Compression of Khalimsky topological spaces

Jeang Min Kang\textsuperscript{a}, Sang-Eon Han\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Yonsei University, 262 Seongsanno, Seodaemun-Gu, Seoul, 126-749, Republic of Korea
\textsuperscript{b}Faculty of Liberal Education, Institute of Pure and Applied Mathematics, Chonbuk National University, Jeonju-City Jeonbuk, 561-756, Republic of Korea

Abstract. Aiming at the study of the compression of Khalimsky topological spaces which is an interesting field in digital geometry and computer science, the present paper develops a new homotopy thinning suitable for the work. Since Khalimsky continuity of maps between Khalimsky topological spaces has some limitations of performing a discrete geometric transformation, the paper uses another continuity (see Definition 3.4) that can support the discrete geometric transformation and a homotopic thinning suitable for studying Khalimsky topological spaces. By using this homotopy, we can develop a new homotopic thinning for compressing the spaces and can write an algorithm for compressing 2D Khalimsky topological spaces.

1. Introduction

Digital geometry is an approach to understanding and compressing qualitative properties of digital images which have been studied in computer science, viewed as non-Hausdorff subspaces of $\mathbb{Z}^n$ with some particular choices of non-Hausdorff topology, $n \in \mathbb{N}$, where $\mathbb{Z}^n$ is the set of points in the Euclidean $n$D space with integer coordinates and $\mathbb{N}$ represents the set of natural numbers. The idea of this subject is that qualitative features of images are often in the center of interest in computer image processing, and further that they provide a useful compression of images.

In digital geometry one of the interesting areas is the Khalimsky $n$D space which is a locally finite space and satisfies the separation axiom $T_0$ instead of the Hausdorff separation axiom if $n \geq 2$. In addition, the Khalimsky 1D space satisfies the separation axiom $T_1$ [21]. Thus the present paper mainly studies subspaces of the Khalimsky $n$D space from the viewpoint of digital geometry.

In relation to the study of discrete objects in $\mathbb{Z}^n$, we have used many tools from combinatorial topology, graph theory, Khalimsky topology and so forth [7–9, 20, 21, 23, 24, 26, 30]. Motivated by Alexandroff spaces in [1], the Khalimsky $n$D space, denoted by $(\mathbb{Z}^n, T^n)$, was established and its study includes the papers [8, 13, 21, 28, 31]. Since the topology $(\mathbb{Z}^n, T^n)$ is established on the Euclidean $n$D space, it is useful to consider a subset $X \subseteq \mathbb{Z}^n$ to be a subspace of $(\mathbb{Z}^n, T^n)$ denoted by $(X, T^n_X)$, $n \geq 1$ [1, 13].

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Email addresses: maniddo992@naver.com (Jeang Min Kang), sehan@jbnu.ac.kr (Sang-Eon Han)
In this regard, both a Khalimsky continuous map and a Khalimsky homeomorphism have been often used in digital geometry [5, 6, 13, 18, 20–22, 24, 28, 31]. But it is well known that the graph $k$-connectivity of $Z^n$ and every topological connectedness as well as Khalimsky connectedness are partially compatible with each other [4, 5, 25]. Besides, a Khalimsky continuous map has some limitations of performing a discrete geometric transformation such as a rotation by 90° and a parallel translation with an odd vector (see Remark 3.3). In order to overcome this difficulty, it can be helpful to take into account a reasonable $k$-adjacency relations of $Z^n$ on $(X, T^n_X)$. Thus, considering a Khalimsky topological space $(X, T^n_X)$ with one of the graph $k$-adjacency relations of (2.1), we call it a space if there is no danger of ambiguity and use the notation $(X, k, T^n_X) := X_{n,k}$. The paper [13] introduced the category (briefly, $CTC$) consisting of both a set $Ob(CTC)$ of $X_{n,k}$ and a set $Mor(X_{m,k}, Y_{n,k})$ of $(k_0, k_1)$-continuous maps between each pair $X_{m,k}$ and $Y_{n,k}$ in $Ob(CTC)$ (see Definition 3.4). The present paper uses the continuity in $CTC$ (see Definition 3.4) instead of both the Khalimsky continuity and the digital continuity in [3, 29] which leads to the development of a homotopy suitable for studying Khalimsky topological spaces.

The paper proposes that an approach to the study of $(X, T^n_X)$ from the viewpoint of $CTC$ can overcome the limitation of Khalimsky continuity mentioned above. This is one of the reasons why we study a Khalimsky topological space with graph $k$-connectivity and continuity of map in $CTC$. Finally, in relation to the compression of Khalimsky topological spaces in $CTC$, we can use a homotopy in $CTC$ and develop a homotopic thinning which can substantially contribute to the compression of the spaces $X_{n,k}$.

This paper is organized as follows. Section 2 provides some basic notions. Section 3 compares a Khalimsky continuous map with continuous maps in $CTC$ and further, it refers to some utilities of the category $CTC$. Section 4 develops a homotopy thinning in $CTC$ and proposes a method of compressing Khalimsky topological spaces in terms of the homotopic thinning in $CTC$. In addition it suggests an algorithm for compressing spaces $X_{2,k}$. Section 5 concludes the paper with a summary and a further work.

2. Preliminaries

Let us now review some basic notions and properties of Khalimsky $nD$ spaces. Khalimsky topology arises from the Khalimsky line. More precisely, Khalimsky line topology on $Z$ is induced from the subbase $\{[2n-1,2n+1]: n \in Z\}$ (see also [21]). Namely, the family of the subset $\{(2n+1), [2m-1, 2m+1]_Z : m, n \in Z\}$ is a basis of the Khalimsky line topology on $Z$ denoted by $(Z, T)$. Indeed, Khalimsky line topology has useful properties. For instance, the Khalimsky line $(Z, T)$ is connected and if one point is removed, then it consists of two components and is finally not connected [21], which is the similar property of the real line with the usual topology $(R, U)$, where $R$ means the set of real numbers. Furthermore, the usual product topology on $Z^n$ induced from $(Z, T)$, denoted by $(Z^n, T^n)$, is called the Khalimsky $nD$ space. In the present paper each space $X \subset Z^n$ will be considered to be a subspace $(X, T^n_X)$ induced from the Khalimsky $nD$ space $(Z^n, T^n)$.

