Remarks on $S_i$-separation axioms

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Abstract. In this paper, a space is a pair $(X, \mathcal{K})$, where $X$ is a set and $\mathcal{K} \subset \exp X$. This paper gives some new characterizations for $S_i$-separation axioms in the space $(X, \mathcal{K})$ ($i = 1, 2$). As some corollaries of these results, some characterizations for $T_i$-separation axioms in the space $(X, \mathcal{K})$ are obtained ($i = 1, 2$).

1. Introduction

In the general theory of topological spaces, separation axioms had played an important role. In a series of papers, the ordinary separation axioms are modified in the way that the role of open sets is given to other classes of sets (see e.g. [2, 3, 5, 7, 8]). Moreover, Arenas et al. [1, 6] studied some weak separation axioms related with Alexandroff topological spaces. In [3], A. Császár discussed some lower separation axioms $T_0, T_1, T_2, S_1$ and $S_2$ in generalized topological spaces, and gave some “nice” characterizations for these separation axioms. Having gained some enlightenment from results on separation axioms obtained by A. Császár in [3], this paper investigates $T_i$-separation axioms and $S_i$-separation axioms ($i = 1, 2$), and obtain some new characterizations for these separation axioms in the space $(X, \mathcal{K})$.

In this paper, a space is a pair $(X, \mathcal{K})$, where $X$ is a set and $\mathcal{K} \subset \exp X$. Throughout this paper, we use the following notations.

**Notation 1.1.** Let $(X, \mathcal{K})$ be a space and $A \subset X$.

1. $\kappa A = \{x : x \in K \in \mathcal{K} \text{ implies } K \cap A \neq \emptyset\}$.
2. $\chi A = \bigcap \{K : A \subset K \in \mathcal{K}\}$.
3. $\overline{\mathcal{K}} A = \bigcap \{K : A \subset K \in \mathcal{K}\}$.

**Remark 1.2.** ([3]) (1) In the sense of Notation 1.1, if no $K \in \mathcal{K}$ satisfies $x \in K$, then $x \in \kappa A$.

(2) In particular, $\chi A = X$ and $\overline{\mathcal{K}} A = X$ if there do not exist sets $K \subset \mathcal{K}$ satisfying $A \subset K$.

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2. Preliminaries

Let us recall $T_1$-separation axioms $(i = 0, 1, 2)$ and $S_i$-separation axioms $(i = 1, 2)$, which come from [3].

Definition 2.1. Let $(X, \mathcal{K})$ be a space.

(1) $T_0$-separation axiom: $x, y \in X$ and $x \neq y$ imply the existence of $K \in \mathcal{K}$ containing precisely one of $x$ and $y$.

(2) $T_1$-separation axiom: $x, y \in X$ and $x \neq y$ imply the existence of $K \in \mathcal{K}$ such that $x \in K$ and $y \notin K$.

(3) $T_2$-separation axiom: $x, y \in X$ and $x \neq y$ imply the existence of $K, K' \in \mathcal{K}$ such that $x \in K$, $y \in K'$ and $K \cap K' = \emptyset$.

(4) $S_1$-separation axiom: If $x, y \in X$ and there exists $K \in \mathcal{K}$ such that $x \in K$ and $y \notin K$, then there exists $K' \in \mathcal{K}$ satisfying $y \in K'$ and $x \notin K'$.

(5) $S_2$-separation axiom: If $x, y \in X$ and there exists $K \in \mathcal{K}$ such that $x \in K$ and $y \notin K$, then there exist $K', K'' \in \mathcal{K}$ satisfying $x \in K'$, $y \in K''$ and $K' \cap K'' = \emptyset$.

Remark 2.2. Some of the consequences of these separation axioms are valid in this generality. In particular, the following hold.

(1) $T_2$-separation axiom $\implies T_1$-separation axiom $\implies T_0$-separation axiom.

(2) $S_2$-separation axiom $\iff T_0$- and $S_1$-separation axiom.

(3) $T_1$-separation axiom $\iff T_0$- and $S_2$-separation axiom.

(4) $T_2$-separation axiom $\iff T_0$- and $S_2$-separation axiom.

The following belong to A. Császár [3].

