Some new classes of \((m, n)\)-hyperrings

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Abstract. The notion of \((m, n)\)-ary hyperring was introduced by Davvaz at the 10th AHA congress [9], as the strong distributive structure. In this article we generalize it, by introducing the notion of \((m, n)\)-ary hyperring with inclusive distributivity. We present construction of \((m, n)\)-ary hyperrings associated with binary relations on semigroup. We also state the condition under which there exists \((m, n)\)-ary hyperring of multiendomorphisms for a starting \(m\)-ary hypergroup \((H, f)\). Finally, we analyze connections between the obtained classes of \((m, n)\)-ary hyperrings.

1. Introduction

The hyperstructure theory was introduced by F. Marty at the 8th Congress of Scandinavian Mathematicians held in 1934. A semihypergroup \((H, \circ)\) is a nonempty set \(H\) equipped with a hyperoperation \(\circ\), that is a map \(\circ : H \times H \to P'(H)\), where \(P'(H)\) denotes the family of all nonempty subsets of \(H\), and for all \((x, y, z) \in H^3\) : \(x \circ (y \circ z) = (x \circ y) \circ z\). A semihypergroup is called a hypergroup in the sense of Marty [16] if for every \(a \in H\) : \(a \circ H = H \circ a = H\). In the above definitions, if \(A, B \in P'(H)\), then \(A \circ B\) is given by:

\[
A \circ B = \bigcup_{a \in A, b \in B} a \circ b
\]

\(x \circ A\) is used for \([x] \circ A\) and \(A \circ x\) for \(A \circ [x]\).

A comprehensive review of the theory of hyperstructures appears in Corsini [4], Corsini and Leoreanu [7] and Vougiouklis [20]. Since 1934, the hyperstructure theory has had applications to several areas of both pure and applied mathematics. About 70 years later, a suitable generalization of a hypergroup, called an \(n\)-ary hypergroup was introduced and studied by Davvaz and Vougiouklis in [12]. Davvaz et al. [11] considered a class of algebraic hypersystems which represent a generalization of semigroups, hypersemigroups and \(n\)-ary semigroups. The properties of this class were investigated in [10] and [11]. The notion of \((m, n)\)-ary hyperring was introduced by Davvaz [9] as a triple \((R, f, g)\) such that \((R, f)\) is an \(m\)-ary hypergroup, \((R, g)\) is an \(n\)-ary hypersemigroup and \(g\) is distributive over \(f\) in the sense of equality. In this article, by an \((m, n)\)-ary hyperring we mean more general structure in the following sense: we let \(g\) to be distributive over \(f\) in the sense of inclusion. A subclass of the \((m, n)\)-hyperrings, called Krasner \((m, n)\)-hyperrings was studied by Mirvakili and Davvaz in [17]. Anvariyeh, Mirvakili and Davvaz [1], considered \((m, n)\)-ary hypermodules on \((m, n)\)-ary hyperring.

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If \((H, \oplus)\) is a commutative binary hypergroup and \(F(H)\) the set of multiendomorphisms of \(H\) i.e. \(F(H) = \{h : H \rightarrow P^r(H) \mid (\forall x, y \in H) h(x \oplus y) \subseteq h(x) \oplus h(y)\}\) then for all pairs \(f, g \in F(H)\) we set:

\[
\begin{align*}
\mathcal{F} & = \{h \in F(H) \mid (\forall x \in H) h(x) \subseteq f(x) \oplus g(x)\} \\
\mathcal{G} & = \{h \in F(H) \mid (\forall x \in H) h(x) \subseteq f(g(x))\}
\end{align*}
\]

It is known that the structure \((F(H), \mathcal{F}, \mathcal{G})\) is a binary hyperring (see Corsini [4], Example 422). In Section 3 of this article, we determine condition under which we can construct the \((m, n)-ary\) hyperring

of multiendomorphisms of \(m\)-ary hypergroup \((H, f)\). We show that we can associate a hyperring of multiendomorphisms with hypergroup \((H, f)\) which is not necessary commutative.

The association between hyperstructures and binary relations had been studied by many authors, for example see Chvalina [2,3], Rosenberg [18], Corsini [5,6], Corsini and Leoreanu [8], and Spartalis [19].

Connections of \(n\)-ary hypergroups with binary relations was studied by Leoreanu and Davvaz in [15]. In Section 4 of this article, we obtain a class of strong distributive \((m, n)-ary\) hyperrings associated with binary relations on semigroup. We investigate their morphisms and we also, establish connection between the constructed \((m, n)-ary\) hyperring \((H, f, g)\) and the hyperring of multiendomorphisms of \(m\)-hypergroup \((H, f)\).

2. Preliminaries

The notion of \((m, n)-ary\) hyperring was introduced by Davvaz [9]. In this section we generalize it, by introducing the notion of \((m, n)-ary\) hyperring with inclusive distributivity and we give several examples of these structures.

We recall the following elementary background from [9].

A mapping \(f : H \times \cdots \times H \rightarrow P^r(H)\), where \(H\) appears \(n\) times and \(P^r(H)\) denotes the set of all non-empty subsets of \(H\), is called an \(n\)-ary hyperoperation and \(n\) is called the arity of this hyperoperation. If \(f\) is an \(n\)-ary hyperoperation defined on \(H\), then \((H, f)\) is called an \(n\)-ary hypergroupoid. We shall use the following abbreviated notation: the sequence \(x_1, x_2, \ldots, x_n\) will be denoted by \(x_i^{(n)}\). For \(j < i\), \(x_j^{(n)}\) is the empty symbol.

