Two convergence theorems to fixed point of a nonexpansive mapping on the unit sphere of a Hilbert space

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Abstract. We consider iterative schemes converging to a fixed point of nonexpansive mapping defined on the unit sphere of a real Hilbert space by using two different types of projection methods.

1. Introduction

Let $T$ be a nonexpansive mapping defined on a subset of a Hilbert space. The problem of finding a fixed point of $T$ is one of the most important problems in nonlinear analysis and it has been investigated by many researchers with various approaches. In particular, approximation of the solution to this problem by generating an iterative sequence has been considered with various schemes, in different settings of underlying spaces, and for more general nonlinear mappings.

Based on the proximal point algorithm proposed and investigated by Martinet \cite{Martinet1970} and Rockafellar \cite{Rockafellar1970}, Solodov and Svaiter \cite{SolodovSvaiter2000} proved strong convergence of an iterative scheme by using metric projections which solves a zero point problem for a maximal monotone operator. This method was applied with a fixed point problem by Nakajo and Takahashi \cite{NakajoTakahashi2003}. Since then, this type of projection method has been modified and generalized to diverse directions.

On the other hand, by modifying this iterative method, Takahashi, Takeuchi, and Kubota \cite{TakahashiTakeuchiKubota2007} proposed another type of projection method which is called shrinking projection method. The original result was proved by the technique used in the theorem of Nakajo and Takahashi, whereas Kimura and Takahashi \cite{KimuraTakahashi2008} proposed different technique to prove a generalized result with the setting of Banach spaces.

In this paper, we consider these two iterative schemes converging to a fixed point of nonexpansive mapping defined on the unit sphere of a real Hilbert space. This space is endowed with the metric defined by a length of minimal great arc and is an example of complete CAT(1) spaces with this metric. To prove the convergence theorems, we use a notion of $\Delta$-convergence which was first proposed by Lim \cite{Lim1984} on general metric spaces and later was investigated in the setting of geodesic spaces.

2. Preliminaries

Let $H$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$. We denote the unit sphere of $H$ by $S_H = \{x \in H : \|x\| = 1\}$. For $x, y \in S_H$, we define a distance between them by $d(x, y) = \arccos \langle x, y \rangle$, that is,
a length of minimal great arc with endpoints $x$ and $y$. Then, $S_H$ is a metric space with $d$. This metric space enjoys a lot of nice properties. In particular, it is a complete CAT(1) space; see [1].

Suppose that a subset $C$ of $S_H$ satisfies that $d(u, v) < \pi/2$ for every $u, v \in C$. Then a subset $K$ in $C$ is said to be convex if for every $x, y \in K, [x, y]$ is included in $K$, where $[x, y]$ is a segment with endpoints $x$ and $y$ joined by a minimal great arc. A point $z \in [x, y]$ satisfying that $d(z, y) = td(x, y)$ is denoted by $z = tx \oplus (1-t)y$. Suppose that $C$ is closed and convex. We know that the subsets

\[ \{ z \in S_H : d(x, z) \leq d(y, z) \} \text{ and } \{ z \in S_H : \cos d(x, y) \cos d(y, z) \geq \cos d(x, z) \} \]

construct a hemisphere for $x, y \in S_H$ if $x \neq y$, and thus the intersection of $C$ and each of them is closed and convex, respectively. We remark that these subsets are not always convex when the underlying space is a general CAT(1) space; see [2].

The following equation, which is easily deduced from the spherical law of cosine, plays an important role in the main results of this paper:

\[ \cos d(x, u) \sin d(y, z) = \cos d(x, y) \sin((1-t)d(y, z)) + \cos d(x, z) \sin((1-t)d(y, z)). \]

We note that, if $y$ and $z$ are not antipodal, then $u = ty \oplus (1-t)z$.