Let us recall basic terminology of the structure of $(Z^n, T^n)$. A point $x = (x_1, x_2, \cdots, x_n) \in Z^n$ is called pure open if all coordinates are odd, and pure closed if each of the coordinates is even [21] and the other points in $Z^n$ is called mixed [21]. In each of the spaces of Figures 1, 2, 3, 4, 5, and 6 a black big dot stands for a pure open point and the symbols ■ and ● mean a pure closed point and a mixed point, respectively.

Since a Khalimsky continuous map $f$ need not preserve the digital connectivity of $Dom(f)$, it is meaningful to study a multi-dimensional Khalimsky topological space $(X, T_X)$ with $k$-connectivity denoted by $X_{n,k}$. Thus let us recall the digital $k$-connectivity of $Z^n$, as follows.

As a generalization of the commonly used $k$-adjacency relations of $Z^2$ and $Z^3$ [26, 29], the $k$-adjacency relations of $Z^n$ were represented in [7] (see also [9]) as follows.

For a natural number $m$ with $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n)$$

and

$$q = (q_1, q_2, \cdots, q_n) \in Z^n,$$

are $k(m, n)$-adjacent if

$\bullet$ there are at most $m$ indices $i$ such that $|p_i - q_i| = 1$ and

$\bullet$ for all other indices $i$ such that $|p_i - q_i| \neq 1, p_i = q_i$. 

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In this operator \( k := k(m, n) \) is the number of points \( q \) which are \( k \)-adjacent to a given point \( p \) according to the numbers \( m \) and \( n \) in \( \mathbb{N} \), where \( \sim \) means equal by definition. Indeed, this \( k(m, n) \)-adjacency is another presentation of the \( k \)-adjacency of [7, 9] (for more details, see [17]). Consequently, this operator leads to the representation of the \( k \)-adjacency relations of \( \mathbb{Z}^n \) [16]:

\[
k := k(m, n) = \sum_{j=n-m}^{n-1} 2^{n-j} C^n_j,
\]

(2.1)

where \( C^n_j = \frac{n!}{j!(n-j)!} \).

For instance, 8-, 32-, 64- and 80-adjacency relations of \( \mathbb{Z}^4 \) are considered and further, 10-, 50-, 130-, 210- and 242-adjacency relations of \( \mathbb{Z}^5 \) are obtained.

Owing to the digital \( k \)-connectivity paradox [26], a set \( X \subset \mathbb{Z}^n \) with one of the \( k \)-adjacency relations of \( \mathbb{Z}^n \) is usually considered in a quadruple \( (\mathbb{Z}^n, k, \bar{k}, X) \), where \( n \in \mathbb{N}, X \subset \mathbb{Z}^n \) is the set of points we regard as belonging to the set depicted, \( k \) represents an adjacency relation for \( X \) and \( \bar{k} \) represents an adjacency relation for \( \mathbb{Z}^n - X \), where \( k \neq \bar{k} \) except \( X \subset \mathbb{Z} \) [29]. But the paper is not concerned with the \( k \)-adjacency of \( X \). We say that the pair \( (X, k) \) is a digital space with \( k \)-adjacency (briefly, (binary) digital space) in \( \mathbb{Z}^n \) and a subset \( (X, k) \) of \( (\mathbb{Z}^n, k) \) is \( k \)-connected if it is not a union of two disjoint non-empty sets not \( k \)-adjacent to each other [26]. In other words, for a set \( (X, k) \) in \( \mathbb{Z}^n \), two distinct points \( x, y \in X \) are called \( k \)-connected if there is a \( k \)-path \( f : [0, m] \rightarrow \mathbb{Z}^n \) whose image is a sequence \( (x_0, x_1, \ldots, x_m) \) consisting of the set of points \( \{f(0) = x_0, f(1) = x_1, \ldots, f(m) = x_m = y\} \) such that \( x_i \) and \( x_{i+1} \) are \( k \)-adjacent, \( i \in [0, m - 1] \). The number \( m \) is called the length of this \( k \)-path [26]. For a digital space \( (X, k) \) and a point \( x \in X \), we say that the maximal \( k \)-connected subset of \( (X, k) \) containing the point \( x \in X \) is the \( k \)-connected component of a point \( x \in X \) [26]. For a digital graph connectivity \( k \), a simple \( k \)-path in \( X \) is the sequence \( (x_i)_{i \in [0, m]} \), where \( x_i \) and \( x_{i+1} \) are \( k \)-adjacent if and only if \( i = j + 1 \) or \( i = j + 1 \) [26]. Further, a simple closed \( k \)-curve with \( l \) elements in \( \mathbb{Z}^n \), denoted by \( SC^n_k \) [10], is the simple \( k \)-path \( (x_i)_{i \in [0, j]} \), where \( x_i \) and \( x_j \) are \( k \)-adjacent if and only if \( j = i + 1 \mod l \) or \( i = j + 1 \mod l \) [26].

3. Comparison of a Khalimsky continuous map and continuous maps in CTC

In this section by comparing a Khalimsky continuous map with a continuous map in CTC, we refer to some limitations of Khalimsky continuity of maps between Khalimsky topological spaces so that we can speak out merits of a Khalimsky topological space with digital \( k \)-connectivity and justify the \((k_0, k_1)\)-continuity of Definition 3.4 which will be used in the paper. In the Khalimsky \( nD \) space \( (\mathbb{Z}^n, T^n) \), as usual, consider a subset \( X \subset \mathbb{Z}^n \) to be a subspace \((X, T^n_X) \) induced from \((\mathbb{Z}^n, T^n) \), where \( T^n_X = \{O \cap X | O \in T^n\} \).

