] \) and \( g, T : X \to X \) such that \( Tx = \frac{x}{12} \) and \( gx = \frac{x}{2} \). Suppose that \( \psi, \phi : [0, \infty) \to [0, \infty) \) are defined as follows \( \psi(t) = \frac{t}{2} \) and \( \phi(t) = \frac{t}{2} \). For \( A_i = [0, 1], (i = 1, 2, \ldots, m) \) all conditions of Theorem 2.3 are satisfied. It is clear that \( x = 0 \) is the common fixed point of \( T \) and \( g \).
References


[31] N. Shobkolaei, E. Karapinar, S. Sedghi, S.M. Vaezpour, Fixed point theory for cyclic $(\phi - \psi)$-contractions on partial metric spaces, (submitted)

