On the solutions of some boundary value problems by using differential transformation method with convolution terms

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Abstract. In this study, we consider some boundary value problems by using the differential transformation method with convolutions term. Further, we also propose a new method to solve the differential equations having singularity by using the convolution. In this new method when the operator has some singularities then we multiply the partial differential operator with continuously differential functions by using the convolution in order to regularize the singularity. Then the differential transform method will be applied to the new partial differential equations that might also have some fractional order.

1. Introduction

In the literature, there are several important partial differential equations in the form of:

\[ P(\mathcal{D})u = f(x, y) \]

and in order to solve, one might have either of the following cases, see Kanwal [6] and Kilicman [7].

(i) The solution \( u \) is a smooth function such that the operation can be performed as in the classical sense and the resulting equation is an identity. Then \( u \) is a classical solution.

(ii) The solution \( u \) is not smooth enough, so that the operation can not be performed but satisfies as a distributions.

(iii) The solution \( u \) is a singular distribution then the solution is a distributional solution.

Now the classification problem for the partial differential equations are well known. That is the classification of second order partial differential equations(PDE’s) is suggested by the classification of the quadratic equations in the analytic geometry, that is the equation

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \]

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is well-defined and continuous. In addition to compactly supported functions and integrable functions, 
\f_{(say \text{ convolution exists, and is also compactly supported and continuous, see }[3]. \text{ More generally, if either function (say } f) \text{ is compactly supported and the other is locally integrable, then the convolution } f \ast g \text{ is well-defined and continuous. In addition to compactly supported functions and integrable functions,}

\text{B}^2 - 4AC.

Now similarly, consider the equation
\[ a(x, y) \frac{\partial^2 u}{\partial x^2} + b(x, y) \frac{\partial^2 u}{\partial x \partial y} + c(x, y) \frac{\partial^2 u}{\partial y^2} + d(x, y) \frac{\partial u}{\partial x} + e(x, y) \frac{\partial u}{\partial y} + f u = G(x, y) \] (2)

under boundary conditions
\[ u(x, 0) = f_1(x) \ast f_2(x), \ u(0, y) = w_1(y) \ast w_2(y) \]
\[ \frac{\partial}{\partial x} u(x, 0) = \frac{d}{dx} (f_1(x) \ast f_2(x)), \ \frac{\partial}{\partial y} u(0, y) = \frac{d}{dy} (w_1(y) \ast w_2(y)) \text{ and } u(0, 0) = 0 \]

where the symbol \( \ast \) convolution and \( a, b, c, d, e \) and \( f \) are coefficients. Then this equation can be written in the following form:
\[ au_{xx} + 2bu_{xy} + cu_{yy} + F(x, y, u, u_x, u_y) = 0 \] (3)

where \( a, b, c, d, e, f \) are of class \( C^2(\Omega) \) and \( \Omega \subseteq \mathbb{R}^2 \) is domain and \( (a, b, c) \neq (0, 0, 0) \) and the expression \( au_{xx} + 2bu_{xy} + cu_{yy} \) is called principal part of equation (3) and since the principal part mainly determines the properties of solution and it is well known that

(i) If \( b^2 - 4ac > 0 \), equation (3) is called a hyperbolic equation.

(ii) If \( b^2 - 4ac < 0 \), equation (3) is called a parabolic equation.

(iii) If \( b^2 - 4ac = 0 \), equation (3) is called an elliptic equation.

However the classification theorem guarantees that every second order linear PDE with constant coefficients can be transformed into exactly one of the above forms.

If this is true at all points, then the equation is said to be hyperbolic, parabolic, or elliptic in the domain. Later we generalize the classification of hyperbolic and elliptic equations by using convolution method where we assume the non constant coefficients are polynomials, see Kılıçman and Eltayeb [8].

\textbf{Question.} Now recall the equation
\( P(D) u = f(x, y) \)
and multiply the differential operator by a function then what will happen to the classification. Since convolution compatible with differentiation then we can ask the question what will happen the new classification problem of the
\( (Q(x, t) \ast P(D)) u = F(x, t) \).