In this paper we mainly study spaces \((X, T^n_X) \) with one of the \( k \)-adjacency relations of \( \mathbb{Z}^n \) that is denoted by \((X, k, T^n_X) := X_{nk} \) [13] and is called a space. In relation to the establishment of various kinds of continuities of maps between spaces \( X_{nk} \) [13], in digital geometry we have often used the following digital \( k \)-neighborhood, denoted by \( N_k(x, \varepsilon) \), [8] (see also [9]) based on the notions of both digital adjacency and a simple \( k \)-path in Section 2.

Definition 3.1. ([8]; see also [9]) Let \((X, k)\) be a digital space, \( X \subset \mathbb{Z}^n \), \( x, y \in X \), and \( \varepsilon \in \mathbb{N} \). By the digital \( k \)-neighborhood \( N_k(x, \varepsilon) \) we denote the set

\[
\{y \in X : d_k(x, y) \leq \varepsilon\} \cup \{x\},
\]

where \( d_k(x, y) \) is the length of a shortest simple \( k \)-path \( x \) to \( y \) in \( X \). Besides, we put \( d_k(x, y) = \infty \) if there is no \( k \)-path from \( x \) to \( y \). Thus, if the \( k \)-component of \( x \) is the singleton \( \{x\} \), then we assume that \( N_k(x, \varepsilon) = \{x\} \) for any \( \varepsilon \in \mathbb{N} \).

Consider digital spaces \((X, k_0)\) in \( \mathbb{Z}^m \) and \((Y, k_1)\) in \( \mathbb{Z}^n \) and further, a map \( f : (X, k_0) \rightarrow (Y, k_1) \). Then the digital continuity of \( f \) in [3] can be equivalently represented in this way [15]:

The map \( f \) is digitally \((k_0, k_1)\)-continuous at a point \( x \in X \) if and only if

\[
f(N_k(x, 1)) \subset N_k(f(x), 1).
\]

(3.1)
By using the digital \((k_0,k_1)\)-continuity of (3.1), we obtain a digital topological category, briefly DTC, consisting of the following two sets [9] (see also [13]):

1. A set of objects \((X,k)\) in \(Z^n\);
2. For every ordered pair of objects \((X,k_0)\) in \(Z^{n_0}\) and \((Y,k_1)\) in \(Z^{n_1}\), a set of all digitally \((k_0,k_1)\)-continuous maps \(f : (X,k_0) \rightarrow (Y,k_1)\) as morphisms.

In DTC, for \([a,b] \subset Z\) with \(a \leq b\) we assume \([a,b]_Z = \{n \in Z | a \leq n \leq b\}\) with 2-adjacency [26]. Let us now recall the Khalimsky topological k-neighborhood which can be used for establishing continuity of maps between the spaces \(X_{n,k}\) in CTC (see Definition 3.4).

**Definition 3.2.** ([8]; see also [13, 20]) Consider a space \(X_{n,k} := X, x, y \in X, \text{ and } \varepsilon \in N\).

1. A subset \(V\) of \(X\) is called a Khalimsky topological neighborhood of \(x\) if there exists \(O_x \in T^n_X\) such that \(x \in O_x \subseteq V\).
2. If a digital \(k\)-neighborhood \(N_k(x,\varepsilon)\) is a Khalimsky topological neighborhood of \(x\) in \((X,T^n_X)\), then this set is called a Khalimsky topological \(k\)-neighborhood of \(x\) with radius \(\varepsilon\) and we use the notation \(N'_k(x,\varepsilon)\) instead of \(N_k(x,\varepsilon)\).

In Figure 1(a), no \(N'_k(a_1,1)\) exists, \(i \in [0,8]\) because the smallest open set containing the point \(a_0\) (resp., \(a_8\)) is the set \([a_1, a_0, a_1, a_2]\) (resp., \([a_0, a_3, a_0, a_0]\)\)). In addition, we can obtain that \(N'_k(a_0,2) = [a_7, a_1, a_0, a_1, a_2]\) and further, \(N'_k(a_8,2) = [a_0, a_1, a_2, a_3, a_0]\)\).

Let us recall Khalimsky (briefly, K)-continuity of maps between Khalimsky topological spaces: As usual, for two Khalimsky topological spaces \((X,T^n_X) := X\) and \((Y,T^n_Y) := Y\) a map \(f : X \rightarrow Y\) is called continuous at the point \(x \in X\) if for any open set \(O_f(y) \subset Y\) containing the point \(f(x)\) there is an open set \(O_x \subset X\) containing the point \(x\) such that \(f(O_x) \subset O_f(y)\). In terms of the Khalimsky continuity of the map \(f\), we obtain the Khalimsky topological category, briefly KTC, consisting of the following two sets [13]:

1. A set of objects \((X,T^n_X)\);
2. For every ordered pair of objects \((X,T^n_X)\) and \((Y,T^n_Y)\) a set of all Khalimsky (briefly, K)-continuous maps \(f : (X,T^n_X) \rightarrow (Y,T^n_Y)\) as morphisms.

![Figure 1: Existence of \(N'_k(x,\varepsilon)\).](image)

Hereafter, \(SC_{k,\varepsilon}^n\) established in Section 2 is assumed to be a subspace of \((Z^n, T^n)\). As already mentioned above, we can observe the following limitations of \(K\)-continuous maps in \(Mor(KTC)\).