In this convention \(f(x_1, \ldots, x_j, y_{j+1}, \ldots, y_r, z_{r+1}, \ldots, z_n)\) may be written as \(f(x_1^{(j)}, y_{j+1}^{(r)}, z_{r+1}^{(n)})\). Similarly, for non-empty subsets \(A_1, \ldots, A_n\) of \(H\) we define:

\[
f(A_1) = f(A_1, ..., A_n) = \cup\{f(x_i^{(n)}) \mid x_i \in A_i, i = 1, ..., n\}.
\]

An \(n\)-ary hyperoperation \(f\) is called associative if:

\[
f(x_1^{(i-1)}, x_2^{(j-1)}, x_3^{(n-i)}) = f(x_1^{(i-1)}, x_2^{(j-1)}, x_3^{(n-i)})
\]

for every \(i, j, n \in [1, \ldots, n]\) and all \(x_1, x_2, \ldots, x_{n-1} \in H\). An \(n\)-ary hypergroupoid with the associative hyperoperation is called an \(n\)-ary hipersemigroup. An \(n\)-ary hypersemigroup \((H, f)\) in which the equation \(b \in f(a_1^{(i-1)}, x, a_n^{(n-i)})\) has a solution \(x \in H\) for every \(a_1^{(i-1)}, a_n^{(n-i)}, b \in H\) and \(1 \leq i \leq n\), is called an \(n\)-ary hypergroup. This condition can be formulated by:

\[
f(a_1^{(i-1)}, H, a_n^{(n-i)}) = H.
\]

An \(n\)-ary hypergroupoid \((H, f)\) is commutative if for all \(\delta \in S_n\) and for every \(a_1^{(i)} \in H\) we have \(f(a_1, ..., a_n) = f(a_0(1), ..., a_0(n))\).

We introduce the following definition of \((m, n)-ary\) hyperring.

**Definition 2.1.** An \((m, n)-ary\) hyperring is an algebraic hyperstructure \((R, f, g)\) which satisfies the following axioms:

1. \((R, f)\) is an \(m\)-ary hypergroup.
2. \((R, g)\) is an \(n\)-ary hypersemigroup.

3. The \(n\)-ary hyperoperation \(g\) is distributive with respect to the \(m\)-ary hyperoperation \(f\) i.e. for every \(a_1^{i-1}, a_{i+1}^n, x_1^n \in R, 1 \leq i \leq n,

\[ g(a_1^{i-1}, f(x_1^n), a_{i+1}^n) \subseteq f(g(a_1^{i-1}, x_1, a_{i+1}^n)), \]

\((R, f, g)\) is called an \(n\)-ary hyperpyring if \(n = m\).

The above definition contains the class of \((m, n)\)-ary hyperrings in the sense of Davvaz. According to [9] an \((m, n)\)-ary hyperpyring is an algebraic hyperstructure \((R, f, g)\) which satisfies the conditions (1), (2) and (3) for every \(a_1^{i-1}, a_{i+1}^n, x_1^n \in R, 1 \leq i \leq n, g(a_1^{i-1}, f(x_1^n), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n)), \)

The \((m, n)\)-ary hyperpyring in the sense of Davvaz will be called strong distributive \((m, n)\)-ary hyperpyring.

**Example 2.2.** a) Let \((R, +, \cdot)\) be a ring and \(\emptyset \neq P \subseteq R\) such that \(RP = R\) and \(Pz = zP\) for all \(z \in R\). If we define an \(m\)-ary hyperoperation \(f\) and an \(n\)-ary hyperoperation \(g\) as follows:

\[
\begin{align*}
 f(x_1^n) &= x_1P + x_2P + \ldots + x_nP \\
 g(x_1^n) &= x_1P_2x_3P_3x_{n-1}P_n
\end{align*}
\]

for any \(x_1^n \in R\) and \(x_i^n \in R\), then it can be verified that \((H, f, g)\) is an \((m, n)\)-ary hyperpyring. b) It is easy to see that if \((R, +, \cdot)\) is a ring with unity \(1\) and \(P = \{1\}\) then \((H, f, g)\) is a strong distributive \((m, n)\)-ary hyperpyring. In this case, \(f(x_1^n) = x_1 + \ldots + x_n\) and \(g(x_1^n) = x_1 \cdot \ldots \cdot x_n\).

**Example 2.3.** Let \((R, +, \cdot)\) be a ring and \(I, J\) be ideals of a ring \(R\). If we set:

\[
\begin{align*}
 f(x_1, x_2) &= x_1 + x_2 + I \\
 g(x_1, x_2) &= x_1 \cdot x_2 + J
\end{align*}
\]

for all \(x_1, x_2 \in R\), then \((R, f, g)\) is \((2, 2)\)-hypiring. If \(I = J\), then obviously \((R, f, g)\) is a strong distributive hyperpyring.

The following definition is a generalization of a suitable definition related to binary hyperrings.

**Definition 2.4.** Let \((R_1, f_1, g_1)\) and \((R_2, f_2, g_2)\) be \((m, n)\)-ary hyperrings. A map \(\varphi : R_1 \rightarrow R_2\) is called an inclusion homomorphism if the following conditions are satisfied:

1) \(\varphi(f_1(a_1^m)) \subseteq f_2(\varphi(a_1), \ldots, \varphi(a_m)) \) for all \(a_1^m \in R_1\)

2) \(\varphi(g_1(a_1^n)) \subseteq g_2(\varphi(a_1), \ldots, \varphi(a_n)) \) for all \(a_1^n \in R_1\)

A map \(\varphi\) is called a good (or strong) homomorphism if in the conditions 1) and 2) the equality is valid.

We recall the following notion and result from [14], [15].

Let \(\rho\) be a binary relation on a non-empty set \(H\). We define a partial \(n\)-ary hypergroupoid \((H, p)\) as follows:

\[
(\forall a \in H), f_p(a, \ldots, a) = \{y(a, y) \in \rho\}
\]

\(n\) times

and

\[
(\forall a_1, a_2, \ldots, a_n \in H), f_p(a_1, a_2, \ldots, a_n) = f_p(a_1, \ldots, a_1) \cup f_p(a_2, \ldots, a_2) \cup \ldots \cup f_p(a_n, \ldots, a_n).
\]

\(n\) times \(n\) times \(n\) times
By a partial $n$-ary hypergroupoid we mean a non-empty set $H$, endowed with a function from 

$$H \times \ldots \times H$$

$n$ times

to the set of subsets of $H$. Notice that $(H, f_\rho)$ is an $n$-ary hypergroupoid if the domain of $\rho$ is $H$.

An element $z \in H$ is called an outer element of $\rho$ if there exists $y \in H$ such that $(y, z) \notin \rho^2$.

It is interesting to see when the above $n$-ary hypergroupoid $(H, f_\rho)$ is an $n$-ary hypergroup.