If $C$ is convex and $K$ is a nonempty closed convex subset of $C$, then for every $x \in C$, there exists unique $y_x \in K$ such that $d(x, y_x) = \inf_{y \in K} d(x, y)$. We define a mapping $P_K : C \to K$ by $P_Kx = y_x$ for $x \in C$ and we call it the metric projection of $C$ onto $K$. We know that, for $x \in C$ and $y \in K$, $y = P_Kx$ if and only if $\cos d(x, y) \cos d(y, z) \geq \cos d(x, z)$ for all $z \in K$.

We introduce the notion of $\Delta$-convergence, which was firstly introduced by Lim [8] in a general metric space setting and was studied on CAT(0) and CAT(1) spaces in [7] and [2], respectively. Let $\{x_n\}$ be a sequence in $S_H$. The asymptotic center of $\{x_n\}$ is a set of point $p \in S_H$ satisfying that

\[ \limsup_{n \to \infty} d(x_n, p) = \inf_{x \in S_H} \limsup_{n \to \infty} d(x_n, x). \]

Since $S_H$ is a complete CAT(1) space, we know from [2, Propositions 4.1, 4.5 and Remark 4.6] that if $\inf_{x \in S_H} \limsup_{n \to \infty} d(x_n, x) < \pi/2$, then the asymptotic center of $\{x_n\}$ consists of exactly one point $[p]$ and that

\[ p \in \bigcap_{k=0}^{\infty} \overline{clco}\{x_n, x_{n+1}, x_{n+2}, \ldots\}, \]

where $\overline{clco}A$ is the intersection of all closed convex subsets of $S_H$ including $A$. We say that a sequence $\{x_n\}$ is $\Delta$-convergent to $x \in S_H$ if $x$ is the unique asymptotic center of any subsequence of $\{x_n\}$. It is obvious that, if $\{x_n\}$ is $\Delta$-convergent to $x \in S_H$, then so is a subsequence of $\{x_n\}$. We know from [2, Corollary 4.4] that every sequence $\{x_n\}$ in $S_H$ has a $\Delta$-convergent subsequence unless

\[ \inf_{x \in S_H} \limsup_{n \to \infty} d(x_n, x) = \frac{\pi}{2}. \]

For more details of $\Delta$-convergence, see [2, 3, 7].

The following two results regarding CAT(1) spaces are proved in [5]. For the sake of completeness, we will show the proofs.

**Lemma 2.1.** (Kimura-Satô [5]) Let $X$ be a complete CAT(1) space, $p \in X$, and $\{x_n\}$ a sequence in $X$ such that $\limsup_{n \to \infty} d(x_n, p) < \pi/2$. Then, for a unique asymptotic center $x \in X$ of $\{x_n\}$,

\[ d(x, p) \leq \limsup_{n \to \infty} d(x_n, p). \]
Proof. Let \( \epsilon \) be a real number satisfying that \( 0 < \epsilon < \pi/2 \) and define

\[
C_\epsilon = \left\{ y \in X : d(y, p) \leq \limsup_{n \to \infty} d(x_n, p) + \epsilon \right\}.
\]

Then, \( C_\epsilon \) is a complete CAT(1) space such that \( x_n \in C_\epsilon \) for sufficiently large \( n \). Then we have that \( x \in C_\epsilon \), that is, \( d(x, p) \leq \limsup_{n \to \infty} d(x_n, p) + \epsilon \). Hence it follows that \( d(x, p) \leq \limsup_{n \to \infty} d(x_n, p) \), which completes the proof. \( \square \)

**Remark 2.2.** If \( \{x_n\} \) is included in some closed convex subset \( C \) of \( S_H \) and is \( \Delta \)-convergent to \( x \in S_H \), then \( x \in C \). In particular,

\[
d(x, y) \leq \limsup_{n \to \infty} d(x_n, y)
\]

for any \( y \in S_H \).