**Remark 3.3.** (1) Let \(f : X_{n,k_0} \rightarrow Y_{n,k_1}\) be a \(K\)-continuous map. Then \(f\) need not map a \(k_0\)-connected subset into a \(k_1\)-connected one [13] (see also the maps \(f\) and \(g\) in Figure 2(a) and (b) of the present paper). More precisely, let us consider the two maps:

\(f : X_1 \rightarrow Y_1\) and \(g : X_2 \rightarrow Y_2\) in Figure 2(a) and (b), respectively. While they are \(K\)-continuous maps, \(f\) (resp. \(g\)) cannot preserve the 4-connectivity (resp. the 8-connectivity).
(2) Regard \( SC_k^{n,l} := (c_i)_{i \in [0,l-1]} \) as a subspace induced from the Khalimsky \( n \)D space \( (Z^n, T^n) \) and consider the self-map \( f : SC_k^{n,l} \rightarrow SC_k^{n,l} \) given by \( f(c_i) = c_i+1(\mod 4) \). Then \( f \) need not be a \( K \)-continuous map. More precisely, assume \( SC_3^{26} := (c_i)_{i \in [0,3]} \) as a subspace of \( (Z^3, T^3) \) (see Figure 1(b)) and the self-map \( f : SC_3^{26} \rightarrow SC_3^{26} \) given by \( f(c_i) = c_i+1(\mod 4) \). Then we can clearly observe that \( f \) cannot be a \( K \)-continuous map because \( O_{c_0} = \{c_3, c_0, c_1\} \), \( O_{c_1} = \{c_1\} \), \( O_{c_2} = \{c_4, c_2, c_1\} \), and \( O_{c_3} = \{c_3\} \), where \( O_x \) means the smallest open set containing the point \( x \).

(3) Let us consider the map \( f : (Z, T) \rightarrow (Z, T) \) given by \( f(t) = t + 1 \) which is a parallel translation with an odd vector. Then we can clearly observe that \( f \) cannot be a \( K \)-continuous map.

In view of Remark 3.3, to study spaces \( X_{n,k} \), we need to use another continuity that can support the digital geometric transformation mentioned in Remark 3.3.

**Definition 3.4.** ([13]) For two spaces \( X_{n,k} := X \) and \( Y_{n,k} := Y \) a function \( f : X \rightarrow Y \) is said to be \((k_0, k_1)\)-continuous at a point \( x \) in \( X \) if \( f(N_{k_0}^*(x, r)) \subset N_{k_1}^*(f(x), s) \), where the number \( r \) is the least element of \( \mathbb{N} \) such that \( N_{k_0}^*(x, r) \) contains an open set including the point \( x \) and \( s \) is the least element of \( \mathbb{N} \) such that \( N_{k_1}^*(f(x), s) \) contains an open set including the point \( f(x) \). Furthermore, we say that a map \( f : X \rightarrow Y \) is \((k_0, k_1)\)-continuous if the map \( f \) is \((k_0, k_1)\)-continuous at every point \( x \in X \).

In Definition 3.4 if such a neighborhood \( N_{k_1}^*(f(x), \epsilon) \) does not exist, then we clearly say that \( f \) cannot be \((k_0, k_1)\)-continuous at the point \( x \). Further, in Definition 3.4 if \( k_0 = k_1 \) and \( n_0 = n_1 \), then we call it a \( k_0 \)-continuous map.

Let us now recall the category [13], denoted by CTC, consisting of the following two sets:

- A set of objects \( X_{n,k} \) denoted by \( \text{Ob}(CTC) \);
- For every ordered pair of spaces \( X_{n,k_0} \) and \( Y_{n,k_1} \) in \( \text{Ob}(CTC) \) a set \( \text{Mor}(X_{n,k_0}, Y_{n,k_1}) \) of \((k_0, k_1)\)-continuous maps as morphisms.

In CTC, for \([a, b] \subset Z \) with \( a \leq b \), \([a, b]_Z = [a \leq n \leq b] \) can be assumed to be a subspace of \((Z, T)\) if it is related to the Khalimsky topology \((Z, T)\). Then \([a, b]_Z \) is called a Khalimsky interval [28] and is briefly denoted by \([a, b]_Z \). For a map \( f : X_{n,k} \rightarrow Y_{n,k} \) let us compare \( K \)-continuity of \( f \) in KTC with \( k \)-continuity of \( f \) in CTC. Indeed, according to the dimension \( n \) and connectedness of \( X_{n,k} \) and \( Y_{n,k} \), some intrinsic features appear.

**Theorem 3.5.** Let \( f : X_{1,2} \rightarrow Y_{1,2} \) be a map. Then \( K \)-continuity of \( f \) in KTC implies \( 2 \)-continuity of \( f \) in CTC. But the converse does not hold.

**Proof.** In the Khalimsky line \((Z, T)\) each point \( t \in Z \) has \( N_2^*(t, 1) = \{t - 1, t, t + 1\} \subset Z \) and further, the point \( t \) is connected to the points \( t - 1 \) and \( t + 1 \). Thus a \( K \)-continuous map \( f : X_{1,2} \rightarrow Y_{1,2} \) implies a \( 2 \)-continuous map in CTC because a \( K \)-continuous map preserves connectedness under the Khalimsky line topology. Let us now examine if the assertion is true or not with the following four cases at every point \( x \in X_{1,2} \) and its image \( f(x) := y \in Y_{1,2} \).

**Case 1** Assume that both \( x \) and \( y \) are pure closed points. Then a \( K \)-continuous map \( f : X_{1,2} \rightarrow Y_{1,2} \) is equivalent to a \( 2 \)-continuous map in CTC because \( N_2^*(x, 1) = O_x \) and \( N_2^*(y, 1) = O_y \), where \( O_x \) means the smallest open set containing \( f \) under the subspace topologies on \( X_{1,2} \) and \( Y_{1,2} \).

**Case 2** Assume that both \( x \) and \( y \) are pure open points so that \( O_x = \{x\} \) and \( O_y = \{y\} \). Since from the hypothesis we obtain that \( f(O_x) \subset O_y \), we clearly observe that \( f(N_2^*(x, 1)) \subset N_2^*(y, 1) \) because \( O_x \subset N_2^*(x, 1) \) and \( O_y \subset N_2^*(y, 1) \) and further, if there is a point \( x_1 \in N_2^*(x, 1) \) with \( x_1 \neq x \) such that \( f(x_1) \notin N_2^*(y, 1) \), then \( f \) cannot be a \( K \)-continuous map at the point \( x_1 \) because \( f \) does not preserve connectedness between the points \( x_1 \) and \( x \).