**Theorem 2.5.** Let $\rho$ be a binary relation with full domain. The $n$-ary hypergroupoid $(H, f_\rho)$ is an $n$-hypergroup if and only if the following conditions hold:

1. $\rho$ has a full range;
2. $\rho \subseteq \rho^2$; 
3. $(x, z) \in \rho^2 \Rightarrow (x, z) \in \rho$ for any outer element $z$ of $\rho$.

3. $(m, n)$-ary hyerring of multiendomorphisms

In this section we determine condition under which we can construct the $(m, n)$-ary hyerring of multiendomorphisms of $m$-ary hypergroup $(H, f)$. We show that we can associate a hyerring of multiendomorphisms with hypergroup $(H, f)$ which is not necessary commutative.

Let $(H, f)$ be an $m$-ary hypergroup.

Before proving the next theorem we introduce the following notation:

$$a_{k1}^m = (a_{k1}, a_{k2}, \ldots, a_{km}), \quad a_{1k}^m = (a_{1k}, a_{2k}, \ldots, a_{mk}),$$

for all $1 \leq k \leq m$.

If $h_1, h_{i+1}, \ldots, h_{m+i-1}$, is the sequence of multiendomorphisms of hypergroup $(H, f)$, and $x \in H$, then we put:

$$f(h_i^{m+i-1}(x)) = f(h_i(x), ..., h_{m+i-1}(x))$$

for all $1 \leq i \leq m$.

If $h_1, ..., h_n$ are multiendomorphisms of hypergroup $(H, f)$ and $x \in H$, then:

$$(h_1, h_2, ..., h_n)(x) = (h_1 \circ \ldots \circ h_n)(x) = h_1(h_2(...(h_{n-1}(h_n(x))))$$

where we take

$$h_i(K) = \bigcup_{k \in K} h_i(k)$$

for any $K \subseteq H$ and $1 \leq i \leq n$.

**Theorem 3.1.** Let $(H, f)$ be an $m$-ary hypergroup such that for all $a_{11}^{m1}, a_{21}^{m1}, \ldots, a_{m1}^{m1} \in H$ it holds:

$$f(f(a_{11}^{m1}), f(a_{22}^{m1}), \ldots, f(a_{m1}^{m1})) = f(f(a_{11}^{m1}), f(a_{22}^{m2}), \ldots, f(a_{m1}^{mmm})).$$

(1)

Let $F(H)$ be the set of multiendomorphisms of hypergroup $(H, f)$ i.e.

$$F(H) = \{ h : H \rightarrow P^r(H) | (\forall a_i^{m} \in H) h(f(a_i^{m})) \subseteq f(h(a_1), ..., h(a_m)) \}.$$

Define an $m$-ary hyperoperation $\oplus$ and an $n$-ary $(n \geq 2)$ hyperoperation $\circ$ on $F(H)$ as follows: For any $h_i^{m} \in F(H)$ set

$$\oplus(h_i^{m}) = \{ h \in F(H) | (\forall x \in H) h(x) \subseteq f(h_1(x), ..., h_m(x)) \}.$$

For any $h_i^{m} \in F(H)$ set

$$\circ(h_i^{m}) = \{ h \in F(H) | (\forall x \in H) h(x) \subseteq f(h_1(x), ..., h_m(x)) \}.$$

The structure $(F(H), \oplus, \circ)$ is an $(m, n)$-ary hyerring.
Proof. For any \( h_i^m \in F(H) \) it holds \( \circledast(h_i^m) \neq \emptyset \), i.e., \( \circledast \) is an \( m \)-ary hyperoperation. Indeed, let \( h : H \to P^m(H) \) be a map defined by:

\[
h(x) = f(h_1(x), ..., h_m(x)), \text{ for all } x \in H.
\]

Then for every \( a_i^m \in H \) it holds:

\[
h(f(a_i^m)) = f(h_1(f(a_i^m)), ..., h_m(f(a_i^m))) \subseteq f(f(h_1(a_1), ..., h_1(a_m)), ..., f(h_m(a_1), ..., h_m(a_m))).
\]

In what follows we shall denote the set \( h_i(a_j) \) by \( A_{ij} \) and the sequence \( A_{i1}, ..., A_{im} \) by \( A_{i1}^m \) for all \( i, j \in \{1, ..., m\} \). So,

\[
h(f(a_i^m)) \subseteq f(f(A_{i1}^{1m}), ..., f(A_{im}^m)) = f(f(f(A_{i1}^{1m}, ..., f(A_{im}^m)), \ldots, f(h_1(a_1), ..., h_m(a_1)), ..., f(h_1(a_m), ..., h_m(a_m))) = f(h(a_1), ..., h(a_m)).
\]

Thus, \( h \in \circledast(h_i^m) \).

Now, we prove that \( m \)-ary hyperoperation \( \circledast \) is associative. Let, \( i, j \in \{1, ..., m\} \) and \( h_{i}^{2m-1} \in F(H) \). Set

\[
L = \bigoplus \bigcup(h_i^{1-1}, h_i^{m+i-1}, h_{m+i-1}^{2m-1}) = \bigcup \bigcup(h_i^{1-1}, h_i^{m+i-1}) \mid h_i \in \bigoplus(h_i^{m+i-1})
\]

\[
= \bigcup \bigcup(h_i^{1-1}, h_i^{m+i-1}) \mid h_i \in F(H) \land (\forall x \in H) h_i(x) \subseteq f(h_i^{m+i-1}(x))
\]

Thus, if \( h_i \in L \) then for all \( x \in H \) it holds:

\[
h_i(x) \subseteq f(h_i^{1-1}(x), f(h_i^{m+i-1}(x)), h_{m+i-1}^{2m-1}(x)).
\]

Conversely, if \( h^\prime_i \) is an element of \( F(H) \) such that

\[
h_i(x) \subseteq f(h_i^{1-1}(x), f(h_i^{m+i-1}(x)), h_{m+i-1}^{2m-1}(x))
\]

for all \( x \in H \), and if we choose \( h^\prime_i \) such that \( h^\prime_i(x) = f(h_i^{m+i-1}(x)) \), for all \( x \in H \), then \( h_i \in \circledast(h_i^{m+i-1}) \) and \( h_i^\prime \in \circledast(h_i^{1-1}, h_i^{m+i-1}) \). i.e., \( h_i^\prime \in L \). So,

\[
L = \{h_i \in F(H) \mid (\forall x \in H) h_i(x) \subseteq f(h_i^{1-1}(x), f(h_i^{m+i-1}(x)), h_{m+i-1}^{2m-1}(x))\}.
\]

On the other hand set:

\[
D = \circledast(h_i^{1-1}, h_i^{m+i-1}, h_{m+i-1}^{2m-1}).
\]

Then,

\[
D = \{h_i \in F(H) \mid (\forall x \in H) h_i(x) \subseteq f(h_i^{1-1}(x), f(h_i^{m+i-1}(x)), h_{m+i-1}^{2m-1}(x))\}.
\]

By the associativity of hyperoperation \( f \), we obtain \( L = D \).