Lemma 2.3. ([3]) Let $(X, \mathcal{K})$ be a space. Given $x, y \in X$, $\kappa[x] \neq \kappa[y]$ if and only if there exists $K \in \mathcal{K}$ containing precisely one of $x$ and $y$. Thus, $\mathcal{K}$ satisfies $T_0$-separation axiom if and only if $\kappa[x] \neq \kappa[y]$ for all $x, y \in X$.

Lemma 2.4. ([3]) Let $(X, \mathcal{K})$ be a space. $\mathcal{K}$ satisfies $S_1$-separation axiom if and only if $x \in K \in \mathcal{K}$ implies $\kappa[x] \subset K$.

Definition 2.5. ([3]) Let $X$ be a set. A mapping $\lambda : \exp X \rightarrow \exp X$ is called an envelope operation (or briefly an envelope) on $X$ if the following hold (We write $\lambda A$ for $\lambda(A)$).

(1) $A \subset \lambda A$ for $A \subset X$.

(2) $\lambda A \subset \lambda B$ for $A \subset B \subset X$.

(3) $\lambda \lambda A = \lambda A$ for $A \subset X$.

Lemma 2.6. ([3]) Let $\kappa : \exp X \rightarrow \exp X$ and $\chi : \exp X \rightarrow \exp X$ be defined as Notation 1.1. Then both $\kappa$ and $\chi$ are envelopes on $X$, and hence the following hold.

(1) $x \in \kappa[x], x \in \chi[x]$ and $\kappa[x] \subset \chi[x]$.

(2) If $x \in \kappa[y]$, then $\kappa[x] \subset \kappa[y]$.

(3) If $x \in \chi[y]$, then $\chi[x] \subset \chi[y]$.

3. The main results

For a space $(X, \mathcal{K})$ and $x \in X$, we write $\mathcal{K}_x = \{K : x \in K \in \mathcal{K}\}$ for the sake of convenience. Consequently, $x \in K \in \mathcal{K}$ if and only if $K \in \mathcal{K}_x$. Thus, “$K \in \mathcal{K}_x$” denotes “$x \in K \in \mathcal{K}$” in this section.

By the definitions of $\kappa, \chi$ and $\overline{x}$, the following remark is obvious.

Remark 3.1. Let $(X, \mathcal{K})$ be a space and $x \in X$. Then the following hold.

(1) $\kappa[x] = \{y : K \in \mathcal{K}_y \text{ implies } x \in K\}$, i.e., $y \in \kappa[x]$ if and only if $x \in K$ for each $K \in \mathcal{K}_y$.

(2) $\chi[x] = \bigcap\{K : K \in \mathcal{K}_y\}$, i.e., $y \in \chi[x]$ if and only if $y \in K$ for each $K \in \mathcal{K}_y$.

(3) $\overline{x}[x] = \bigcap\{K : K \in \mathcal{K}_x\}$.

(4) $y \notin \kappa[x]$ if and only if there exists $K \in \mathcal{K}_x$ such that $x \notin K$.

(5) $y \notin \chi[x]$ if and only if there exists $K \in \mathcal{K}_x$ such that $y \notin K$.

(6) $y \notin \overline{x}[x]$ if and only if there exists $K \in \mathcal{K}_x$ such that $y \notin xK$. 
Lemma 3.2. Let \((X, \mathcal{H})\) be a space and \(x \in X\). Then \(\kappa[x] = X - \bigcup\{K : K \in \mathcal{H} - \mathcal{H}_x\}\).

Proof. Let \(y \in \kappa[x]\). By Remark 3.1(1), \(x \in K\) for each \(K \in \mathcal{H}_y\). So, for each \(K \in \mathcal{H}\), \(y \not\in K\) if \(K \not\in \mathcal{H}_x\). That is, for each \(K \in \mathcal{H} - \mathcal{H}_x\), \(y \not\in K\). It follows that \(y \not\in \bigcup\{K : K \in \mathcal{H} - \mathcal{H}_x\}\), and so \(y \in X - \bigcup\{K : K \in \mathcal{H} - \mathcal{H}_x\}\). On the other hand, let \(y \in X - \bigcup\{K : K \in \mathcal{H} - \mathcal{H}_x\}\). Then we have \(y \in \kappa[x]\) by reversing the proof above. This proves that \(\kappa[x] = X - \bigcup\{K : K \in \mathcal{H} - \mathcal{H}_x\}\). \(\square\)

Lemma 3.3. Let \((X, \mathcal{H})\) be a space and \(x, y \in X\). Then the following are equivalent.