**Case 3** Assume that \( x \) is a pure closed point and \( y \) is a pure open point. Owing to both the hypothesis of the \( K \)-continuity of \( f \) and the fact that \( N_2^*(x, 1) = O_x \), we obtain that \( f(N_2^*(x, 1)) \subset O_y = \{y\} \), which implies that \( f(N_2^*(x, 1)) \subset N_2^*(y, 1) \) because \( O_y \subset N_2^*(y, 1) \).
(Case 4) Assume that $x$ is a pure open point and $y$ is a pure closed point. Owing to the hypothesis of the $K$-continuity of $f$, we obtain that $f(O_x) \subset O_y \subset N_2(y, 1)$. Further, we clearly observe that $f(N_2^*(x, 1)) \subset N_2^*(y, 1)$. If not, suppose that there is a point $x_1 \in N_2^*(x, 1)$ with $x_1 \neq x$ such that $f(x_1) \notin N_2^*(y, 1)$. Then the map $f$ cannot be a $K$-continuous map at the point $x_1$ because $f$ does not preserve connectedness between the points $x_1$ and $x$.

In view of the above four cases, we can conclude that a $K$-continuous map $f : X_{1,2} \to Y_{1,2}$ implies a 2-continuous map in CTC.

But the converse does not hold as demonstrated by the following example: Consider the map $f$ in Remark 3.3(3) which is a parallel translation with an odd vector. While $f$ is a 2-continuous map in CTC, as already mentioned in Remark 3.3(3), it cannot be a $K$-continuous map. \quad \square

**Remark 3.6.** Regardless of connectedness and disconnectedness of the two spaces $X_{1,2}$ and $Y_{1,2}$, the assertion of Theorem 3.2 is valid.

Unlike Theorem 3.2, if $n \geq 2$, then we obtain the following:

**Theorem 3.7.** Let $f : X_{n,k} \to Y_{n,k}$ be a map, $n \geq 2$. Then none of $K$-continuity of $f$ and $k$-continuity of $f$ in CTC implies the other.

**Proof.** Assume that $f : X_{n,k} \to Y_{n,k}$ is a $K$-continuous map on $X_{n,k}$, $n \geq 2$. Namely, for each point $x \in X_{n,k}$ and its image $f(x) \in Y_{n,k}$ under the map $f$ we can obtain that for the smallest open set $O_{f(x)} \subset Y$ containing the point $f(x)$ there is the smallest open set $O_x \subset X$ containing the point $x$ such that

$$f(O_x) \subset O_{f(x)}. \quad (3.2)$$

But the property (3.2) need not imply the property

$$f(N_2^*(x, \delta)) \subset N_2^* (f(x), \epsilon) \quad (3.3)$$

for some $\delta$ and $\epsilon$ in $\mathbb{N}$ (see Cases 1-1, 1-2 and 2 below), which means that $K$-continuity of $f$ does not imply $k$-continuity of $f$ in CTC.

Conversely, assume that $f : X_{n,k} \to Y_{n,k}$ satisfies the property (3.3). Then we can observe that the property (3.3) need not imply the property (3.2) (see Cases 1-3, 1-4 and 2 below), which means that $k$-continuity of $f$ in CTC does not propose $K$-continuity of $f$.

(Case 1) In case $n = 2$, by using several similar described in 2D Khalimsky spaces, we can prove the assertion.

(Case 1-1) In Figure 2(a) assume the map $f : X_1 := \{x_i | i \in [0, 3]\} \to Y_1 := \{y_i | i \in [0, 1]\}$ defined by $f(x_0) = y_0$ and $f(x_2) = y_1$. While the map $f$ is $K$-continuous on $X_1$, it cannot be $4$-continuous in CTC (see the point $x_2$). More precisely, since the smallest open set containing $x_2$ is the set $\{x_2, x_3\}$, we obtain that $N_2^*(x_2, 1) = \{x_1, x_2, x_3\}$ and further, $N_2^*(y_1, 1) = \{y_1\}$. Since $f(N_2^*(x_2, 1))$ cannot be a subset of $N_2^*(y_1, 1)$, we can conclude that $f$ cannot be $4$-continuous at the point $x_2$.

(Case 1-2) In Figure 2(b) consider the map $g : X_2 := \{x_i | i \in [0, 2]\} \to Y_2 := \{y_i | i \in [0, 2]\}$ given by $g(x_0) = y_0$ and $g(x_1) = y_2$. Indeed, the map $g$ is $K$-continuous on $X_2$ because $O_{y_0}$ and $O_{y_1}$ are the sets $\{x_0\}$ and $\{x_1, x_2\}$, respectively. But it cannot be $8$-continuous in CTC (see the points $x_0$ and $x_1$) because $g(N_2^*(x_0, 1))$ cannot be a subset of $N_2^*(y_0, 1) = \{y_0\}$, where $N_2^*(x_0, 1) = \{x_0, x_1\}$.

(Case 1-3) In Figure 2(c) assume the map $h : X_3 := \{x_i | i \in [0, 2]\} \to Y_3 := \{y_i | i \in [0, 1]\}$ defined by $h(x_0) = y_0$ and $h(x_2) = \{y_1\}$. While the map $h$ is $8$-continuous in CTC, it cannot be $K$-continuous on $X_3$ (see the point $x_1$) because $O_{y_1}$ is the total set $X_3$ and for each $y_i \in Y_3$ we obtain that $O_{y_i} = \{y_i\}$, $i \in [0, 1]$.

(Case 1-4) In Figure 2(d) consider the map $i : X_4 := \{x_i | i \in [0, 2]\} \to Y_4 := \{y_i | i \in [0, 1]\}$ given by $i(x_0) = y_0$ and $i(x_2) = \{y_1\}$. While the map $i$ is $4$-continuous in CTC because $N_2^*(y_0, 1) = N_2^*(y_1, 1) = Y_4$ and $N_2^*(x_2, 1) = \{x_0, x_2\}$, it cannot be $K$-continuous on $X_4$ (see the point $x_1$) because $O_{x_1}$ is the total set $X_4$ which implies that $N_2^*(x_1, 1) = X_4$.

