A commutator approach to Buzano’s inequality

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Abstract. Using a 2×2 matrix trick, an inequality involving commutators of certain Hilbert space operators as an operator version of Buzano’s inequality, which is in turn a generalization of the Cauchy–Schwarz inequality, is presented. Also a version of the inequality in the framework of Hilbert C∗-modules is stated and a special case in the context of C∗-algebras is presented.

1. Introduction and preliminaries

In [4], Buzano obtained the following extension of the celebrated Cauchy–Schwarz inequality in a real or complex inner product space H:

\[ |(a,x)(x,b)| \leq \frac{1}{2} (|(a,b)| + ||a|| ||b||) ||x||^2 \quad (a,b,x \in H) \]

When \( a = b \) this inequality becomes the Cauchy–Schwarz inequality

\[ |(a,x)|^2 \leq ||a|| ||x||^2. \]

For a real inner product space, Richard [18] independently obtained the following stronger inequality:

\[ |(a,x)(x,b) - \frac{1}{2} (a,b)||x||^2| \leq \frac{1}{2} ||a|| ||b|| ||x||^2 \quad (a,b,x \in H). \]

Dragomir [5] showed that this inequality (for real or complex case) is valid with coefficients \( \frac{1}{|1 - \alpha|} \) instead of \( \frac{1}{2} \), where a non-zero number \( \alpha \) satisfies the equality \( |1 - \alpha| = 1 \). As an application of this inequality, Fujii and Kubo [9] found a bound for roots of algebraic equations. During developing the operator theory and its applications, the authors of [6] have recently extended some numerical inequalities to operator inequalities.

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Some mathematicians have also investigated the operator versions of the Cauchy–Schwarz inequality or its reverse; see [7, 8, 12, 16, 19].

In the next section an operator version of Buzano’s inequality is introduced as a commutator inequality and in the last section we state a suitable version of it for Hilbert $C^*$-modules including $C^*$-algebras.

In this paper, $\mathcal{B}(\mathcal{H})$ stands for the $C^*$-algebra of all bounded operators on a complex separable Hilbert space $\mathcal{H}$ equipped with the usual operator norm $\| \cdot \|$. If $\mathcal{H}$ is finite-dimensional with $\dim \mathcal{H} = n$, we identify $\mathcal{B}(\mathcal{H})$ with the full matrix algebra $M_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$.

If $x, y \in \mathcal{H}$, the rank-one operator $x \otimes y$ is defined by

$$(x \otimes y)z = (z, y)x \quad (z \in \mathcal{H})$$

For a compact operator $T \in \mathcal{B}(\mathcal{H})$, the singular values of $T$ are defined to be the eigenvalues of the positive operator $|T| = (T^*T)^{1/2}$, enumerated as $s_1(T) \geq s_2(T) \geq \cdots$ with their multiplicities counted. If $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$, we use the direct sum notation $S \oplus T$ for the block-diagonal operator

$$\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$$

defined on $\mathcal{H} \oplus \mathcal{H}$.

It can be easily seen that the set of singular values of $S \oplus T$ is the union of those of $S$ and $T$. In particular, the operator norm of $S \oplus T$ is the maximum of the norm of $S$ and $T$. For $A, B, X \in \mathcal{B}(\mathcal{H})$, the operator $AX - XA$ is called a commutator and the operator $AX - XB$ is said to be a generalized commutator. There are several results related to the singular values and unitarily invariant norms of (generalized) commutators, see [3, 11, 13, 14] and references therein. Recall that a norm $\| \cdot \|$ on $M_n$ is said to be unitarily invariant if $\|UAV\| = \|A\|$ for all $A \in M_n(\mathbb{C})$ and all unitary matrices $U, V \in M_n(\mathbb{C})$.

The notion of Hilbert $C^*$-module is a generalization of that of Hilbert space. Let $\mathcal{A}$ be a $C^*$-algebra, and let $\mathcal{X}$ be a complex linear space, which is a right $\mathcal{A}$-module satisfying $\lambda(\alpha a) = x(\lambda a) = (\lambda x)a$ for all $x \in \mathcal{X}, a \in \mathcal{A}, \lambda \in \mathbb{C}$. The space $\mathcal{X}$ is called a (right) pre-Hilbert $C^*$-module over $\mathcal{A}$ if there exists an $\mathcal{A}$-inner product $(\cdot, \cdot) : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying

(i) $(x, y) \geq 0$ (i.e. $(x, x)$ is a positive element of $\mathcal{A}$) and $(x, y) = 0$ if and only if $x = 0$;

(ii) $(x, \lambda y + z) = \lambda (x, y) + (x, z)$;

(iii) $(x, ya) = (x, y)a$;

(iv) $(x, y)^* = (y, x)$;

for all $x, y, z \in \mathcal{X}, \lambda \in \mathbb{C}, a \in \mathcal{A}$.