For example,
\( (Q(x, t) \ast P(\text{Elliptic})) u = F(x, t) \)

when it will be elliptic and on what conditions. Similarly,
\( (Q(x, t) \ast P(\text{Hyperbolic})) u = F(x, t) \)

when it will be Hyperbolic and on what conditions.

In the convolution of \( f \) and \( g \) if \( f \) and \( g \) are compactly supported continuous functions, then their convolution exists, and is also compactly supported and continuous, see [3]. More generally, if either function (say \( f \)) is compactly supported and the other is locally integrable, then the convolution \( f \ast g \) is well-defined and continuous. In addition to compactly supported functions and integrable functions,
functions that have sufficiently rapid decay at infinity can also be convolved. An important feature of the convolution is that if $f$ and $g$ both decay rapidly, then $f \ast g$ also decays rapidly. Further, the convolution is also a finite measure, whose total variation satisfies

$$ \| f \ast g \| \leq \| f \| \cdot \| g \| $$

then it follows that

$$ \| f \ast \mathcal{P}(D) \| \leq \| f \| \cdot \| \mathcal{P}(D) \|. $$

Further, the convolution and related operations are found in many applications of engineering and mathematics, yet in the literature there was no systematic way to generate a partial differential equations by using the equations with constant coefficients, the most of the partial differential equations with variable coefficients depend on nature of particular problems.

**Question.** Since the classification depends upon the signature of the eigenvalues of the coefficient matrix what will be the effect on the eigenvalues?

(i) Elliptic: The eigenvalues are all positive or all negative.

(ii) Parabolic: The eigenvalues are all positive or all negative, save one that is zero.

(iii) Hyperbolic: There is only one negative eigenvalue and all the rest are positive, or there is only one positive eigenvalue and all the rest are negative.

(iv) Ultrahyperbolic: There is more than one positive eigenvalue and more than one negative eigenvalue, and there are no zero eigenvalues. There is only limited theory for ultrahyperbolic equations, see Courant and Hilbert [1].

**Question.** What is the relation between solutions? In the next we can see the relation between solutions before and after convolution.

Consider the differential equation in the form of

$$ y''' - y'' + 4y' - 4y = 2 \cos(2t) - \sin(2t) \tag{4} $$

$$ y(0) = 1, \ y'(0) = 4, \ y''(0) = 1. $$

Then, by taking the Sumudu transform, we obtain:

$$ Y(u) = \frac{u^3 (2u + 1)}{(4u^2 + 1)(1 - u + 4u^2 - 4u^3)} + \frac{(u^2 + 3u + 1)}{(1 - u + 4u^2 - 4u^3)}. \tag{5} $$

Replacing the complex variable $u$ by $\frac{1}{s}$, Eq. (5) turns to:

$$ Y\left(\frac{1}{s}\right) = \frac{s(s + 2)}{(s^2 + 4)(s^3 + 4)(s - 1)} + \frac{s(s^2 + 3s + 1)}{(s^3 + 4)(s - 1)}. \tag{6} $$

Now in order to obtain the inverse Sumudu transform for Eq.(6), we use

$$ S^{-1}(Y(s)) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}Y\left(\frac{1}{s}\right) \frac{ds}{s} = \sum \text{residues} \left[ e^{st}Y\left(\frac{1}{s}\right) \right]. $$

Thus, the solution of Eq. (4) is given by:

$$ y(t) = \frac{13}{8} \sin(2t) - \frac{1}{4} t \cos(2t) + e^t. $$
Now, if we consider to multiply the left hand side of Eq. (4) by the non constant coefficient \( t^2 \), then Eq. (4) becomes
\[
t^2 \cdot (y''' - y'' + 4y' - 4y) = 2 \cos(2t) - \sin(2t)
\]
\[
y(0) = 1, \quad y'(0) = 4, \quad y''(0) = 1.
\]