Let \( i \in \{1, ..., m\} \) and \( h, h_i^{1-1}, h_i^{m+i-1} \in F(H) \). We prove that equation

\[
h \in \circledast(h_i^{1-1}, h_i^{m+i-1})
\]

has a solution \( h_i \in F(H) \). If we set \( h_i(x) = H \) for all \( x \in H \), then \( h_i \in F(H) \) and for all \( x \in H \) it holds:

\[
f(h_i^{1-1}(x), h_i(x), h_i^{m+i-1}(x)) = H \supseteq h(x).
\]

So, \( h \in \circledast(h_i^{1-1}, h_i, h_i^{m+i-1}) \). Thus, \( (F(H), \circledast) \) is an \( m \)-ary hypergroup.

Now we prove that \( (F(H), \circledast) \) is an \( n \)-ary hypersemigroup. Let \( h_i^m \in F(H) \). For all \( x \in H, h_n(x) \neq \emptyset \). Hence,

\[
(h_1 h_2 ... h_n)(x) \neq \emptyset.
\]
Let $h : H \to P'(H)$ be a map defined by $h(x) = (h_1, \ldots, h_n)(x)$. We want to prove that $h \in \otimes(h_i^n)$ i.e. that $\otimes$ is an $n$-ary hyperoperation. For any $a_i^n \in H$ it holds:

$$h(f(a_i^n)) = (h_1h_2\ldots h_n)(f(a_i^n)) = (h_1h_2\ldots h_n)(h_n(f(a_i^n))) \subseteq (h_1h_2\ldots h_n-1)(f(h(a_1), \ldots, h(a_n)))$$
$$\subseteq (h_1h_2\ldots h_n-2)(f(h(a_1), \ldots, h(a_n), h(a_2), \ldots, h(a_n))) \subseteq \cdots \subseteq$$
$$\subseteq f \{ (h_1h_2\ldots h_n)(a_1), \ldots, (h_1h_2\ldots h_n)(a_n) \} = f(h(a_1), \ldots, h(a_n)).$$

So, $h \in \otimes(h_i^n)$.

Let us prove that $\otimes$ is associative. Let $i, j \in \{1, \ldots, n\}$ and $h_{ij}^{2n-1} \in F(H)$. Set

$$L = \circlearrowleft (h_i^{j-1}, \circlearrowleft (h_i^{j+1}, h_{ij}^{2n-1})$$
and

$$D = \circlearrowleft (h_i^{j-1}, \circlearrowleft (h_i^{j+1}, h_{ij}^{2n-1}).$$

Then

$$L = \bigcup \{ \circlearrowleft (h_i^{j-1}, h', h_{ij}^{2n-1}) \mid h' \in F(H) \land (\forall x \in H) h'(x) \subseteq (h_1, \ldots, h_{n+1-1})(x) \}. $$

So, if $h'' \in L$ then $h''(x) \subseteq (h_1, h_{2n-1})(x)$, for all $x \in H$. On the other hand if $h'' \in F(H)$ and $h''(x) \subseteq (h_1, h_{2n-1})(x)$ for all $x \in H$, then we choose $h'' \in F(H)$ such that $h''(x) = (h_1, h_{n+1-1})(x)$ and consequently we obtain $h'' \in \otimes(h_i^{j-1}, h', h_{ij}^{2n-1})$ where $h' \in \otimes(h_i^{j+1})$. Thus, $h'' \in L$. So,

$$L = \{ h'' \in F(H) \mid (\forall x \in H) h''(x) \subseteq (h_1, h_{2n-1})(x) \}. $$

Similarly,

$$D = \{ h'' \in F(H) \mid (\forall x \in H) h''(x) \subseteq (h_1, h_{2n-1})(x) \}. $$

Thus, $L = D$.

Now we prove that the $n$-ary hyperoperation $\otimes$ is distributive with respect to the $m$-ary hyperoperation $\oplus$.

Let $h_i^{j-1}, h_i^{j+1}, g_i^n \in F(H), 1 \leq i \leq n$. Set

$$L = \circlearrowleft (h_i^{j-1}, \bigoplus (g_i^n), h_i^{j+1}) = \bigcup \{ \circlearrowleft (h_i^{j-1}, h', h_i^{j+1}) \mid h' \in \oplus(g_i^n) \}$$
$$= \bigcup \{ \circlearrowleft (h_i^{j-1}, h', h_i^{j+1}) \mid h' \in F(H) \land (\forall x \in H) h'(x) \subseteq f(g_1(x), \ldots, g_m(x)) \}.$$ 

So, if $k \in L$ then for all $x \in H$, it holds:

$$k(x) \subseteq (h_1, h_{j-1})(f((g_1h_{i+1}h_n)(x), \ldots, (g_mh_{i+1}h_n)(x)))$$
$$\subseteq (h_1, h_{j-2})(f((h_{i+1}g_1h_{i+1}h_n)(x), \ldots, (h_{i+1}g_mh_{i+1}h_n)(x)))$$
$$\subseteq \cdots \subseteq f((h_1, h_{j-1}g_1h_{i+1}h_n)(x), \ldots, (h_1, h_{j-1}g_mh_{i+1}h_n)(x)).$$