We can define a norm on $\mathcal{X}$ by $\|\cdot\| : = \|x, x\|^{1/2}$, where the latter norm denotes that in the $C^*$-algebra $\mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module is called a (right) Hilbert $C^*$-module over $\mathcal{A}$ (or a (right) Hilbert $\mathcal{A}$-module) if it is complete with respect to its norm. Any inner product space can be regarded as a pre-Hilbert $C^*$-module and any $C^*$-algebra $\mathcal{A}'$ is a Hilbert $C^*$-module over itself via $(a, b) = \alpha b$ ($a, b \in \mathcal{A}'$). For more information about $C^*$-algebras and Hilbert $C^*$-modules see [17] and [15], respectively.

2. The Hilbert space case

To establish singular value inequalities for Hilbert space operators, we need the following lemma, which is an immediate consequence of the Maximin principle (see, e.g., [2, p. 75] or [10, p. 27]).

**Lemma 2.1.** Suppose that $X, Y, Z \in \mathcal{B}(\mathcal{H})$. If $Y$ is compact, then

$$s_j(XYZ) \leq \|X\|\|Z\|s_j(Y)$$

for all $j = 1, 2, \ldots$

Now we state our main result.

**Theorem 2.2.** Let $A, B, X \in \mathcal{B}(\mathcal{H})$ be such that $A$ is invertible and it commutes with $X$. Suppose that, for some Hilbert space $\mathcal{H}$, $\overline{A} = A \oplus A' \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, $\overline{B} = B \oplus 0 \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ and $\overline{X} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is any compact extension of $X$. Then, for $j = 1, 2, \ldots$,

$$s_j(A\overline{X} - \overline{XB}) \leq \max\{1, \|I - A^{-1}B\|\|\overline{A}\|\} s_j(\overline{X}).$$
If $\mathcal{H} = \{0\}$, then
\[
s_j(AX -XB) \leq \|A - B\| s_j(X) \leq \|I - A^{-1}B\| \|A\| s_j(X).
\]

**Proof.** Since $\overline{X}$ leaves $\mathcal{H}$ invariant, we can write
\[
\overline{A} = \begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{X} = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}.
\]

It follows from Lemma 2.1 and
\[
\overline{AX} - \overline{XB} = \begin{bmatrix} AX -XB \\ 0 \end{bmatrix} \begin{bmatrix} AY \\ A'Z \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix} \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} I - A^{-1}B \\ 0 \end{bmatrix}
\]

that
\[
s_j(\overline{AX} - \overline{XB}) \leq \max\{1, \|I - A^{-1}B\|\} \|\overline{A}\| s_j(\overline{X}).
\]

If $\mathcal{H} = \{0\}$, then $s_j(AX -XB) = s_j(X(A - B)) \leq \|A - B\| s_j(X)$ by Lemma 2.1.

**Corollary 2.3.** Let $A, B, X$ be $n \times n$ matrices such that $A$ is invertible and it commutes with $X$. Suppose that $A = A \oplus A'$, $B = B \oplus 0$ and $\overline{X}$ is any extension of $X$ to $\mathbb{C}^n \oplus \mathbb{C}^m$ for some $m$. Then

(i) $\|\overline{AX} - \overline{XB}\| \leq \max\{1, \|I - A^{-1}B\|\} \|\overline{A}\| \|\overline{X}\|$ for every unitarily invariant norm $\| \cdot \|$ on $\mathbb{C}^{n+m}$.

(ii) $\|AX - XB\| \leq \max\{1, \|I - A^{-1}B\|\} \|\overline{A}\| \|\overline{X}\|$ for some unitary matrix $U \in M_{n+m}(\mathbb{C})$.

**Proof.** It follows from Theorem 2.2 that, for each $k = 1, 2, \ldots, n + m$,
\[
\sum_{j=1}^{k} s_j(\overline{AX} - \overline{XB}) \leq \sum_{j=1}^{k} \max\{1, \|I - A^{-1}B\|\} \|\overline{A}\| s_j(\overline{X}).
\]

The Ky Fan dominance theorem (see, e.g., [2, p. 93]) then completes the proof of (i).

The assertion (ii) follows from the fact that for positive matrices $S, T$ the inequalities $s_j(S) \leq s_j(T)$ ($1 \leq j \leq n + m$) are equivalent to $S \leq UTU^*$ for some unitary matrix $U$.

The next result may be considered as a slight generalization of [13, Lemma 3] in the case when $X \in \mathcal{B}(\mathcal{H})$ is a compact operator leaving invariant the range of a projection $P \in \mathcal{B}(\mathcal{H})$.