In view of the above four cases, if $n = 2$, then the proof is completed.

(Case 2) In case $n \geq 3$, by using a method similar to that used in the proof of the case that $n = 2$, we can prove the assertion. \quad \square
In relation to the proof of Theorem 3.4, we can observe that the two spaces $X_2$ and $Y_2$ in Figure 2(b) are not connected. In addition, we can also observe that $Y_3$ in Figure 2(c) cannot be connected either. Thus, in Theorem 3.4 if we further assume that $Dom(f)$ and $Codom(f)$ are connected, then we can obtain the result different from Theorem 3.4.

**Theorem 3.8.** Let $f : X_{n,k} \to Y_{n,k}$ be a map such that $k = 3^n - 1$, $n \geq 2$, and let both $X_{n,k}$ and $Y_{n,k}$ be connected. Then $k$-continuity of $f$ implies $(3^n - 1)$-continuity of $f$ in CTC. But the converse does not hold.

**Proof.** Assume that $f : X_{n,3^{n-1}} \to Y_{n,3^{n-1}}$ is a $K$-continuous map. For every connected space $A_{n,k} \subset X_{n,k}$, $k = 3^n - 1$ its image $f(A_{n,k})$ is connected in $Y_{n,k}$. Owing to this property, for every point $x \in X_{n,k}$ and its image $f(x) \in Y_{n,k}$ there are $O_x \subset X_{n,k}$ and $O_f(x) \subset Y_{n,k}$ such that $f(O_x) \subset O_f(x)$ and further, $N_{3^{n-1}}(x, 1) \subset X_{n,k}$ and $N_{3^{n-1}}(f(x), 1) \subset Y_{n,k}$ such that $O_x \subset N_{3^{n-1}}(x, 1)$ and $O_f(x) \subset N_{3^{n-1}}(f(x), 1)$. Finally, we obtain that

$$f(N_{3^{n-1}}(x, 1)) \subset N_{3^{n-1}}(f(x), 1).$$

If not, suppose that there is a point $x \in N_{3^{n-1}}(x, 1) \setminus O_x$ such that $f(x) \notin N_{3^{n-1}}(f(x), 1)$, then the map $f$ cannot be a $K$-continuous map at the point $x$, which contradicts the hypothesis.

Let us now prove that the converse does not hold in terms of the following example. Consider the space $SC_{3^{n-1}} \subset (c_i)_{i \in [0,3]}$, in which two points are pure open and the others are pure closed, and the self-map $f : SC_{3^{n-1}} \to SC_{3^{n-1}}$ given by $f(c_i) = c_{i+1 \pmod{4}}$. Then we can observe that $f$ is $(3^n - 1)$-continuous from the viewpoint of CTC. But it cannot be a $K$-continuous map. For instance, consider the space $SC_{26} := (c_i)_{i \in [0,3]}$ in Remark 3.3(2) and the self-map $f : SC_{26} \to SC_{26}$. While $f$ is 26-continuous in CTC, it cannot be a $K$-continuous map.

In Theorem 3.5 if we replace the $(3^n - 1)$-adjacency by a $k$-adjacency of (2.1) with $k \neq 3^n - 1$, then the assertion of Theorem 3.5 cannot be true.

**Theorem 3.9.** In Theorem 3.5, if $k \neq 3^n - 1$, then unlike the assertion of Theorem 3.5, we obtain the following: Let $f : X_{n,k} \to Y_{n,k}$ be a map with $k \neq 3^n - 1$ and $n \geq 2$ such that $X_{n,k}$ and $Y_{n,k}$ are connected. Then none of $K$-continuity of $f$ and $k$-continuity of $f$ in CTC implies the other.

**Proof.** (Case 1) In case $n = 2$, let us prove that $K$-continuity of $f$ need not imply 4-continuity of $f$ in CTC. Assume the self-map $f : X_{2,4} \to X_{2,4}$ in Figure 3(a) given by $f([x_0]) = [x_0]$ and $f([x_1, x_2]) = [x_2]$. While the map $f$ is a $K$-continuous map, it cannot be a 4-continuous map in CTC (see the point $x_1$).
Conversely, let us prove that 4-continuity in CTC need not imply K-continuity in KTC with the following example: Consider the self-map $f : X_{3,4} \rightarrow X_{3,4}$ in Figure 3(b) given by $f(x_i) = x_{i+1(\mod 8)}$. While $f$ is a 4-continuous map in CTC, it cannot be a K-continuous map.

(Case 2) In case $n = 3$, let us now prove that K-continuity need not imply k-continuity in CTC, $k \neq 26$.

(Case 2-1) In case $k = 18$, assume the map $f : X_{3,18} \rightarrow Y_{3,18}$ in Figure 3(c) given by $f(\{x_0, x_1\}) = \{y_0\}$ and $f(\{x_2, x_3\}) = \{y_2\}$. Then we observe that both $X_{3,18}$ and $Y_{3,18}$ are connected, and $f$ is K-continuous. But it cannot be 18-continuous at the point $x_2$ because $f(N_{18}^*(x_2, 1))$ is not a subset of $N_{18}^*(f(x_2), 1)$, where $N_{18}^*(x_2, 1) = [x_1, x_2, x_3]$ and $N_{18}^*(f(x_2), 1) = [y_1, y_2]$ because $x_2 = [x_2, x_3]$ and $O_{f(x_2)} = \{f(x_2)\}$ with $f(x_2) = y_2$. In view of this investigation, we can observe that K-continuity of $f$ in KTC cannot support 18-continuity of $f$ in CTC.