By applying a similar method, we obtain the solution of Eq. (7) in the form:
\[
y_1(t) = \cos(2t) - t \sin(2t) + \frac{3}{2} \sin(2t).
\]

Now in order to see the effect of the convolutions we can see the difference as \( \|y - y_1\| \) on \([0, 1]\), see the detail [11]. Note that in particular if we consider \( \delta \cdot (y''' - y'' + 4y' - 4y) = y''' - y'' + 4y' - 4y \) then we have the original solution. We also note that some certain integro-differential equations can be represented as a convolution equation. For example,
\[
x^2 y^{(4)} - xy^{(3)} + xy' - 5y = 60x^3 - \frac{142}{7} + \int_{-1}^{1} t y(t) dt; \quad -1 \leq x \leq 1
\]
\[
y(-1) = 3, \quad y(0) = 4, \quad y'(0) = 0, \quad y^{(3)} \left( -\frac{1}{2} \right) = 15.
\]

Then we can see that the right hand side of the equation (8) can be shown as the convolution.

2. Differential Transformation Method

Differential Transform Method can easily be applied to linear or nonlinear problems and reduces the size of computational work. With this method exact solutions may be obtained without any need of further difficult computation and it is a useful tool for analytical and numerical solutions, see [5]-[16].

In this study, we consider some boundary value problems by using the differential transformation method with convolutions term. Further, we also propose a new method to solve the differential equations having singularity by using the convolution. In this new method when the operator has some singularities then we multiply the partial differential operator with continuously differential functions by using the convolution in order to remove the singularity. Then the differential transform method will be applied to the new partial differential equations that might also have some fractional order. We also study the existence, uniqueness as well as the smoothness of the new equations.

Suppose that the function \( y(x) \) is continuously differentiable in the interval \((x_0 - r, x_0 + r)\) for \( r > 0 \) then we have the following definition.

**Definition 2.1.** The differential transform of the function \( y(x) \) for the \( k^{th} \) derivative is defined as follows:
\[
Y(k) = \frac{1}{k!} \left[ \frac{d^k y(x)}{dx^k} \right]_{x=x_0}
\]

where \( y(x) \) is the original function and \( Y(k) \) is the transformed function. The inverse differential transform of \( Y(k) \) is defined as
\[
y(x) = \sum_{k=0}^{\infty} (x - x_0)^k Y(k).
\]

Note that, the substitution of (2.1) into (2.2) yields the following equation:
\[
y(x) = \sum_{k=0}^{\infty} (x - x_0)^k \frac{1}{k!} \left[ \frac{d^k y(x)}{dx^k} \right]_{x=x_0}
\]
which is the Taylor’s series for \( y(x) \) at \( x = x_0 \).

The following theorem was proved in [12].
Theorem 2.2. The general differential transformation for non-linear \( n \)-th order BVPs,
\[
y^{(n)}(x) = e^{-x^2} \ast (y(x))^m
\]
is given by
\[
Y(n + k) = \left(\frac{k!}{(n + k)!}\right) \sum_{k=0}^{n} \frac{(-\lambda)^k}{k_1!} \left(\prod_{i=2}^{m} Y(k_i - k_{i-1}) Y(k - k_m)\right).
\] (13)

Now by replacing the \(-\lambda\) by \(\lambda\) in the previous theorem and using the definition of \(\cosh\) function we can have the following theorem.

Theorem 2.3. The general differential transformation for non-linear \( n \)-th order BVPs,
\[
y^{(m)}(x) = \cosh(\lambda x) \ast (y(x))^m
\]
is given by
\[
Y(n + k) = \left(\frac{k!}{(n + k)!}\right) \sum_{k=0}^{n} \frac{(-\lambda)^k}{k_1!} \left(\prod_{i=2}^{m} Y(k_i - k_{i-1}) Y(k - k_m)\right).
\] (14)

The proof of the theorem is similar to the proof of Theorem 2.2.