1. \(K \cap \{x, y\} = \{x, y\}\) for each \(K \in \mathcal{H}_y\).
2. \(y \in \chi[x]\).
3. \(x \in \kappa[y]\).
4. \(\mathcal{H}_x \subset \mathcal{H}_y\).
5. \(\chi[y] \subset \chi[x]\).
6. \(\kappa[x] \subset \kappa[y]\).

Proof. (1) \(\Rightarrow\) (2): Let \(K \cap \{x, y\} = \{x, y\}\) for each \(K \in \mathcal{H}_y\). Then \(y \in K\) for each \(K \in \mathcal{H}_y\). By Remark 3.1(2), \(y \in \chi[x]\).

(2) \(\Rightarrow\) (3): It holds from Remark 3.1(1),(2).

(3) \(\Rightarrow\) (4): Let \(x \in \kappa[y]\). By Lemma 3.2, \(x \in X - \bigcup\{K : K \in \mathcal{H} - \mathcal{H}_y\}\), i.e., \(x \not\in \bigcup\{K : K \in \mathcal{H} - \mathcal{H}_y\}\). So, if \(K \in \mathcal{H} - \mathcal{H}_x\), then \(x \not\in K\). Consequently, if \(K \in \mathcal{H}_y\), then \(K \in \mathcal{H}_y\). This proves that \(\mathcal{H}_x \subset \mathcal{H}_y\).

(4) \(\Rightarrow\) (5): Let \(\mathcal{H}_x \subset \mathcal{H}_y\). Then \(\chi[y] = \bigcap\{K : K \in \mathcal{H}_y\} \subset \bigcap\{K : K \in \mathcal{H}_x\} = \chi[x]\).

(5) \(\Rightarrow\) (1) Let \(\chi[y] \subset \chi[x]\). For each \(K \in \mathcal{H}_y\), since \(y \in \chi[y] \subset \chi[x]\), \(y \in K\). Note that \(x \in K\). It follows that \(K \cap \{x, y\} = \{x, y\}\).

(6) \(\Rightarrow\) (3) Let \(\kappa[x] \subset \kappa[y]\). Then \(x \in \kappa[x] \subset \kappa[y]\) \(\square\)

Lemma 3.4. Let \((X, \mathcal{H})\) be a space. Then the following are equivalent.

1. \((X, \mathcal{H})\) satisfies \(S_1\)-separation axiom.
2. For every pair \(x, y \in X\), \(x \not\in \chi[y]\) implies \(y \not\in \chi[x]\).
3. For every pair \(x, y \in X\), \(x \not\in \chi[y]\) implies \(y \in \chi[x]\).
4. For every pair \(x, y \in X\), \(x \not\in \kappa[y]\) implies \(y \not\in \kappa[x]\).
5. For every pair \(x, y \in X\), \(x \not\in \kappa[y]\) implies \(y \in \kappa[x]\).

Proof. (1) \(\Rightarrow\) (2): Assume that \((X, \mathcal{H})\) satisfies \(S_1\)-separation axiom. Let \(x, y \in X\) and \(x \not\in \chi[y]\). Then there exists \(K \in \mathcal{H}\) such that \(y \in K\) and \(x \not\in K\). Since \((X, \mathcal{H})\) satisfies \(S_1\)-separation axiom, there exists \(K' \in \mathcal{H}\) such that \(x \in K'\) and \(y \not\in K'\). So \(y \not\in \chi[x]\).

(2) \(\Rightarrow\) (1): Assume that (2) holds. Let \(x, y \in X\) and let there exist \(K \in \mathcal{H}\) such that \(x \in K\), \(y \not\in K\). Then \(y \not\in \chi[x]\). Since (2) holds, \(x \not\in \chi[y]\). So there exists \(K' \in \mathcal{H}\) such that \(y \in K'\) and \(x \not\in K'\). This proves that \((X, \mathcal{H})\) satisfies \(S_1\)-separation axiom.

(2) \(\iff\) (3): It is clear.

(4) \(\iff\) (5): It is clear.

(5) \(\Rightarrow\) (3): Assume that (3) holds. Let \(x, y \in X\) and \(x \in \kappa[y]\). By Lemma 3.3, \(y \in \chi[x]\), and so \(x \in \chi[y]\). By Lemma 3.3 again, \(y \in \kappa[x]\).