**Lemma 2.3.** (Kimura-Satô [5]) Let \( X \) be a complete CAT(1) space and suppose that \( d(u, v) < \pi/2 \) for every \( u, v \in X \). Let \( \{x_n\} \) be a sequence in \( X \). Suppose that \( \{x_n\} \) is \( \Delta \)-convergent to \( x \in X \) and \( \{d(x_n, p)\} \) converges to \( d(x, p) \) for some \( p \in X \). Then \( \{x_n\} \) converges to \( x \).

Proof. If \( x = p \), then the lemma is trivial so that we may suppose that \( x \neq p \). Let \( \{\Delta(x_n, \overline{x}_n, y_n)\} \) be comparison triangles in \( S^2 \) for \( n \in \mathbb{N} \) with an identical geodesic segment \([\overline{x}, \overline{x}]\). Then we have that \( d_{S}^e(\overline{x}, \overline{y}) = d(x, p), d_{S}(x_n, \overline{y}) = d(x_n, p), \) and \( d_{S}(x_n, x) = d(x_n, x) \) for all \( n \in \mathbb{N} \). Let \( d = \sup_{n \in \mathbb{N}} d_{S}(x_n, \overline{y}) \). Then since \( \lim_{n \to \infty} d(x_n, p) = d(x, p) < \pi/2 \), we have that \( d < \pi/2 \). Let \( U = [\overline{x}, \overline{y}] \subset S^2 : d_{S}(\overline{y}, \overline{p}) \leq d \). Then \( U \) is a nonempty closed subset of \( S^2 \) including \( \{x_n\} \). Thus the metric projection \( P \subset [\overline{x}, \overline{y}] \) of \( U \) onto a closed convex set \([\overline{x}, \overline{y}] \subset U \) is well defined.

Let \( \{x_n\} \) be an arbitrary subsequence of \( \{x_n\} \) converging to \( y \in U \subset S^2 \). Then, by assumption we have that

\[
d_{S}(\overline{y}, \overline{p}) = \lim_{i \to \infty} d_{S}(\overline{y}, \overline{x}_n, \overline{p}) = \lim_{i \to \infty} d(x_n, p) = d(x, p) = d_{S}(\overline{x}, \overline{p}).
\]

Since \( P \) is continuous, we have that \( \{P\overline{x}_n\} \) converges to \( P\overline{y} \in S^2 \). Let \( z \in [p, x] \subset X \) be a point corresponding to \( \overline{z} = P\overline{y} \in [\overline{p}, \overline{x}] \subset S^2 \). Using the CAT(1) inequality, we have that

\[
\limsup_{i \to \infty} d(x, x_n) = \limsup_{i \to \infty} d_{S}(\overline{x}, \overline{x}_n)
\]

\[
\geq \limsup_{i \to \infty} d_{S}(P\overline{x}_n, \overline{x}_n) = \limsup_{i \to \infty} d_{S}(\overline{z}, \overline{x}_n)
\]

\[
\geq \limsup_{i \to \infty} d(z, x_n).
\]

By the uniqueness of the asymptotic center of \( \{x_n\} \), we obtain that \( z = x \) and thus \( \overline{z} = \overline{x} \). Since

\[
d_{S}(\overline{x}, \overline{y}) = d_{S}(\overline{x}, \overline{y}) = d_{S}(P\overline{y}, \overline{y}) \leq d_{S}(t\overline{x} \oplus (1-t)\overline{p}, \overline{y}),
\]

for every \( t \in [0, 1] \subset \mathbb{R} \), we have that \( \cos d_{S}(\overline{x}, \overline{y}) \geq \cos d_{S}(t\overline{x} \oplus (1-t)\overline{p}, \overline{y}) \). It follows that

\[
\cos d_{S}(\overline{x}, \overline{y}) \sin d_{S}(\overline{x}, \overline{p}) \geq \cos d_{S}(\overline{y}, t\overline{x} \oplus (1-t)\overline{p}) \sin d_{S}(\overline{x}, \overline{p})
\]