3. Convolutions

The convolutions are very important in the development of differential equation, difference equation therefore the main purpose of this section is to generalize the classical definition of the convolution product to distributions in \( D' \). The classical definition for the convolution product of two functions \( f \) and \( g \) is as follows:

Definition 3.1. Let \( f \) and \( g \) be functions. Then the convolution product \( f \ast g \) is defined by the equation
\[
(f \ast g)(x) = \int_{-\infty}^{\infty} f(t)g(x - t) \, dt
\]
for all points \( x \) for which the integral exists.

We note that the convolution operator differs from the other multiplication operator in that \( 1 \ast f \neq f \) and \( f \ast f \neq f^2 \), for example, the square of the any number is positive for the ordinary function this also true, however, the convolution of a function \( f \) with itself might be negative. We also note that convolution is more often taken over an infinite range,
\[
(f \ast g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) \, d\tau = \int_{-\infty}^{\infty} g(\tau)f(t - \tau) \, d\tau.
\]
Thus it follows easily from the definition that if \( f \ast g \) exists then \( g \ast f \) exists and
\[
f \ast g = g \ast f.
\] (16)

Now if we let \( f, g, \) and \( h \) be arbitrary functions and \( a \) a constant then convolution has the following properties:
\[
\begin{align*}
f \ast (g \ast h) &= (f \ast g) \ast h \\
f \ast (g + h) &= (f \ast g) + (f \ast h) \\
a(f \ast g) &= (af) \ast g = f \ast (ag) \\
\int_{a}^{b} \int_{a}^{b} f(t) \, dt \, dx &= \int_{a}^{b} (x - t) f(t) \, dt
\end{align*}
\]
also gives a convolution.

Similarly, if \((f * g)'\) and \(f * g'\) (or \(f' * g\)) exist, then it can be shown that
\[
(f * g)' = f * g' \quad \text{(or \(f' * g\))}.
\]

That is, derivative of a convolution satisfies a very important property of convolutions that derivatives of a convolution may be placed on either factor but not both. In the one-variable case,
\[
\frac{d}{dx} (f * g) = \frac{df}{dx} * g = f * \frac{dg}{dx}
\]
where \(d/dx\) is the derivative. This means the derivatives of a function \(f\) can be expressed as convolutions, using the derivatives of the \(\delta\) distribution which is strange but useful:
\[
f = \delta * f, \quad f' = \delta' * f, \quad f'' = \delta'' * f
\]
Thus if the \(n\)-th order linear differential equation has constant coefficients, we may write it as \(f * x = b\) by introducing the distribution
\[
f = \delta^{(n)} + a_{n-1}\delta^{(n-1)} + \ldots + a_{3}\delta^{(3)} + a_{2}\delta^{(2)} + a_{1}\delta' + a_{0}\delta
\]
Further, if we have a function such that \(f * g = \delta\), we will obtain a special solution of the inhomogeneous equation as \(g * b\).

More generally, in the case of functions of several variables, an analogous formula holds with the partial derivative:
\[
\frac{\partial}{\partial x_i} (f * g)(x) = \frac{\partial f}{\partial x_i} * g = f * \frac{\partial g}{\partial x_i}
\]
A particular consequence of this is that the convolution can be viewed as a “smoothing” operation: the convolution of \(f\) and \(g\) is differentiable as many times as \(f\) and \(g\) are together. In fact, we can say that because of this rule that the convolutions are important in the solutions to differential equations are often given by convolutions where one factor is the given function and the other is a special kernel.

Further there is no algebra of functions possesses an identity for the convolution. The lack of identity is typically not a major inconvenience, since most collections of functions on which the convolution is performed can be convolved with a delta distribution or, at the very least admit approximations to the identity. The linear space of compactly supported distributions does, however, admit an identity under the convolution. Specifically,
\[
f * \delta = f
\]
where \(\delta\) is the delta distribution.

The convolution operation can be extended to generalized functions. If \(f\) and \(g\) are generalized functions such that at least one of them has compact support, and if \(\phi\) is a test function, then \(f * g\) is defined by
\[
(f * g, \phi) = (f(x) \times g(y), \phi(x + y))
\]
where \(\times\) is the direct product of \(f\) and \(g\), that is, the functional on the space of test functions of two independent variables given by for every infinitely-differentiable function of compact support.