(5) \(\Rightarrow\) (3): The proof is similar to that of (3) \(\Rightarrow\) (5) \(\square\)

Theorem 3.4.1. Let \((X, \mathcal{H})\) be a space. Then the following are equivalent.

1. \((X, \mathcal{H})\) satisfies \(S_1\)-separation axiom.
2. For each \(x \in X\), \(\chi[x] = \kappa[x]\).
3. For each \(x \in X\), \(\chi[x] \subset \kappa[x]\).
4. For each \(x \in X\), \(\kappa[x] \subset \chi[x]\).
Proof. (1) \implies (2): Assume that \((X, \mathcal{X})\) satisfies \(S_1\)-separation axiom. Let \(x \in X\). If \(y \notin \kappa(x)\), then there exists \(K \in \mathcal{X}_y\) such that \(x \notin K\). Since \((X, \mathcal{X})\) satisfies \(S_1\)-separation axiom, there exists \(K' \in \mathcal{X}_y\) such that \(y \notin K'\), and so \(y \notin \chi(x)\). This proves that \(\chi(x) \subset \kappa(x)\). On the other hand, if \(y \notin \chi(x)\), then there exists \(K \in \mathcal{X}_y\) such that \(y \notin K\). Since \((X, \mathcal{X})\) satisfies \(S_1\)-separation axiom, there exists \(K' \in \mathcal{X}_y\) such that \(x \notin K'\), and so \(y \notin \kappa(x)\). This proves that \(\kappa(x) \subset \chi(x)\). Consequently, \(\chi(x) = \kappa(x)\).

(2) \implies (3): It is clear.

(3) \implies (4): Assume that (3) holds. Let \(x \in X\). If \(y \in \kappa(x)\), then \(x \in \chi[y]\) from Lemma 3.3. Since \(\chi[y] \subset \kappa[y]\), \(x \in \kappa[y]\). By Lemma 3.3 again, \(y \in \chi(x)\). This proves that \(\kappa(x) \subset \chi(x)\).

(4) \implies (1): Assume that (4) holds. Let \(x \in K \in \mathcal{X}\), then \(\kappa(x) \subset \chi(x)\). By the definition of \(\chi(x), \chi(x) \subset K\). So \(\kappa(x) \subset \chi(x) \subset K\). By Lemma 2.4, \((X, \mathcal{X})\) satisfies \(S_1\)-separation axiom.

Proposition 3.5. Let \((X, \mathcal{X})\) be a generalized topological space. If \((X, \mathcal{X})\) satisfies \(S_1\)-separation axiom, then \(\{\kappa(x) : x \in X\}\) constitutes a partition of \(X\).

The following theorem improve Proposition 3.5 by omitting “generalized topological” in Proposition 3.5.

Theorem 3.5.1. Let \((X, \mathcal{X})\) be a space. Then the following are equivalent.

1. \((X, \mathcal{X})\) satisfies \(S_1\)-separation axiom.
2. \(\{\kappa(x) : x \in X\}\) constitutes a partition of \(X\).
3. \(\{\chi(x) : x \in X\}\) constitutes a partition of \(X\).

Proof. (1) \implies (2): Assume that \((X, \mathcal{X})\) satisfies \(S_1\)-separation axiom. Let \(x, y \in X\) and \(\kappa(x) \cap \kappa(y) \neq \emptyset\). Then there exists \(z \in \kappa(x) \cap \kappa(y)\). By Lemma 3.4, \(x \in \kappa[z]\), since \(z \in \kappa[x]\). So \(\kappa[z] \subset \kappa[x]\) and \(\kappa[x] \subset \kappa[z]\) by Lemma 3.6. It follows that \(\kappa[x] = \kappa[z]\). Similarly, \(\kappa[y] = \kappa[z]\). Thus \(\kappa[x] = \kappa[y]\). This proves that \(\{\kappa(x) : x \in X\}\) constitutes a partition of \(X\).

(2) \implies (3): Assume that (2) holds. Let \(x, y \in X\) and \(\chi[x] \cap \chi[y] \neq \emptyset\). Then there exists \(z \in \chi[x] \cap \chi[y]\) by Lemma 2.6. \(\chi[z] \subset \chi(x)\) since \(z \in \chi(x)\). On the other hand, \(x \in \chi[z]\) by Lemma 3.3. Thus \(x \in \kappa[x]\) by Lemma 3.3. So \(\kappa[x] \subset \kappa[z]\), and hence \(z \in \kappa[z] = \kappa[x]\). So \(x \in \kappa[x]\). It follows that \(\chi(x) \subset \chi[x]\). This proves that \(\chi(x) = \kappa[x]\). Similarly, \(\chi[y] = \kappa[z]\). Consequently, \(\chi[x] = \chi[y]\). So \(\chi[x] : x \in X\) constitutes a partition of \(X\).