\[
= \cos d_{S}(\overline{y}, \overline{x}) \sin(td_{S}(\overline{x}, \overline{p})) + \cos d_{S}(\overline{y}, \overline{p}) \sin((1-t)d_{S}(\overline{x}, \overline{p}))
\]

\[
= \cos d_{S}(\overline{x}, \overline{y}) \sin(td_{S}(\overline{x}, \overline{p})) + \cos d_{S}(\overline{x}, \overline{p}) \sin((1-t)d_{S}(\overline{x}, \overline{p})).
\]
Thus we have that
\[
\cos d_{S^2}(\overline{x}, \overline{y}) \geq \frac{\cos d_{S^2}(\overline{x}, \overline{p}) \sin((1-t)d_{S^2}(\overline{x}, \overline{p}))}{\sin d_{S^2}(\overline{x}, \overline{p}) - \sin(td_{S^2}(\overline{x}, \overline{p}))} = \frac{2\cos d_{S^2}(\overline{x}, \overline{p}) \sin \left(\frac{1-t}{2}d_{S^2}(\overline{x}, \overline{p})\right) \cos \left(\frac{1-t}{2}d_{S^2}(\overline{x}, \overline{p})\right)}{2\cos \left(\frac{1+t}{2}d_{S^2}(\overline{x}, \overline{p})\right) \sin \left(\frac{1-t}{2}d_{S^2}(\overline{x}, \overline{p})\right)} = \frac{\cos d_{S^2}(\overline{x}, \overline{p}) \cos \left(\frac{1-t}{2}d_{S^2}(\overline{x}, \overline{p})\right)}{\cos \left(\frac{1+t}{2}d_{S^2}(\overline{x}, \overline{p})\right)}.
\]

Letting \( t \to 1 \), we have that \( \cos d_{S^2}(\overline{x}, \overline{y}) \geq 1 \), that is, \( \overline{x} = \overline{y} \). Since any convergent subsequence \( [\overline{x}_n] \) of a bounded sequence \( [\overline{x}_n] \) in \( S^2 \) has a limit \( \overline{x} \), we have that \( [\overline{x}_n] \) converges to \( \overline{x} \). Thus we have that \( d(x_n, x) = d_{S^2}(\overline{x}_n, \overline{x}) \to 0 \) as \( n \to \infty \), and hence \( [x_n] \) converges to \( x \in X \). \( \square \)

Since the unit sphere of a Hilbert space is a complete CAT(1) space, these results are valid for our setting.

3. Strong convergence of iterative sequences

In this section, we show two convergence theorems of iterative sequences which approximate a solution to the problem of finding a fixed point of a nonexpansive mapping. A mapping \( T \) defined on a metric space \( X \) is said to be nonexpansive if \( d(Tx, Ty) \leq d(x, y) \) for every \( x, y \in X \).

**Theorem 3.1.** Let \( S_H \) be the unit sphere of a real Hilbert space \( H \) with the metric \( d \) defined by a length of minimal great arc, and \( C \) a closed convex subset of \( S_H \) such that \( d(u, v) < \pi/2 \) for every \( u, v \in C \). Let \( T : C \to C \) be a nonexpansive mapping such that the set of fixed points \( F = \{ z \in C : Tz = z \} \) is nonempty. For a given initial point \( x_0 \in C \), generate a sequence \( \{x_n\} \) as follows:

\[
\begin{align*}
C_{n+1} &= \{ z \in C : d(Tx_n, z) \leq d(x_n, z) \}, \\
Q_{n+1} &= \{ z \in C : \cos d(x_0, x_n) \cos d(x_n, z) \geq \cos d(x_0, z) \}, \\
x_{n+1} &= P_{C_{n+1} \cap Q_{n+1}} x_n,
\end{align*}
\]

for each \( n \in \mathbb{N} \). Then \( \{x_n\} \) is well defined and converges to \( P_K x_0 \in C \), where \( P_K : C \to K \) is the metric projection of \( C \) onto a nonempty closed convex subset \( K \) of \( C \).