Similarly, double convolution between two continuous functions \(F(x, y)\) and \(G(x, y)\) given by
\[
F_1(x, y) * * F_2(x, y) = \int_0^y \int_0^x F_1(x - \theta_1, y - \theta_2)F_2(\theta_1, \theta_2)d\theta_1d\theta_2.
\]
for further details and properties of the double convolutions and derivatives, we refer to [9] and [13].
Consider the following diffusion equation:

$$u_t = u_{xx} + \delta(x-a)k(u(x,t))$$

where $0 \leq x \leq l$, $0 \leq a \leq l$, with zeros boundary and initial conditions. All we know that this is used to model physical scenarios where the energy that is being put into the system which is highly spatially localized.

**Question.** Is solution of such PDE possible? In a classical way or in other way (Delta function is not real function)? Or maybe solution of such PDE exists in a normal sense (Delta function is a limit of so called delta sequences, which are sequences of ordinary functions)?

Some of second-order linear partial differential equations can be classified as Parabolic, Hyperbolic or Elliptic however if a PDE has coefficients which are not constant, it is rather a mixed type. In many applications of partial differential equations the coefficients are not constant in fact they are a function of two or more independent variables and possible dependent variables. Therefore the analysis to describe the solution may not be hold globally for equations with variable coefficients that we have for the equations having constant coefficients.

On the other side there are some very useful physical problems where its type can be changed. One of the best known example is for the transonic flow, where the equation is in the form of

$$
\left(1 - \frac{v^2}{c^2}\right) \phi_{xx} - \frac{2uv}{c^2} \phi_{xy} + \left(1 - \frac{c^2}{v^2}\right) \phi_{yy} + f(\phi) = 0
$$

where $u$ and $v$ are the velocity components and $c$ is a constant, see [14].

Similarly, partial differential equations with variable coefficients are also used in finance, for example, the arbitrage-free value $C$ of many derivatives

$$\frac{\partial C}{\partial \tau} + s \frac{\sigma^2(s, \tau)}{2} \frac{\partial^2 C}{\partial s^2} + b(s, \tau) \frac{\partial C}{\partial s} - r(s, \tau)C = 0
$$

with three variable coefficients $\sigma(s, \tau)$, $b(s, \tau)$ and $r(s, \tau)$. In fact this partial differential equation holds whenever $C$ is twice differentiable with respect to $s$ and once with respect to $\tau$, see [15].

However, in the literature there was no systematic way to generate a partial differential equations with variable coefficients by using the equations with constant coefficients, the most of the partial differential equations with variable coefficients depend on nature of particular problems.

**Question.** How to generate a PDE with variable coefficients from the PDE with constants coefficients.

In order to answer the above questions we extend the classification of partial differential equations to the further by using the convolutions products.

For example, if we consider the wave equation in the following example

$$
\begin{align*}
  u_{tt} - u_{xx} &= G(x,t) \quad (x,t) \in \mathbb{R}^2 \\
  u(x,0) &= f_1(x), \quad u_t(x,0) = g_1(x) \\
  u(0,t) &= f_2(t), \quad u_x(0,t) = g_2(t)
\end{align*}
$$

(19)

Now, if we consider to multiply the left hand side equation of the above equation by non-constant coefficient $Q(x,t)$ by using the double convolution with respect to $x$ and $t$ respectively, then the equation becomes

$$
\begin{align*}
  Q(x,t) \ast \ast (u_{tt} - u_{xx}) &= G(x,t) \quad (t,x) \in \mathbb{R}^2 \\
  u(x,0) &= f_1(x), \quad u_t(x,0) = g_1(x) \\
  u(0,t) &= f_2(t), \quad u_x(0,t) = g_2(t)
\end{align*}
$$

(20)