On the other hand,

$$D = \bigoplus (\bigcirc (h_i^{j-1}, g_1^n), \ldots, \bigcirc (h_i^{j-1}, g_m^n), (h_i^{j+1})) = \bigcup \{ (k_1, \ldots, k_m) \mid k_j \in \oplus(h_i^{j-1}, g_j^n), j \in \{1, 2, \ldots, m\} \}.$$ 

Let $k \in L$. Choose $k_1, \ldots, k_m \in F(H)$ such that for all $j \in \{1, 2, \ldots, m\}$

$$k_j(x) = (h_1, h_{j-1}g_1h_{i+1}h_n)(x), \text{ for all } x \in H.$$ 

Then $k_j \in \otimes(h_i^{j-1}, g_j^n)$ and $k \in \oplus(k_1, \ldots, k_m)$. Thus, $k \in D$. So, $L \subseteq D$. 

\[\Box\]
Remark 3.2. If \((H, f)\) is an \(m\)-ary hypergroup that satisfies condition (1) then for any \(n \geq 2\), there exists \((m, n)\)-ary hyperring \((F(H), \oplus, \odot)\). The structure \((F(H), \oplus, \odot)\) will be called \((m, n)\)-ary hyperring of multiendomorphisms of \(m\)-ary hypergroup \((H, f)\).

Remark 3.3. If \((H, f)\) is a commutative binary hypergroup, then \((H, f)\) satisfies condition (1) of previous theorem. Thus, the binary hyperring of multiendomorphisms of commutative binary hypergroup \((H, f)\) is a special case of \((m, n)\)-ary hyperring constructed in Theorem 3.1. But, the following example shows that there also exist noncommutative hypergroups, that satisfy condition (1), implying that we can associate a hyperring of multiendomorphisms with noncommutative hypergroup.

Example 3.4. If \(H = \{x, y, z\}\) and \(f\) is defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
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<tbody>
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<td>(x)</td>
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<td>(y)</td>
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<td>(z)</td>
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then \((H, f)\) is a noncommutative binary hypergroup which satisfies condition (1).

4. \((m, n)\)-ary hyperrings associated with binary relations

In this section we construct a class of \((m, n)\)-ary hyperrings associated with binary relations on semigroup. Then, we investigate their morphisms and we also, establish connection between the constructed \((m, n)\)-ary hyperring \((H, f, g)\) and the hyperring of multiendomorphisms of \(m\)-hypergroup \((H, f)\).

Theorem 4.1. Let \((H, \cdot)\) be a semigroup equipped with binary relations \(\rho_1\) and \(\rho_2\) such that \(\rho_1 \subseteq \rho_2\). Let \(\rho_i\) \((i = 1, 2)\) be a reflexive and transitive relation such that for all \(a, b, x \in H\),

\[
(a, b) \in \rho_i \text{ implies } (a \cdot x, b \cdot x) \in \rho_i \text{ and } (x \cdot a, x \cdot b) \in \rho_i.
\]

We define an \(m\)-ary hyperoperation \(f\) and an \(n\)-ary hyperoperation \(g\) on \(H\), as follows:

\[
f(a_1^n) = \left\{ z \mid (a_1, z) \in \rho_1 \lor (a_2, z) \in \rho_1 \lor \cdots \lor (a_m, z) \in \rho_1 \right\}
\]

for any \(a_1^n \in H\), and \(g(a_1^n) = \left\{ z \mid a_1 \cdot a_2 \cdot \cdots \cdot a_n \rho_2 z \right\}\) for any \(a_1^n \in H\). The structure \((H, f, g)\) is a strong distributive \((m, n)\)-ary hyperring.

Proof. Since \(\rho_1\) is reflexive and transitive relation, then by Theorem 2.5, \((H, f)\) is an \(m\)-ary hypergroup.

Now we prove that \((H, g)\) is an \(n\)-ary hypersemigroup. Since \(\rho_2\) is reflexive, then for any \(a_1^n \in H\) it holds \(g(a_1^n) \neq \emptyset\) i.e. \(g\) is an \(n\)-ary hyperoperation. Let \(i, j \in \{1, \ldots, n\}\) and \(a_1^{2n-1} \in H\).

Set

\[
L = g(a_1^{i-1}, g(a_1^{n+i-1}, a_2^{2n-1})) = \bigcup \left\{ g(a_1^{i-1}, z, a_2^{2n-1}) \mid z \in g(a_1^{n+i-1}) \right\}
\]

and

\[
D = g(a_1^{i-1}, g(a_1^{n+i-1}, a_2^{2n-1})) = \bigcup \left\{ g(a_1^{i-1}, \delta, a_2^{2n-1}) \mid \delta \in g(a_1^{n+j-1}) \right\}.
\]

Suppose \(w \in L\). Then there exists \(z \in g(a_1^{n+i-1})\) such that \(w \in g(a_1^{i-1}, z, a_2^{2n-1})\). Thus, \((a_1 \cdot \cdots \cdot a_{n+i-1}, z) \in \rho_2\) and \((a_1 \cdot \cdots \cdot a_{n+i-1} \cdot z \cdot a_{n+i+1} \cdot \cdots \cdot a_{2n-1}, \omega) \in \rho_2\). By the condition (2) we have

\[
(a_1 \cdot \cdots \cdot a_{n+i-1} \cdot a_1 \cdot \cdots \cdot a_{n+i} \cdot a_{n+i+1} \cdot \cdots a_{2n-1} \cdot z \cdot a_{n+i+1} \cdot \cdots \cdot a_{2n-1}) \in \rho_2.
\]
Hence, there exists $k \in \mathbb{D}$ such that $y \in g(a_{i+1}^{-1})$ and $w \in g(a_{i}^{-1}, \delta, a_{n+1}^{-1})$, i.e. $w \in D$. So, $L \subseteq D$. Similarly, we obtain $D \subseteq L$.