**Proof.** We first show that \( \{x_n\} \) is well defined by induction. An initial point \( x_0 \in C \) is given. Letting \( C_0 = Q_0 = C \), we have that they are closed and convex such that \( F \subseteq C_0 \cap Q_0 \). Suppose that \( x_0, x_1, \ldots, x_k \) are defined and both \( C_k \) and \( Q_k \) are closed convex subsets of \( C \) such that \( F \subseteq C_k \cap Q_k \) for fixed \( k \in \mathbb{N} \). By definition, it is obvious that \( C_{k+1} \) is closed. We also know that \( C_{k+1} \) is convex since \( C \) is convex and \( \{ z \in S_H : d(Tx_n, z) \leq d(x_n, z) \} \) is a hemisphere of \( S_H \) and thus it is also convex. Similarly, since \( \{ z \in S_H : \cos d(x_0, x_n) \cos d(x_n, z) \geq \cos d(x_0, z) \} \) is also a hemisphere of \( S_H \), \( Q_{k+1} \) is closed and convex. Let \( z \in F \). Since \( T \) is nonexpansive, we get that \( d(Tx_n, z) \leq d(x_n, z) \) and thus \( z \in C_{k+1} \). This implies that \( F \subseteq C_{k+1} \).

To prove \( F \subseteq Q_{k+1} \), it is sufficient to show that \( C_k \cap Q_k \subseteq Q_{k+1} \). For any \( z \in C_k \cap Q_k \) and \( t \in [0, 1] \),

\[
(tz \oplus (1-t)x_k) = tz \oplus (1-t)P_{C_k \cap Q_k} x_0 \in C_k \cap Q_k.
\]
It follows that
\[ 2 \cos d(x_0, x_k) \cos \left( \frac{1 - t}{2} d(x_k, z) \right) \sin \left( \frac{t}{2} d(x_k, z) \right) = \cos d(x_0, x_k) (\sin d(x_k, z) - \sin ((1 - t) d(x_k, z))) = \cos d(x_0, x_k) \sin d(x_k, z) - d(x_0, x_k) \sin((1 - t) d(x_k, z)) \geq \cos d(x_0, x_k) \sin(t d(x_k, z)) = 2 \cos d(x_0, z) \cos \left( \frac{t}{2} d(x_k, z) \right) \sin \left( \frac{t}{2} d(x_k, z) \right). \]

If \( z \neq x_k \), then dividing by \( \sin(t d(x_k, z)/2) \) and letting \( t \to 0 \), we have that \( z \in Q_{k+1} \). If \( z = x_k \), then obviously \( z \in Q_{k+1} \). Thus we have that \( C_k \cap Q_k \subset Q_{k+1} \). Therefore, \( C_{k+1} \cap Q_{k+1} \) is a nonempty closed convex subset of \( C \) and hence \( x_{k+1} = P_{C_{k+1} \cap Q_{k+1}} x_0 \) is well defined.

Next, we show that \( \lim_{n \to \infty} d(Tx_n, x_n) = 0 \). For every \( n \in \mathbb{N} \), we have that
\[ d(x_0, x_n) = d(x_0, P_{C_n \cap Q_n} x_0) \leq d(x_0, P_F x_0) < \frac{\pi}{2}. \]

It follows that \( \sup_{n \in \mathbb{N}} d(x_0, x_n) < \pi/2 \). Moreover, from the definition of \( Q_{n+1} \), we get that
\[ d(x_0, x_n) = d(x_0, P_{Q_{n+1}} x_0) \leq d(x_0, P_{C_{n+1} \cap Q_{n+1}} x_0) = d(x_0, x_{n+1}) \]
for \( n \in \mathbb{N} \). Thus \( \{\cos d(x_0, x_n)\} \) is a nonincreasing real sequence so that it has a limit such that
\[ a = \lim_{n \to \infty} \cos d(x_0, x_n) > \cos \frac{\pi}{2} = 0. \]