(21)
Thus the relationship between the solutions partial differential equations with constant coefficients and non constant coefficients was studied in [8]. Note that in particular case, if \( \lim_{n \to \infty} Q_n(x, t) = \delta(x, t) \) then will be an approximate identity which plays a significant role in convolution algebra as the same role as a sequence of function approximations to the Dirac delta function that is the identity element for convolution. Then the properties of a approximate identities are related to its behavior under the operation of convolution. Such as:

(i) Smoothing property: For any distribution \( f \), the following sequence of convolutions indexed by the real number

\[
\tau_n = f \ast \delta_n
\]

where \( \ast \) denotes the convolution, is a sequence of smooth functions.

(ii) The Approximation of identity: For any distribution \( f \), the following sequence of convolutions indexed by the real number converges to \( \tau_n \)

\[
\lim_{n \to \infty} \tau_n = \lim_{n \to \infty} f \ast \delta_n = f \in \mathcal{D}'(\mathbb{R}^2)
\]

(iii) Support of convolution: For any distribution \( f \),

\[
\text{supp} \tau_n = \text{supp}(f \ast \delta_n) \subset \text{supp} f + \text{supp} \delta_n
\]

where \( \text{supp} \) indicates the support in the sense of distributions.

Further, delta functions often arise as convolution semigroups. This amounts to the further constraint that the convolution of \( \delta_n \) with \( \delta_m \) must satisfy

\[
\delta_n \ast \delta_m = \delta_{n+m}
\]

for all \( n, m > 0 \). Thus the convolution semigroups in \( L^1 \) that form a delta function(nascent) are always an approximation to the identity in the above sense, however the semigroup condition is quite a strong restriction. In fact semigroups approximating the delta function arise as fundamental solutions or Green’s functions to physically motivated elliptic or parabolic partial differential equations.

Note that there is no general method that can solve all type of the differential equations, each might require different methods and techniques.

Now let us consider the general linear second order partial differential equation with non-constant coefficients in the form of

\[
a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = 0
\]

and almost linear equation in two variable

\[
a u_{xx} + b u_{xy} + c u_{yy} + F(x, y, u, u_x, u_y) = 0
\]

where \( a, b, c \), are polynomials and defined by

\[
a(x, y) = \sum_{|\alpha|=1}^{n} a_{\alpha} x^\alpha, \quad b(x, y) = \sum_{|\alpha|=1}^{m} b_{\alpha} x^\alpha y^\beta, \quad c(x, y) = \sum_{|\alpha|=1}^{n} c_{\alpha} x^\alpha y^\beta
\]

and \( (a, b, c) \neq (0, 0, 0) \) where the expression \( a u_{xx} + b u_{xy} + c u_{yy} \) is called the principal part of Eq (23), since the principal part mainly determines the properties of solution. Throughout this paper we also use the following notations

\[
|a|_m = \sum_{|\alpha|=1}^{n} |a_{\alpha}|, \quad |b|_m = \sum_{|\alpha|=1}^{m} |b_{\alpha}| \quad \text{and} \quad |c|_m = \sum_{|\alpha|=1}^{n} |c_{\beta}|.
\]
Now in order to generate a new PDEs we convolute Eq (23) by a polynomial with single convolution as $p(x) \ast^c x$ where $p(x) = \sum_{i=1}^{m} p_i x^i$ then Eq (23) becomes

$$p(x) \ast^c [a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + F(x, y, u, u_x, u_y)] = 0$$

where the symbol $\ast^c$ indicates single convolution with respect to $x$, and we shall classify Eq (24) instead of Eq (23) by considering and examining the function

$$D(x, y) = (p(x) \ast^c b(x, y))^2 - (p(x) \ast^c a(x, y))(p(x) \ast^c c(x, y))$$

From Eq (25), one can see that if $D$ is positive then Eq (24) is called Hyperbolic, if $D$ is negative then Eq (24) is called Elliptic, otherwise it is parabolic.