(3) \implies (1): Assume that (3) holds. Let \(x, y \in X\) and \(y \notin \chi[x]\). By Lemma 3.4, it suffices to prove that \(\kappa[x] \subset \chi[y]\). Since \(y \notin \chi[x]\), so \(\chi[x] \neq \chi[y]\), and hence \(\chi[x] \cap \chi[y] = \emptyset\). Since \(x \in \chi[x]\), so \(x \notin \chi[y]\).

Theorem 3.5.2. Let \((X, \mathcal{X})\) be a space. Then the following are equivalent.

1. \((X, \mathcal{X})\) satisfies \(S_2\)-separation axiom.
2. \(\chi[x] \subset \kappa[x]\) for each \(x \in X\).
3. \(\chi[x] \subset \kappa[x]\) for each \(x \in X\).

Proof. (1) \implies (2): Assume that \((X, \mathcal{X})\) satisfies \(S_2\)-separation axiom. Let \(x \in K \in \mathcal{X}\) and \(y \in \chi[x]\). It suffices to prove that \(y \in K\). In fact, if \(y \notin K\), then there exist \(K', K'' \in \mathcal{X}\) such that \(x \in K', y \in K'' \) and \(K' \cap K'' = \emptyset\). Thus \(y \notin \kappa[K']\). Note that \(K' \in \mathcal{X}\) since \(y \notin \chi[x]\). This is a contradiction.

(2) \implies (1): Assume that (2) holds. Let \(x, y \in X\) and let there exist \(K \in \mathcal{X}\) such that \(x \in K\), \(y \notin K\). Then \(y \notin \chi[x]\). So there exists \(K' \in \mathcal{X}\) such that \(y \notin \kappa[K']\). It follows that there exists \(K'' \in \mathcal{X}\) such that \(K' \cap K'' = \emptyset\). This proves that \((X, \mathcal{X})\) satisfies \(S_2\)-separation axiom.

(1) \implies (3): Assume that \((X, \mathcal{X})\) satisfies \(S_2\)-separation axiom. Let \(x \in X\). By Lemma 3.3, we only need to prove \(\chi[x] \subset \kappa[x]\). Let \(y \in \chi[x]\). It suffices to prove that \(y \in \kappa[x]\). In fact, if \(y \notin \kappa[x]\), then there exists \(K \in \mathcal{X}\) such that \(x \notin K\). And so there exist \(K', K'' \in \mathcal{X}\) such that \(x \in K', y \in K'' \) and \(K' \cap K'' = \emptyset\). Thus \(y \notin \kappa[K']\). Note that \(K' \in \mathcal{X}\). So \(y \notin \chi[x]\). This is a contradiction.
(3) \(\implies\) (1): Assume that (3) holds. Let \(x, y \in X\) and let there exist \(K \in \mathcal{K}\) such that \(x \in K, y \notin K\). Then \(x \notin \kappa[y]\), and hence \(x \notin \overline{y}(y)\). So there exists \(K' \in \mathcal{K'y}\) such that \(x \notin \kappa K'\). It follows that there exists \(K'' \in \mathcal{K'}x\) such that \(K' \cap K'' = \emptyset\). This proves that \((X, \mathcal{K})\) satisfies \(S_2\)-separation axiom. \(\Box\)

Taking Lemma 3.4 and Theorem 3.5.1 into account, the following question is interesting.

**Question 3.6.** Let \((X, \mathcal{K})\) be a space. Are the following equivalent.

1. \((X, \mathcal{K})\) satisfies \(S_2\)-separation axiom.
2. For every pair \(x, y \in X\), \(x \notin \overline{y}(y)\) implies \(y \notin \overline{x}(x)\).
3. \(\overline{y}(x) : x \in X\) constitutes a partition of \(X\).

The following answer the above question.

**Proposition 3.7.** Let \((X, \mathcal{K})\) be a space. Then, for every pair \(x, y \in X\), \(x \notin \overline{y}(y)\) implies \(y \notin \overline{x}(x)\).