Since \( x_{n+1} \in Q_{n+1} \), we have that \( \cos d(x_0, x_n) \cos d(x_n, x_{n+1}) \geq \cos d(x_0, x_{n+1}) \) for \( n \in \mathbb{N} \). Letting \( n \to \infty \), we have that
\[ a \cos \lim_{n \to \infty} d(x_n, x_{n+1}) \geq a > 0 \]
and hence \( \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \). On the other hand, since \( x_{n+1} \in C_{n+1} \), it holds that \( d(Tx_n, x_{n+1}) \leq d(x_n, x_{n+1}) \). These facts imply that
\[ \lim_{n \to \infty} d(Tx_n, x_n) \leq \lim_{n \to \infty} (d(Tx_n, x_{n+1}) + d(x_{n+1}, x_n)) \leq 2 \lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \]

Let \( \{x_n\} \) be an arbitrary subsequence of \( \{x_n\} \). Then, since \( \sup_{n \in \mathbb{N}} d(x_0, x_n) < \pi/2 \), there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \), simply denoted by \( \{w_j\} \), which is \( \Delta \)-convergent to some \( w_\infty \in C \). We know that \( w_\infty \) is a fixed point of \( T \). Indeed, it holds that
\[ \limsup_{j \to \infty} d(w_j, Tw_\infty) \leq \limsup_{j \to \infty} (d(w_j, Tw_j) + d(Tw_j, Tw_\infty)) \leq \limsup_{j \to \infty} (d(w_j, Tw_j) + d(w_j, w_\infty)) = \limsup_{j \to \infty} d(w_j, w_\infty), \]
and since \( w_\infty \) is a unique asymptotic center of \( \{w_j\} \), we have that \( Tw_\infty = w_\infty \). Further, since \( w_j = x_{n_j} = P_{C_{n_j} \cap Q_{n_j}} x_0 \in C_{n_j} \cap Q_{n_j} \) and \( F \subset C_{n_j} \cap Q_{n_j} \) for \( j \in \mathbb{N} \), we have that
\[ d(x_0, P_F x_0) \leq d(x_0, w_\infty) \leq \lim_{j \to \infty} d(x_0, w_j) = \lim_{j \to \infty} d(x_0, P_{C_{n_j} \cap Q_{n_j}} x_0) \leq d(x_0, P_F x_0). \]
Thus \( \lim_{j \to \infty} d(x_0, w_j) = d(x_0, w_\infty) \) and, by Lemma 2.3, we obtain that \( \{w_j\} \) converges to \( w_\infty \). The inequalities above also imply that \( d(x_0, P_F x_0) = d(x_0, w_\infty) \), and hence \( w_\infty = P_F x_0 \). Consequently, we have that \( \{x_n\} \) converges to \( P_F x_0 \), which is the desired result. 

The second iterative scheme is known as the shrinking projection method, which was introduced by Takahashi, Takeuchi, and Kubota [13] in the setting where the underlying space is a Hilbert space. Various generalized results to Banach spaces have been proposed, and recently, Kimura [4] showed convergence of this scheme defined on a real Hilbert ball. Employing the method of its proof, we obtain the following result.

**Theorem 3.2.** Let \( S_H \) be the unit sphere of a real Hilbert space \( H \) with the metric \( d \) defined by a length of minimal great arc, and \( C \) a closed convex subset of \( S_H \) such that \( d(u, v) < \pi/2 \) for every \( u, v \in C \). Let \( T : C \to C \) be a nonexpansive mapping such that the set of fixed points \( F = \{z \in C : Tz = z\} \) is nonempty. For a given initial point \( x_0 \in C \) and \( P_0 = C \), generate a sequence \( \{x_n\} \) as follows:

\[
C_{n+1} = \{z \in C : d(Tx_n, z) \leq d(x_n, z)\} \cap C_n,
\]

\[
x_{n+1} = P_{C_{n+1}}x_0,
\]

for each \( n \in \mathbb{N} \). Then \( \{x_n\} \) is well defined and converges to \( P_F x_0 \in C \), where \( P_K : C \to K \) is the metric projection of \( C \) onto a nonempty closed convex subset \( K \) of \( C \).