Now we prove that $n$-ary hyperoperation $g$ is strong distributive with respect to the $m$-ary hyperoperation $f$. Let $i \in \{1, \ldots, n\}$ and $a_{1}, \ldots, a_{n+1}, x^{m} \in H$. Set

$$L = \{g(a_{i}^{-1}, f(x^{m}_{i})), a_{n+1}^{-1}) \mid w \in f(x^{m}_{i})\}$$

and

$$D = f(g(a_{i}^{-1}, x_{1}, a_{n+1}^{-1}), \ldots, g(a_{i}^{-1}, x_{m}, a_{n+1}^{-1})) = \{f(\delta_{1}, \ldots, \delta_{m}) \mid \delta_{1} \in g(a_{1}^{-1}, x_{1}, a_{n+1}^{-1}), \ldots, \delta_{m} \in g(a_{1}^{-1}, x_{m}, a_{n+1}^{-1})\}.$$

If $y \in L$, then there exists $w \in f(x^{m}_{i})$ such that $y \in g(a_{i}^{-1}, w, a_{n+1}^{-1})$. Thus, there exists $k \in \{1, \ldots, m\}$ such that $(x_{k}, w) \in \rho_{1} \subseteq \rho_{2}$ and $(a_{1} \cdot \ldots \cdot a_{i-1} \cdot w \cdot a_{i+1} \cdot \ldots \cdot a_{n}, y) \in \rho_{2}$. By condition (2) we obtain

$$(a_{1} \cdot \ldots \cdot a_{i-1} \cdot x_{k} \cdot a_{i+1} \cdot \ldots \cdot a_{n}, a_{1} \cdot \ldots \cdot a_{i-1} \cdot w \cdot a_{i+1} \cdot \ldots \cdot a_{n}) \in \rho_{2},$$

while $(a_{1} \cdot \ldots \cdot a_{i-1} \cdot w \cdot a_{i+1} \cdot \ldots \cdot a_{n}, y) \in \rho_{2}$. Since $\rho_{2}$ is transitive we obtain $y \in g(a_{i}^{-1}, x_{k}, a_{n+1}^{-1})$.

So, if we choose $\delta_{1}, \ldots, \delta_{m}$ such that $\delta_{1} \in g(a_{1}^{-1}, x_{1}, a_{n+1}^{-1})$ for $l \in \{1, 2, \ldots, m\}$ and $\delta_{k} = y$, then $y \in f(\delta_{1}, \ldots, \delta_{m})$, i.e., $y \in D$.

Suppose now $y \in D$. Then there exist $\delta_{1} \in g(a_{1}^{-1}, x_{1}, a_{n+1}^{-1})$, $\ldots$, $\delta_{m} \in g(a_{1}^{-1}, x_{m}, a_{n+1}^{-1})$ such that $y \in f(\delta_{1}, \ldots, \delta_{m})$.

Hence, there exists $k \in \{1, \ldots, m\}$ such that $(\delta_{k}, y) \in \rho_{1} \subseteq \rho_{2}$ while $(a_{1} \cdot \ldots \cdot a_{i-1} \cdot x_{k} \cdot a_{i+1} \cdot \ldots \cdot a_{n}, \delta_{k}) \in \rho_{2}$. Since $\rho_{2}$ is transitive we obtain $(a_{1} \cdot \ldots \cdot a_{i-1} \cdot x_{k} \cdot a_{i+1} \cdot \ldots \cdot a_{n}, y) \in \rho_{2}$ i.e. $y \in g(a_{i}^{-1}, x_{k}, a_{n+1}^{-1})$. As $x_{k} \in f(x^{m}_{i})$, we have $y \in L$.

Therefore, $D = L$. \qed

Throughout the following text the quadruple $(H, \cdot, \rho_{1}, \rho_{2})$ will denote a semigroup $(H, \cdot)$ equipped with binary relations $\rho_{1}$ and $\rho_{2}$ such that $\rho_{1}$ and $\rho_{2}$ satisfy the conditions of Theorem 4.1. By an $(m, n)$-ary hyperring associated with $(H, \cdot, \rho_{1}, \rho_{2})$ we mean an $(m, n)$-ary hyperring $(H, f, g)$ constructed in Theorem 4.1.

**Theorem 4.2.** Let $(H, f, g)$ be an $(m, n)$-ary hyperring associated with $(H, \cdot, \rho_{1}, \rho_{2})$ and $(F(H), \oplus, \odot)$ be an $(m, n)$-ary hyperring of multidentomorphisms of the $m$-ary hypergroup $(H, f)$.

If we define a mapping $\varphi : (H, f, g) \rightarrow (F(H), \oplus, \odot)$ by $\varphi(a) = h_{a}$ for all $a \in H$, where $h_{a} : H \rightarrow P^{*}(H)$ is defined by:

$$h_{a}(x) = f(\underbrace{a, \ldots, a, x}_{m-1 \text{ times}}), \text{ for all } x \in H,$$

then the following holds:

1. $\varphi(f(a^{m}_{n})) \subseteq \bigoplus_{\varphi(a_{1})} \cdots \varphi(a_{m})$, for all $a^{m}_{n} \in H$.
2. If

$$(a \cdot b, w) \in \rho_{2} \Rightarrow (a, w) \in \rho_{1} \text{ or } (b, w) \in \rho_{1}$$

for any triple of elements $a, b, w \in H$, then

$$\varphi(g(a^{n}_{n})) \subseteq \bigodot_{\varphi(a_{1})} \cdots \varphi(a_{n})$$

for any $a^{n}_{n} \in H$.
3. If $\rho_{1}$ is an order, then $\varphi$ is injective.
Proof. First notice that for any \(a_{11}^{m}, a_{21}^{m}, ..., a_{m1}^{m} \in H\), it holds:
\[
f(a_{11}^{m}), ..., f(a_{m1}^{m}) = \{ z \mid \exists x, l \in \{1, ..., m\}, (a_{lj}, z) \in \rho_1 \} = f(f(a_{11}^{m}), ..., f(a_{m1}^{m})).
\]

Thus, by Theorem 3.1, there exists an \((m, n)\)-ary hyperpoint \((F(H), \oplus, \odot)\).

Now we verify that \(h_a \in F(H)\), for any \(a \in H\). Let \(a_{1}^{m} \in H\). Set
\[
L = h_a(f(a_{11}, ..., a_{m})) = \bigcup \{ h_a(x) \mid x \in f(a_{11}, ..., a_{m}) \} = f(a_{11}, ..., f(a_{m})).
\]
and
\[
D = f(h_a(a_{1}), ..., h_a(a_{m})) = \bigcup \{ f(x_1, ..., x_m) \mid x_j \in f(a_{1}, ..., a_{j}), j = 1, ..., m \}.
\]

Let \(z \in L\). We have the following possibilities:

(i) If \((a, z) \in \rho_1\), we put \(x_1 = z_1 = ... = z_m = a\) and then \(z \in f(x_1, ..., x_m)\) and
\[
x_j \in f(a_{1}, ..., a_{j}), j = 1, ..., m-1
\]
for all \(j \in \{1, ..., m\}\). So, \(z \in D\).