Conversely, let us prove that 18-continuity in CTC need not imply K-continuity in KTC with the following example: Consider the self-map $f : W_{3,18} \rightarrow W_{3,18}$ in Figure 3(d) given by $f(w_i) = w_{i+1(\mod 4)}$. While $f$ is an 18-continuous map in CTC, it cannot be a K-continuous map.

(Case 2-2) In case $k = 6$, let us prove that K-continuity of $f$ in KTC need not imply 6-continuity of $f$ in CTC: Assume the map $f : X_{3,6} \rightarrow Y_{3,6}$ in Figure 3(e) given by $f(\{x_0\}) = \{y_0\}$ and $f(\{x_1, x_2, x_3\}) = \{y_1\}$. Then we can observe that both $X_{3,6}$ and $Y_{3,6}$ are connected and further, $f$ is K-continuous. But it cannot be 6-continuous at the point $x_1$ because $f(N_{6}^*(x_1, 2))$ is not a subset of $N_{6}^*(f(x_1), 1)$, where $N_{6}^*(x_1, 2)$ is the smallest 6-neighborhood of the point $x_1$ so that $N_{6}^*(x_1, 2)$ is the total set $X_{3,6}$.

Conversely, let us prove that 6-continuity of a map $f$ in CTC need not imply K-continuity of $f$ in KTC with the following example: Consider the self-map $f : Z_{3,6} \rightarrow Z_{3,6}$ in Figure 3(f) given by $f(z_i) = z_{i+1(\mod 12)}$. Since each point $z_i \in Z_{3,6}$ has $N_{6}^*(z_i, 1) = [z_i, z_{i+1(\mod 12)}, z_i, z_{i+1(\mod 12)}]$, $f$ is a 6-continuous map in CTC. But $f$ cannot be a K-continuous map.

In view of this investigation, in case $n = 3$ if $k \neq 26$, then it turns out that none of K-continuity of $f$ in KTC and k-continuity of $f$ in CTC implies the other.

(Case 3) In case $n \geq 4$ and $k \neq 3^n - 1$ with the hypothesis that $X_{n,k}$ and $Y_{n,k}$ are connected, by using the similar methods used for proving Cases 1 and 2 above, we can clearly prove that none of K-continuity of $f$ in KTC and k-continuity of $f$ in CTC implies the other.

As already mentioned in Remark 3.3, while K-continuous maps in CTC have some limitations of proceeding the digital geometric transformation related to a rotation, a parallel translation and so forth, we can observe that a $(k_0, k_1)$-continuous map $f : X_{n_0, k_0} \rightarrow Y_{n_1, k_1}$ in CTC overcomes the limitations (see Remark 3.10), which invokes strong merits of CTC.

**Remark 3.10.** (1) For a $(k_0, k_1)$-continuous map $f : X_{n_0, k_0} \rightarrow Y_{n_1, k_1}$ in CTC, regardless of connectedness or disconnectedness of $\text{Dom}(f)$, $f$ maps a $k_0$-connected subset into a $k_1$-connected one if $k_i = 3^n - 1, n_i \in \mathbb{N}, i \in \{0, 1\}$. More precisely, in a Khalimsky topological space $(X, T_k^X)$ since each point $x \in X_{n_0, k_0}$ (resp. $y \in Y_{n_1, k_1}$) has $N_{k_1}^*(x, 1)$ (resp. $N_{k_1}^*(y, 1)$) which is equal to $N_{3^n-1}(x, 1)$ (resp. $N_{3^n-1}(y, 1)$), we can clearly observe that a $(3^n - 1, 3^n - 1)$-continuous map in CTC is equivalent to a digitally $(3^n - 1, 3^n - 1)$-continuous map in DTC.

(2) Assume $SC_{k}^{n_1} := \{c_i\}_{i \in \{0, \ldots, n_1 - 1\}_{2}}$ to be a subspace of $(Z^n, T^n)$, where $k = 3^n - 1$. Consider the self-map $f : SC_{k}^{n_1} \rightarrow SC_{k}^{n_1}$ in Remark 3.3(2) given by

$$f(c_i) = c_{i+1(\mod 1)}.$$  

While $f$ need not be a K-continuous map, $f$ is a k-continuous map in CTC because each point $c_i \in SC_{k}^{n_1}$ has $N_{k}^{*}(c_i, 1) \subset SC_{k}^{n_1}$. If $k \neq 3^n - 1$, then the map of (3.4) need not be a k-continuous map in CTC.

For instance, consider the set $A_{2,4} := \{a_i\}_{i \in \{0, \ldots, 11\}_{12}}$ in Figure 1(a). Assume the set $A_{2,4}$ with Khalimsky topology. Then the space can be regarded as $SC_{k}^{n_1, 12}$ in CTC or CTC. Let us consider the self-map $f : A_{2,4} \rightarrow A_{2,4}$ defined by $f(a_i) = a_{i+1(\mod 12)}$ from the viewpoint of KTC or CTC. Then we can observe that $f$ is neither K-continuous nor 4-continuous (see the point $a_0$).
If for $SC^n_k := (c_i)_{i \in [0,l-1]_Z}$ in CTC each point $c_i \in SC^n_k := (c_i)_{i \in [0,l-1]_Z} \subset (Z^n,T^n)$ has $N_k(c_i,1) \subset SC^n_k$, then the self-map defined by the map of (3.4) is always a $k$-continuous map.

(3) Consider the map $f : (Z,T) \rightarrow (Z,T)$ given by $f(t) = t + (2m + 1)$ which is a parallel translation with an odd vector, $m \in N$ (see Remark 3.3(3)). Then we can clearly observe that $f$ is a 2-continuous map in CTC because each point $t \in Z$ has $N_k^*(t,1) = [t-1,t+1]$.

In view of Remark 3.10, by using the continuity of Definition 3.4, we can study Khalimsky topological spaces without the limitations of $K$-continuity of $f$ in CTC discussed in Remark 3.3.