**Proof.** Let \(x, y \in X\). If \(x \notin \overline{y}(y)\), then there exists \(K \in \mathcal{K}\) such that \(x \notin \kappa K\), and hence there exists \(K' \in \mathcal{K}\) such that \(K' \cap K = \emptyset\). Thus, \(y \notin \kappa K'\). So \(y \notin \overline{x}(x)\). \(\Box\)

**Proposition 3.8.** Let \((X, \mathcal{K})\) be a space. If \((X, \mathcal{K})\) satisfies \(S_2\)-separation axiom, then \(\overline{y}(x) : x \in X\) constitutes a partition of \(X\).

**Proof.** Assume that \((X, \mathcal{K})\) satisfies \(S_2\)-separation axiom. By Remark 2.2(2) and Proposition 3.6, \(\{x : y \in X\} \) constitutes a partition of \(X\). Also, by Theorem 3.5.2, \(\overline{y}(x) = \{x : y \in X\}\) for each \(x \in X\). So \(\overline{y}(x) : x \in X\) constitutes a partition of \(X\). \(\Box\)

**Example 3.9.** There exists a space \((X, \mathcal{K})\) such that \(\overline{y}(x) : x \in X\) constitutes a partition of \(X\), and \((X, \mathcal{K})\) does not satisfy \(S_1\)-separation axiom.

Put \(X = \{a, b, c, d\}\) and \(\mathcal{K} = \{\{a, \{a, b\}, \{a, c, d\}\}\}.\) It is not difficult to check that \(\kappa K = \{a, b\}\) if \(K = \{a, \{a, b\}\}\), and \(\kappa K = \{c, d\}\) if \(K = \{c, d\}\). So \(\overline{y}(x) = \{a, b\}\) if \(x \in \{a, b\}\), and \(\overline{y}(x) = \{c, d\}\) if \(x \in \{c, d\}\). Then \(\overline{y}(x) : x \in X\) constitutes a partition \(\{\{a, b\}, \{c, d\}\}\) of \(X\). Since \(\kappa[a] = \{a, b\}\) and \(\chi[a] = \{a, \kappa[a] \neq \chi[a]\}, so (X, \mathcal{K}) does not satisfy \(S_1\)-separation axiom.

As some applications of Theorem 3.4.1, Theorem 3.5.1 and Theorem 3.5.2, we give some characterizations of \(T_i\)-separation axiom \((i = 1, 2)\).

**Theorem 3.9.1.** Let \((X, \mathcal{K})\) be a space. Then the following are equivalent.

1. \((X, \mathcal{K})\) satisfies \(T_1\)-separation axiom.
2. For every pair \(x, y \in X\), \(x \neq y\) implies \(\kappa[x] \cap \kappa[y] = \emptyset\).
3. For each \(x \in X\), \(\kappa[x] = \{x\}\).
4. For each \(x, y \in X\), \(x \neq y\) implies \(\chi[x] \cap \chi[y] = \emptyset\).
5. For each \(x \in X\), \(\chi[x] = \{x\}\).

**Proof.** (1) \(\implies\) (2): Assume that \((X, \mathcal{K})\) satisfies \(T_1\)-separation axiom. Let \(x, y \in X\) and \(x \neq y\). By Remark 2.2(3), \((X, \mathcal{K})\) satisfies \(T_0\) and \(S_1\)-separation axiom. So \(\kappa[x] \neq \kappa[y]\) from Lemma 2.3, and hence \(\kappa[x] \cap \kappa[y] = \emptyset\) from Theorem 3.5.1.

(2) \(\implies\) (3): Assume that (2) holds. Let \(x \in X\), then \(x \in \kappa[x]\). For each \(y \in X - \{x\}\), since \(y \in \kappa[y]\) and \(\kappa[x] \cap \kappa[y] = \emptyset\), \(y \notin \kappa[x]\). It follows that \(\kappa[x] = \{x\}\).

(3) \(\implies\) (1): Assume that (3) holds. For every pair \(x, y \in X\), if \(x \neq y\), then \(\kappa[x] = \{x\} \neq \{y\} = \kappa[y]\), So \((X, \mathcal{K})\) satisfies \(T_0\)-separation axiom from Lemma 2.3. On the other hand, for each \(x \in X\), \(\kappa[x] = \{x\} \subseteq \chi[x]\). By Theorem 3.4.1, \((X, \mathcal{K})\) satisfies \(S_1\)-separation axiom. Thus, \((X, \mathcal{K})\) satisfies \(T_1\)-separation axiom from Remark 2.2.(3).