**Corollary 2.4.** Let $P \in \mathcal{B}(\mathcal{H})$ be a non-zero projection on a subspace $\mathcal{K}$ of $\mathcal{H}$, and let $X \in \mathcal{B}(\mathcal{H})$ be a compact operator which leaves $\mathcal{K}$ invariant. Suppose that $C \in \mathcal{B}(\mathcal{H})$ is a contraction satisfying $PC = CP = 0$. Then, for $a \in \mathcal{C}$ and $j = 1, 2, \ldots$,
\[
s_j((P + C)X - aXP) \leq \max\{1, \|I - a\|\} s_j(X).
\]

**Proof.** Since the restriction of the operator $P + C$ to the subspace $\mathcal{K}$ is the identity operator, Theorem 2.2 can be applied for the operators $\overline{A} = P + C$, $\overline{B} = aP$ and $\overline{X} = X$.

Suppose that $x, a, b \in \mathcal{H}$ and $\|x\| = 1$. Set $P = x \otimes x$, $C = 0$ and $X = x \otimes b$ in inequality (1). Then
\[
\|PXa - aXPa\| \leq \max\{1, \|1 - a\|\} \|X\| a.
\]

Since
\[
\|PXa - aXPa\| = \|(x \otimes x)(x \otimes b)a - a(x \otimes b)(x \otimes x)a\|
\]
\[
= \|(a, b)(x, x)x - a(a, x)(x, b)x\|
\]
\[
= |a, b| - a(a, x)(x, b)|
\]
and $\|X\| = \|b\|$, we obtain that
\[
|a, b| - a(a, x)(x, b) \leq \max\{1, \|1 - a\|\} \|b\| a.
\]
If \( x \) is an arbitrary non-zero vector in an inner product space \( \mathcal{H} \), by completing the space we can assume that \( \mathcal{H} \) is a Hilbert space. Then an application of the last inequality for the unit vector \( \frac{x}{\|x\|} \) proves the following version of Buzano’s inequality. It allows us to regard inequality (1) as an operator version of Buzano’s inequality.

**Corollary 2.5.** Let \( x, a, b \) be vectors in an inner product space \( \mathcal{H} \) and \( \alpha \in \mathbb{C} \). Then

\[
|\langle a, b \rangle \|x\|^2 - \alpha \langle a, x \rangle \langle x, b \rangle| \leq \max\{1, |1 - \alpha| \} \|b\| \|\alpha\| \|x\|^2.
\]

(2)

**Remark 2.6.** An easy inspection of the proof of Dragomir’s result [5, Theorem 3.3] shows that he in fact proved inequality (2).

The following result is a slight generalization of [5, Theorem 3.7] (and of Corollary 2.5).

**Theorem 2.7.** Let \( \{e_i\}_{i=1}^\infty \) be an orthonormal family in a Hilbert space \( \mathcal{H} \), and let \( \{\lambda_i\}_{i=1}^\infty \) be a bounded sequence of complex numbers. Then

\[
\left| \sum_{i=1}^\infty \lambda_i \langle a, e_i \rangle \langle e_i, b \rangle - \langle a, b \rangle \right| \leq \max\{1, \sup_{i \geq 1} |1 - \lambda_i| \} \|b\| \|a\|
\]

for all \( a, b \in \mathcal{H} \). If \( \{e_i\}_{i=1}^\infty \) is an orthonormal basis of \( \mathcal{H} \), then

\[
\left| \sum_{i=1}^\infty \lambda_i \langle a, e_i \rangle \langle e_i, b \rangle - \langle a, b \rangle \right| \leq \sup_{i \geq 1} |1 - \lambda_i| \|b\| \|a\|.
\]

**Proof.** The series \( \sum_{i=1}^\infty \lambda_i \langle a, e_i \rangle \langle e_i, b \rangle \) converges absolutely, since

\[
\sum_{i=1}^\infty |\lambda_i \langle a, e_i \rangle \langle e_i, b \rangle| \leq \sup_{i \geq 1} |\lambda_i| \left( \sum_{i=1}^\infty |\langle a, e_i \rangle|^2 \right)^{1/2} \left( \sum_{i=1}^\infty |\langle e_i, b \rangle|^2 \right)^{1/2} \leq \sup_{i \geq 1} |\lambda_i| \|a\| \|b\|
\]

by the Cauchy–Schwarz inequality in the sequence space \( l^2 \) and by Bessel’s inequality. Therefore, it is enough to show that, for each positive integer \( n \), we have

\[
\left| \sum_{i=1}^n \lambda_i \langle a, e_i \rangle \langle e_i, b \rangle - \langle a, b \rangle \right| \leq \max\{1, |1 - \lambda_1|, \ldots, |1 - \lambda_n| \} \|b\| \|a\|.
\]