First of all, we compute and examine the coefficients of principal part of Eq (24) as follow

$$A_1(x, y) = p(x) \ast^c a(x, y) = \sum_{i=1}^{m} p_i x^i \ast^c \sum_{j=1}^{n} a_{ij} x^j y^j$$

by using single convolution definition and integration by parts, thus we obtain the first coefficient of Eq (24) in the form of

$$A_1(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} p_i a_{ij} \sum_{l=1}^{i} \frac{1}{(\alpha + 1)(\alpha + 2)\ldots(\alpha + i + 1)} x^{\alpha + i + 1} y^j$$

similarly, for the coefficients of the second part in Eq (24) we have

$$B_1(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} p_i b_{ij} \sum_{l=1}^{i} \frac{1}{(\zeta + 1)(\zeta + 2)\ldots(\zeta + i + 1)} x^{\zeta + i + 1} y^j$$

also the last coefficient of Eq (24) given by

$$C_1(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} p_i c_{ij} \sum_{l=1}^{i} \frac{1}{(k + 1)(k + 2)\ldots(k + i + 1)} x^{k + i + 1} y^j$$

then one can easily set up

$$D_1(x, y) = B_1^2(x, y) - A_1(x, y)C_1(x, y)$$

then there are several cases and the classification of partial differential equations with polynomials coefficients depend on very much the signs of the coefficients, see Eltayeb and Kılıçman [2]. In fact, this analysis can also be carried out for $\ast^d$ and as well as the double convolution. In order to make the comparison we provide the following example, for details we refer to Kılıçman and Eltayeb in [10].

Now how to generate a PDE with variable coefficients by using the convolutions. For example in particular case we can have

$$x^3 \ast^c x^2 y^3 u_{xx} + x^3 \ast^c x^2 y^2 u_{xy} + x^3 \ast^c x^2 y^5 u_{yy} = f(x, y) \ast^c g(x, y).$$

The first coefficients of Eq (30) given by

$$A_1(x, y) = x^3 \ast^c x^2 y^3 = y^3 \int_{0}^{x} (x - \theta)^3 \theta^2 d\theta = \frac{1}{60} y^3 x^6.$$
Similarly, the second coefficient given by

\[ B_1(x, y) = x^3 \ast y^7 = \frac{1}{140} y^4 x^7. \]  

(32)

By the same way we get the last coefficients of Eq (30)

\[ C_1(x, y) = x^3 \ast y^8 = \frac{1}{280} y^5 x^8. \]  

(33)

By using Eqs (31), (32) and (33) we obtain

\[ D_1(x, y) = -\frac{1}{117600} y^8 x^{14} \]  

(34)

We can easily see from Eq (34) that Eq (30) is an elliptic equation for all \((x_0, y)\).

In the same way, if we multiply the Eq (22) by polynomial with single convolution as \(h(y) \ast y^\beta\) where \(h(y) = \sum_{j=1}^{n} y^j\) then Eq (22) becomes

\[ h(y) \ast y^\beta \left[ a(x, y) u_{xx} + b(x, y) u_{xy} + c(x, y) u_{yy} + f(x, y, u, u_x, u_y) \right] = 0 \]  

(35)

where the symbol \( \ast \) indicates single convolution with respect to \(y\), and we shall classify Eq (35). First of all, let us compute the coefficients of Eq (35) by using definition of single convolution with respect to \(y\) and integral by part we obtain the first coefficient of Eq (35) as follow

\[ A_2(x, y) = h(y) \ast y^\beta a(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{l=1}^{m} \frac{j! x^\gamma y^\beta x^j y^i}{(l + 1)(l + 2)...(l + j + 1)}. \]  

and the second coefficients of Eq (35) given by

\[ B_2(x, y) = h(y) \ast y^\beta b(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{l=1}^{m} \frac{j! x^\gamma y^\beta x^j y^i}{(\eta + 1)(\eta + 2)...(\eta + i + 1)}. \]  

Similarly, the last coefficients of Eq (35) given by

\[ C_2(x, y) = h(y) \ast y^\beta c(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{l=1}^{m} \frac{j! x^\gamma y^\beta x^j y^i}{(l + 1)(l + 2)...(l + i + 1)}. \]  