**Proof.** Since \( T \) is nonexpansive, we have that a hemisphere \( \{z \in C : d(Tu, z) \leq d(u, z)\} \) includes the set of fixed point \( F \) for every \( u \in C \). Therefore, by induction, we may show that \( \{x_n\} \) is well defined and \( \{C_n\} \) is a sequence of nonempty closed convex subsets such that \( F \subset \bigcap_{n=1}^{\infty} C_n \). Since \( x_n = P_{C_n}x_0 \) and \( P_{C_n} \in F \subset C_n \), we have that

\[
d(x_0, x_n) = d(x_0, P_{C_n}x_0) \leq d(x_0, P_F x_0) < \frac{\pi}{2}
\]

for every \( n \in \mathbb{N} \) and thus \( \sup_{n \in \mathbb{N}} d(x_0, x_n) < \pi/2 \).

Let \( \{x_{n_j}\} \) be any arbitrary subsequence of \( \{x_n\} \). Then, since \( \sup_{n \in \mathbb{N}} d(x_0, x_n) < \pi/2 \), there exists a subsequence \( \{x_{n_{j_k}}\} \) (simply denoted by \( \{w_j\} \), which is \( \Delta \)-convergent to some \( w_\infty \in C \). Let \( k \in \mathbb{N} \) be arbitrarily fixed. Then, there exists \( j_0 \in \mathbb{N} \) such that \( n_{j_k} > k \). Since \( w_j \in C_{j_k} \) for every \( j \geq j_0 \) and \( C_{j_k} \) is closed and convex, by Remark 2.2, we have that \( w_\infty \in C_{j_k} \). Hence we have that \( w_\infty \in \bigcap_{n=1}^{\infty} C_n \). Let \( y_0 = P_{\bigcap_{n=0}^{\infty} C_n}x_0 \). Then we have that

\[
d(x_0, y_0) = d(x_0, P_{\bigcap_{n=0}^{\infty} C_n}x_0) \leq \lim_{j \to \infty} d(x_0, P_{C_{n_j}}x_0) \leq d(x_0, P_F x_0) < \frac{\pi}{2}
\]

and thus \( d(x_0, y_0) = d(x_0, w_\infty) = \lim_{j \to \infty} d(x_0, w_j) \). By Lemma 2.3, \( \{w_j\} \) converges to \( w_\infty \) and \( w_\infty = y_0 \). Since every subsequence of \( \{x_{n_k}\} \) of \( \{x_{n_{j_k}}\} \) converges to \( y_0 \), we obtain that \( \{x_n\} \) converges to \( y_0 = P_{\bigcap_{n=0}^{\infty} C_n}x_0 \). Since \( y_0 \in C_n \) for every \( n \in \mathbb{N} \), we have that

\[
\lim_{n \to \infty} d(x_n, y_0) = \lim_{n \to \infty} d(Tx_n, y_0) = 0
\]

and thus \( \{x_n\} \) also converges to \( y_0 \). Using the continuity of \( T \), we obtain that \( Ty_0 = y_0 \), that is \( y_0 \in F \). Therefore we have that

\[
d(x_0, y_0) = d(x_0, P_{\bigcap_{n=0}^{\infty} C_n}x_0) \leq d(x_0, P_F x_0) \leq d(x_0, y_0),
\]

which means that \( y_0 \) is the unique nearest point to \( x_0 \) in \( F \). Hence \( y_0 = P_F x_0 \), which is the desired result. 

\[\square\]
References