Set \( A := \sum_{i=1}^n e_i \otimes e_i, B := \sum_{i=1}^n \lambda_i e_i \otimes e_i \) and \( X := \sum_{i=1}^n e_i \otimes b \). Consider the closed subspace \( \mathcal{H} \) spanned by vectors \( e_1, \ldots, e_n \). Note that \( A \) is the identity operator on \( \mathcal{H} \), and \( B \) leaves \( \mathcal{H} \) invariant and it is zero on the orthogonal complement of \( \mathcal{H} \). Also, \( X \) is an operator from \( \mathcal{H} \) to \( \mathcal{H} \), so its restriction to \( \mathcal{H} \) commutes with \( A \). By Theorem 2.2, we have

\[
\|AX - XB\| \leq \max\{1, \|A\|^{-1} B\|X\| \} \|A\| \|X\|.
\]

and so, for each \( a \in \mathcal{H} \),

\[
\|(AX - XB)a\| \leq \max\{1, |1 - \lambda_1|, \ldots, |1 - \lambda_n| \} \left\| \sum_{i=1}^n e_i \right\| \|b\| \|a\|.
\]

Since

\[
(AX - XB)a = \left( \sum_{i=1}^n e_i \right) \left( \langle a, b \rangle - \sum_{i=1}^n \lambda_i \langle a, e_i \rangle \langle e_i, b \rangle \right),
\]

we obtain the desired inequality. When \( \{e_i\}_{i=1}^\infty \) is a basis of \( \mathcal{H} \), we can omit the number 1 in the maximum by the last assertion of Theorem 2.2. \( \Box \)
3. The Hilbert C^*-module case

The following theorem is Buzano’s inequality in the context of Hilbert C^*-modules.

**Theorem 3.1.** Let \( \mathcal{H} \) be a Hilbert C^*-module. If \( x, y, z \in \mathcal{H} \) such that \( \langle x, z \rangle \) commutes with \( \langle z, z \rangle \), then

\[
|2(x, z)(z, y) - \langle z, z \rangle(x, y)| \leq |x||y||z|^2.
\]

**Proof.** For \( x, y, z \in \mathcal{H} \), we have

\[
|2(x, z)(z, y) - \langle z, z \rangle(x, y)| = |\langle 2z(x, z), y \rangle - \langle x(z, z), y \rangle|
\]

\[
\leq \|2z(x, z) - x(z, z)\| |y|
\]

and

\[
\|2z(x, z) - x(z, z)\|^2 = \|\langle 2z(x, z), y \rangle - \langle x(z, z), y \rangle\|
\]

\[
\leq \|z\|^4 |x|^2.
\]

(3)

Now (3) follows from (4) and (5). \( \square \)

Using Theorem 3.1 and the fact that, in a C^*-algebra, the relation \( |c| \leq M \) is equivalent to the condition that \( |cd| \leq M|d| \) for all \( d \), we get

**Corollary 3.2.** If \( a, b \in \mathcal{A} \) are elements of a C^*-algebra such that \( a^*b \) commutes with \( b^*a \), then

\[
|2a^*bb^* - b^*ba^*| \leq |a||b||b|^2.
\]

The following provides a non-trivial example.

**Example 3.3.** Let \( \mathcal{H} \) be a separable complex Hilbert space and let \( \{e_i\}_{i=1}^\infty \) be an orthonormal basis for \( \mathcal{H} \). Define the operator \( u : \mathcal{H} \rightarrow \mathcal{H} \) by

\[
u(u) = \begin{cases} 
    e_i, & i \leq n \\
    0, & i > n 
\end{cases}
\]

Then the adjoint operator \( u^* \) is defined by \( u^*(e_i) = \begin{cases} 
    e_i, & 2 \leq i \leq n + 1 \\
    0, & i > n + 1 \text{ or } i = 1
\end{cases} \). If \( \mathcal{H}_1, \mathcal{H}_2 \) are the subspaces generated with \( \{e_1, \ldots, e_n\} \) and \( \{e_2, \ldots, e_{n+1}\} \), respectively, then \( u^*u \) is the projection onto \( \mathcal{H}_1 \) and \( uu^* \) is the projection onto \( \mathcal{H}_2 \). For all \( v \in B(\mathcal{H}) \), we clearly have \( vv = 0 \) on \( \mathcal{H}_1^\perp \). Therefore, if \( v(\mathcal{H}_2) \subseteq \mathcal{H}_1 \), then \( vu \) commutes with \( u^*u \), so that we have

\[
\|2vu - u^*uv\| \leq \|v\|.
\]

References