(ii) If there exists \(u \in f(a_{1}, ..., a_{m})\) such that \((u, z) \in \rho_1\), then, there exists \(i \in \{1, ..., m\}\) such that \((a_i, u) \in \rho_1\) and \((u, z) \in \rho_1\). By transitivity of \(\rho_1\), we have \((a_i, z) \in \rho_1\). If we put \(x_i = z\) and \(x_1 = ... = x_{i-1} = x_{i+1} = ... = x_m = a\), then \(z \in f(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_m)\) and
\[
x_j \in f(a_{1}, ..., a_{j}), j = 1, ..., m-1
\]
for all \(j \in \{1, ..., m\}\).

So, \(z \in D\).

Thus, \(h_a \in F(H)\).

(1) Let \(a_{1}^{m} \in H\). Set:
\[
L = \phi(f(a_{1}^{m})) = \{ h_w \mid (a_{1}, w) \in \rho_1 \lor \ldots \lor (a_{m}, w) \in \rho_1 \}
\]
and
\[
D = \bigoplus \{ \phi(a_{1}), ..., \phi(a_{m}) \} = \{ h \in F(H) \mid (\forall x \in H) h(x) \subseteq f(h_{a_{1}}(x), ..., h_{a_{m}}(x)) \}.
\]

Let \(h_w \in L\) and \(x \in H\). Then
\[
h_w(x) = f(w, ..., w, x) = \{ z \mid (w, z) \in \rho_1 \lor (x, z) \in \rho_1 \}.
\]

Suppose \(z \in h_w(x)\). We have two possibilities:

(i) If \((w, z) \in \rho_1\), since \(h_w \in L\), then there exists \(j \in \{1, ..., m\}\) such that \((a_j, w) \in \rho_1\), and by the transitivity of \(\rho_1\) we have \((a_j, z) \in \rho_1\), i.e., \(z \in f(a_{1}, ..., a_{j}, x) = h_{a_{j}}(x)\).

Since \(h_{a_{j}}(x) \subseteq f(h_{a_{1}}(x), ..., h_{a_{m}}(x))\), then \(z \in f(h_{a_{1}}(x), ..., h_{a_{m}}(x))\).

(ii) If \((x, z) \in \rho_1\), then \(z \in f(x, ..., x)\). Since \(x \in h_{a_{1}}(x), ..., x \in h_{a_{m}}(x)\), then \(z \in f(h_{a_{1}}(x), ..., h_{a_{m}}(x))\).
So, \( h_\omega(x) \subseteq f(h_\omega(x), ..., h_\omega(x)) \), for all \( x \in H \) i.e. \( h_\omega \in D \). Thus \( L \subseteq D \).

(2) Let \( a^\omega_i \in H \). First, notice that for any \( x \in H \) and \( i \in \{1, ..., n\} \) it holds \( h_\omega(x) \subseteq (h_\omega, ..., h_\omega)(x) \) and \( h_\omega(x) \subseteq (h_\omega, ..., h_\omega, x)(x) \).

Indeed, since \( y \in h_\omega(y) \) for all \( y \in H \) and \( 1 \leq j \leq n \), then \( h_\omega(x) \subseteq h_\omega, ..., h_\omega(x) \) and \( h_\omega, ..., h_\omega(x) \subseteq h_\omega, ..., h_\omega, x(x) \).

Thus, \( h_\omega(x) \subseteq (h_\omega, ..., h_\omega)(x) \). So, after finite number of steps we obtain:

\[
h_\omega(x) \subseteq (h_\omega, ..., h_\omega)(x).
\]  

(4)

For the second inclusion we proceed in a similar way.

As \( x \in h_\omega(x) \) then \( h_\omega(x) \subseteq (h_\omega, ..., h_\omega)(x) \). Thus, \( x \in (h_\omega, ..., h_\omega)(x) \) implying that \( h_\omega(x) \subseteq (h_\omega, ..., h_\omega, h_\omega)(x) \).

After finite number of steps we obtain

\[
h_\omega(x) \subseteq (h_\omega, ..., h_\omega)(x).
\]  

(5)

From (4) and (5) it follows \( h_\omega(x) \subseteq h_\omega, ..., h_\omega(x) \). Now, set

\[
L = \varphi(\sigma(a^n_i)) = \{ h_b \mid (a_1 \cdot ... \cdot a_n, b) \in \rho_2 \}
\]

and

\[
D = \bigcirc \{ \varphi(a_1), ..., \varphi(a_n) \} = \{ h \in F(H) \mid (\forall x \in H) h(x) \subseteq (h_\omega, ..., h_\omega)(x) \}.
\]

Let \( h_b \in L \) and \( x \in H \). Then

\[
h_b(x) = f(b, ..., b, x) = \{ z \mid (b, z) \in \rho_1 \lor (x, z) \in \rho_1 \}.
\]

If \( z \in h_b(x) \) we have the following possibilities:

(i) If \( (b, z) \in \rho_1 \), since \( h_b \in L \), then \( (a_1 \cdot ... \cdot a_n, b) \in \rho_2 \). As \( \rho_1 \subseteq \rho_2 \), by transitivity of \( \rho_2 \) we have \( (a_1 \cdot ... \cdot a_n, z) \in \rho_2 \).

By the condition (3), there exists \( i \in \{1, ..., n\} \) such that \( (a_i, z) \in \rho_1 \) i.e. \( z \in h_\omega(x) \subseteq (h_\omega, ..., h_\omega)(x) \).

(ii) If \( (x, z) \in \rho_1 \), then \( z \in h_\omega(x) \subseteq (h_\omega, ..., h_\omega)(x) \).

Thus, \( h_b(x) \subseteq (h_\omega, ..., h_\omega)(x) \), for all \( x \in H \) i.e. \( h_b \in D \).