(1) \(\implies\) (4): Assume that \((X, \mathcal{K})\) satisfies \(T_1\)-separation axiom. Then \((X, \mathcal{K})\) satisfies \(S_1\)-separation axiom from Remark 2.2.(3). Let \(x, y \in X\) and \(x \neq y\). Then \(\chi[x] = \kappa[x]\) and \(\chi[y] = \kappa[y]\) from Theorem 3.4.1. By the above (1) \(\implies\) (2), \(\kappa[x] \cap \kappa[y] = \emptyset\). It follows that \(\chi[x] \cap \chi[y] = \emptyset\).

(4) \(\implies\) (5) \(\implies\) (1): The proof can be completed by a similar way as in the proof of (2) \(\implies\) (3) \(\implies\) (1), so we omit it. \(\Box\)
Theorem 3.9.2. Let \((X, \mathcal{X})\) be a space. Then the following are equivalent.

1. \((X, \mathcal{X})\) satisfies \(T_2\)-separation axiom.
2. \(x, y \in X\) and \(x \neq y\) imply the existence of \(K \in \mathcal{X}\) such that \(x \in K\) and \(y \not\in \kappa K\).
3. \(x, y \in X\) and \(x \neq y\) imply the existence of \(K \in \mathcal{X}\) such that \(x \in K \subset \kappa K \subset X - \{y\}\).
4. For each \(x \in X\), \(\overline{x} = \{x\}\).
5. For every pair \(x, y \in X\), \(x \neq y\) implies \(\overline{x} \cap \overline{y} = \emptyset\).

Proof. (1) \(\implies\) (2): Assume that \((X, \mathcal{X})\) satisfies \(T_2\)-separation axiom. Let \(x, y \in X\) and \(x \neq y\). By Remark 2.2(1),(4), \((X, \mathcal{X})\) satisfies \(S_2\)- and \(T_1\)-separation axiom. By Theorem 3.5.2 and Theorem 3.9.1, \(\overline{x} = \kappa x = \{x\}\), and hence \(y \not\in \overline{x}\). Thus, there exists \(K \in \mathcal{X}\) such that \(x \in K\) and \(y \not\in \kappa K\).

(2) \(\implies\) (3): Assume that (2) holds. Let \(x, y \in X\) and \(x \neq y\). Then there exists \(K \in \mathcal{X}\) such that \(x \in K\) and \(y \not\in \kappa K\). Thus \(x \in K \subset \kappa K \subset X - \{y\}\).

(3) \(\implies\) (4): Assume that (3) holds. Let \(x \in X\). If \(y \in X\) and \(x \neq y\), then there exists \(K \in \mathcal{X}\) such that \(x \in K \subset \kappa K \subset X - \{y\}\). Thus, \(K \in \mathcal{X}_1\) and \(y \not\in \kappa K\), so \(y \not\in \overline{x}\). This proves that \(\overline{x} = \{x\}\).

(4) \(\implies\) (1): Assume that (4) holds. Let \(x \in K \in \mathcal{X}\). Then \(\overline{x} = \{x\} \subset K\). So \((X, \mathcal{X})\) satisfies \(S_2\)-separation axiom. By Theorem 3.5.2, in addition, \(\{x\} \subset \chi x \subset \overline{\{x\}} = \{x\}\), so \(\chi x = \{x\}\). By Theorem 3.9.1, \((X, \mathcal{X})\) satisfies \(T_1\)-separation axiom. It follows that \((X, \mathcal{X})\) satisfies \(T_2\)-separation axiom from Remark 2.2(1),(4).

(4) \(\implies\) (5): Assume that (4) holds. Let \(x, y \in X\) and \(x \neq y\). Then \(\overline{x} \cap \overline{y} = \{x\} \cap \{y\} = \emptyset\).

(5) \(\implies\) (4): Assume that (5) holds. Let \(x \in X\), then \(x \in \overline{x}\). For each \(y \in X - \{x\}\), since \(y \in \overline{\{x\}}\) and \(\overline{x} \cap \overline{\{x\}} = \emptyset\), \(y \not\in \overline{x}\). It follows that \(\overline{x} = \{x\}\). \(\square\)

References