In particular, let us classify the following example

\[ y^7 \ast y^2 x^2 y^3 u_{xx} + y^7 \ast x^2 y^4 u_{xy} + y^7 \ast x^2 y^5 u_{yy} = f(x, y) \ast y^\beta g(x, y). \]  

(36)

the symbol \( \ast \) means single convolution with respect to \(y\). We follow the same technique that used above, then the first coefficient of Eq (36) given by

\[ A_2(x, y) = y^7 \ast x^2 y^3 = \frac{1}{1320} x^2 y^{11}, \]  

(37)

the second coefficient of (36) given by

\[ B_2(x, y) = y^7 \ast x^3 y^4 = \frac{1}{3860} x^3 y^{12} \]  

(38)
and the last coefficient given by
\[ C_2(x, y) = y^7 \ast y^4 y^5 = \frac{1}{10296} x^4 y^{13}. \] (39)

On using Eqs (25), (37), (38) and (39) we have
\[ D_2(x, y) = \frac{1}{101930400} x^6 y^{24}. \] (40)

We can easily see from Eq (40) that Eq (36) is an also elliptic equation for all \((x, y_0)\).

Now, consider the wave equation
\[
\begin{align*}
F_{tt} - F_{xx} &= -3e^{2x+t} && (x, t) \in \mathbb{R}^2_+ \\
F(x, 0) &= e^{2x} + c', \ F_1(x, 0) = e^{2x} + e^x \\
F(0, t) &= 2e', \ F_0(0, t) = 3e'.
\end{align*}
\] (41)

By taking the double integral transform for equation (41) and single Sumudu transform of equations (42) and (43) with \(u, v\) as transform variables for \(x, t\) respectively, on using equations (25) and (29), after some little arrangements, we obtain
\[
F(u, v) = \frac{u^2 [2 - 3u] (v + 1)}{(1 - u) (1 - 2u) [u^2 - v^2]} - \frac{v^2 (2 + 3u)}{(1 - v) [u^2 - v^2]} - \frac{3u^2 v^2}{(1 - 2u) (1 - v) [u^2 - v^2]}.
\] (44)

Now, we consider to multiply the left hand side equation of (41) by non-constant coefficient \(x^3t^4 \ast \ast\) where the symbol \(\ast \ast\) means a double convolution with respect to \(x\) and \(t\) respectively, then equation (41) becomes
\[
\begin{align*}
xu^2 \ast \ast (F_{tt} - F_{xx}) &= -3e^{2x+t} && (x, t) \in \mathbb{R}^2_+ \\
F(x, 0) &= e^{2x} + c', \ F_1(x, 0) = e^{2x} + e^x \\
F(0, t) &= 2e', \ F_0(0, t) = 3e'.
\end{align*}
\] (45)

Similarly, we apply the double Sumudu transform technique for equation (45) and single Sumudu transform for the equations (46) and (47) we obtain
\[
F(u, v) = \frac{u^2 [2 - 3u] (v + 1)}{(1 - u) (1 - 2u) [u^2 - v^2]} - \frac{v^2 (2 + 3u)}{(1 - v) [u^2 - v^2]} - \frac{3}{2v (1 - 2u) (1 - v) [u^2 - v^2]}. \] (48)

Now, by taking double inverse Sumudu transform for both sides of equation (48) we obtain the solution of equation (45) as follows
\[
F_1(x, t) = \frac{17}{4} e^{-2i+2x} - \frac{45}{4} e^{2i+2x} + e^{i+x} + 2e^{2i+2x}.
\]

Problem. Now in order to make the classification of the following equation
\[
(Q(x, t) \ast \ast \mathcal{P}(D)) u = f(x, t)
\]
then we might have some certain remarks. For example, what is the necessary and sufficient condition that the new classification of the PDE will remain the same as before, that is, what kind of function \(Q(x, t)\) we should have that the results will be invariant?

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