(3) Let \( \rho_1 \) be an order on \( H \). Suppose \( a, b \in H \) and \( \varphi(a) = \varphi(b) \) i.e. \( h_a = h_b \).

Then, \( h_a(a) = h_b(a) \) and \( h_a(b) = h_b(b) \). Thus,

\[
f(a_1, ..., a, b) = f(b_1, ..., b, a) \quad \text{and} \quad f(a_1, ..., a, b) = f(b_1, ..., b).
\]

Since,

\[
f(a_1, ..., a, b) = f(b_1, ..., b, a),
\]

then \( f(a, ..., a) = f(b, ..., b) \).

From \( a \in f(a, ..., a) \) it follows \( a \in f(b, ..., b) \), i.e., \( (b, a) \in \rho_1 \). Similarly, it is proved \( (a, b) \in \rho_1 \). As \( \rho_1 \) is an order, we obtain \( a = b \). \( \Box \)
Example 4.3. Notice that \((N, \cdot, \leq, \leq)\) satisfies the conditions of Theorem 4.1. Thus, there exists an \((m, n)\)-ary hyperring \((N, f, g)\) associated with \((N, \cdot, \leq, \leq)\).

For all \(k^n_1 \in N\) and \(k^n_2 \in N\) we have

\[
f(k^n_1) = \{ k \in N | \min[k_1, \ldots, k_m] \leq k \} \quad \text{and} \quad g(k^n_2) = \{ k \in N | k_1 \cdot \ldots \cdot k_n \leq k \}.
\]

It is easy to see that \((N, \cdot, \leq, \leq)\) satisfies the conditions of Theorem 4.2. So, there exists an inclusion morphism of \((N, f, g)\) into \((F(N), \oplus, \odot)\).

Definition 4.4. Let the triples \((H_1, \rho_1, \rho_2)\) and \((H_2, \delta_1, \delta_2)\) denote the nonempty set \(H_1\) equipped with binary relations \(\rho_1, \rho_2\) and nonempty set \(H_2\) with binary relations \(\delta_1, \delta_2\).

(a) The map \(\alpha : H_1 \to H_2\) is said to be isotone if

\[
x \rho; y \implies \alpha(x) \delta; \alpha(y),
\]

for all \(x, y \in H_1\) and \(i \in \{1, 2\}\).

(b) The map \(\alpha : H_1 \to H_2\) is said to be strongly isotone if

\[
\alpha(x) \delta; y \iff (\exists x' \in H_1) x \rho; x' \land \alpha(x') = y,
\]

for all \((x, y) \in H_1 \times H_2\) and \(i \in \{1, 2\}\).

Theorem 4.5. Let \((H_1, f_1, g_1)\) be an \((m, n)\)-ary hyperring associated with \((H_1, \cdot, \rho_1, \rho_2)\) and \((H_2, f_2, g_2)\) be an \((m, n)\)-ary hyperring associated with \((H_2, \cdot, \delta_1, \delta_2)\).

(1) If \(\alpha : (H_1, \cdot) \to (H_2, \cdot)\) is an isotone homomorphism of semigroups \((H_1, \cdot)\) and \((H_2, \cdot)\) then \(\alpha : (H_1, f_1, g_1) \to (H_2, f_2, g_2)\) is an inclusion homomorphism.

(2) If \(\alpha : (H_1, \cdot) \to (H_2, \cdot)\) is a strongly isotone homomorphism, then \(\alpha : (H_1, f_1, g_1) \to (H_2, f_2, g_2)\) is a strong homomorphism.

Proof. (1) Let \(\alpha : (H_1, \cdot) \to (H_2, \cdot)\) be an isotone homomorphism and \(x^n_1 \in H_1\). If \(w \in \alpha(f_1(x^n_1))\), then there exists \(z \in H_1\) such that \(w = \alpha(z)\) and \((x_i, z) \in \rho_1\) for some \(i \in \{1, \ldots, m\}\).

Since, \(\alpha\) is isotone then \((\alpha(x_i), \alpha(z) = w) \in \delta_1\) and so \(w \in f_2(\alpha(x_1), \ldots, \alpha(x_m))\). Thus \(\alpha(f_1(x^n_1)) \subseteq f_2(\alpha(x_1), \ldots, \alpha(x_m))\).

Now, let \(y^n_2 \in H_2\). To prove that \(\alpha(g_1(y^n_2)) \subseteq g_2(\alpha(y_1), \ldots, \alpha(y_n))\) we can proceed similarly as in the proof of Theorem 3.2. (ii) in [19].

Therefore, \(\alpha : (H_1, f_1, g_1) \to (H_2, f_2, g_2)\) is an inclusion homomorphism.

(2) Let \(\alpha : (H_1, \cdot) \to (H_2, \cdot)\) be a strongly isotone homomorphism. Since \(\alpha\) is isotone, then by (1) we obtain that \(\alpha : (H_1, f_1, g_1) \to (H_2, f_2, g_2)\) is an inclusion homomorphism.

Thus, for any \(x^n_1 \in H_1\), it holds \(\alpha(f_1(x^n_1)) \subseteq f_2(\alpha(x_1), \ldots, \alpha(x_m))\).

Suppose \(w \in f_2(\alpha(x_1), \ldots, \alpha(x_m))\). Then \((\alpha(x_i), w) \in \delta_1\) for some \(i \in \{1, \ldots, m\}\). Since \(\alpha\) is strongly isotone, then there exists \(z \in H_1\) such that \((x_i, z) \in \rho_1\) and \(\alpha(z) = w\). Thus, \(w = \alpha(z) \in \alpha(f_1(x^n_1))\). Therefore,

\[
f_2(\alpha(x_1), \ldots, \alpha(x_m)) \subseteq \alpha(f_1(x^n_1)).
\]

Thus, \(\alpha(f_1(x^n_1)) = f_2(\alpha(x_1), \ldots, \alpha(x_m))\).

Now, let \(y^n_2 \in H_1\). In similar way as in the proof of Theorem 3.3. (i) in [19], we prove that

\[
\alpha(g_1(y^n_2)) = g_2(\alpha(y_1), \ldots, \alpha(y_n)).
\]

This completes the proof. \(\Box\)
References