In relation to the classification of spaces $X_{n,b}$, we can also observe some utilities of CTC. Let us now recall a Khalimsky homeomorphism in CTC and a $(k_0,k_1)$-homeomorphism in CTC, as follows. In CTC we can say that for two Khalimsky spaces $(X,T^n_k) := X$ and $(Y,T^n_l) := Y$ a map $h : X \rightarrow Y$ is a Khalimsky (briefly, $K$-) homeomorphism if $h$ is a $K$-continuous bijection and further, $h^{-1} : Y \rightarrow X$ is $K$-continuous [13].

In view of Remarks 3.3 and 3.10, we need to use another homeomorphism in CTC for classifying spaces $X_{n,b}$ in CTC. In relation to the classification of the spaces $X_{n,b}$, by using the continuity of Definition 3.4, we can establish the following:

**Definition 3.11.** ([13]) In CTC, for two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$ a function $f : X \rightarrow Y$ is said to be a $(k_0,k_1)$-homeomorphism if

1. $f$ is bijective, and
2. $f$ is a $(k_0,k_1)$-continuous map and further, $f^{-1}$ is a $(k_1,k_0)$-continuous map.

Then we say that the space $X$ is $(k_0,k_1)$-homeomorphic to $Y$.

In Definition 3.11 if $k_0 = k_1$ and $n_0 = n_1$, then we use the terminology $k_0$-homeomorphism instead of $(k_0,k_1)$-homeomorphism. By comparing a $K$-homeomorphism with a $k$-homeomorphism, we can observe some merits of a $k$-homeomorphism.

**Example 3.12.** (1) Let us consider the map $f : X \rightarrow Y$ in Figure 4(a) defined by $f(x_i) = y_i, i \in [0,7]_Z$. Even though the spaces $X$ and $Y$ have the same cardinality, the bijection $f$ cannot be a $K$-homeomorphism (see

![Figure 3: Comparison of $K$-continuity of CTC and $k$-continuity of CTC, $k \neq 3^n - 1$.](image)

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**Figure 3:** Comparison of $K$-continuity of CTC and $k$-continuity of CTC, $k \neq 3^n - 1$. 
Meanwhile, we can observe that the map \( f \) is an \( 8 \)-homeomorphism in CTC because each point \( x \in X \) (resp. \( y \in Y \)) has \( N'_8(x, 1) \) (resp. \( N'_8(y, 1) \)).

(2) Consider the map \( g : Z := (z_i)_{i \in [0,3]_2} \rightarrow W := (w_i)_{i \in [0,3]_2} \) in Figure 4(b) given by \( g(z_i) = w_i, i \in [0,3]_2 \). Then we can observe that the bijection \( g \) can be a \( K \)-homeomorphism. But we may undergo an eccentric situation at the points \( w_2 \) and \( w_3 \) which are the images of \( g(z_2) \) and \( g(z_3) \), respectively. More precisely, while \( z_3 \in N_8(z_2, 1) \), we can obtain \( g(z_3) \notin N_8(g(z_2), 1) \) in which we can do an unusual experience. Meanwhile, we can observe that the map \( g \) cannot be an \( 8 \)-homeomorphism in CTC in which a \( k \)-homeomorphism in CTC is substantially helpful tool for classifying Khalimsky topological spaces.

(3) In Figure 4(c) consider the map \( h : B := (b_i)_{i \in [0,5]_2} \rightarrow C := (c_i)_{i \in [0,5]_2} \) in \( \mathbb{Z}^2 \) given by \( h(b_i) = c_i, i \in [0,5]_2 \). While the bijection \( h \) cannot be a \( K \)-homeomorphism, it can be an \( 18 \)-homeomorphism in CTC. More precisely, we can observe that \( (B, T^3_B) \) has a base

\[ \{[b_0], [b_1, b_2], [b_3, b_4], [b_2, b_4] \} \]

and \( (C, T^3_C) \) has a base

\[ \{[c_0, c_1, c_5], [c_2, c_3, c_4], [c_1, c_5], [c_2, c_4] \} \].

While \( h \) cannot be a \( K \)-homeomorphism, it is an \( 18 \)-homeomorphism because each point \( b_i \in B \) (resp. \( c_i \in C \)) has \( N'_{18}(b_i, 1) \) (resp. \( N'_{18}(c_i, 1) \)).

![Figure 4: Comparison of two spaces with the same cardinality up to a k-homeomorphism in CTC.](image)

In view of Example 3.8, by comparing a \( K \)-homeomorphism in KTC with a \( k \)-homeomorphism in CTC, we can observe that the latter has strong merits of classifying spaces \( X_{n,k} \).

4. Compressing method of Khalimsky topological spaces in CTC

In DTC established in Section 3 a digital \( k \)-homotopy has contributed to the classification of digital spaces \( (X,k) \) in terms of a digital fundamental group and a discrete deck transformation group (or an automorphism group) [2, 3, 9–12, 24, 27]. Similarly, in CTC we can consider a homotopy for studying the spaces \( X_{n,k} \). Thus, in this section we study a \((k_0, k_1)\)-homotopy which is suitable for studying spaces \( X_{n,k} \) (see
Definition 4.1. For a space \(X_{n,k_0}\) and its subspace \(A_{n,k_0}\), consider a space pair \((X_{n,k_0}, A_{n,k_0}) := (X, A)_{n,k_0}\).

For two space pairs \((X, A)_{n,k_0}\) and \((Y, B)_{n,k_1}\), we say that \(f : (X, A)_{n,k_0} \rightarrow (Y, B)_{n,k_1}\) is \((k_0, k_1)\)-continuous if \(f : X_{n,k_0} \rightarrow Y_{n,k_1}\) is \((k_0, k_1)\)-continuous and \(f(A_{n,k_0}) \subseteq B_{n,k_1}\). As a Khalimsky topological analogue of the \((k_0, k_1)\)-homotopy in DTC [12], in CTC we can establish a \((k_0, k_1)\)-homotopy relative to \((briefly, rel.)A_{n,k_0}\) in terms of the \((k_0, k_1)\)-continuity of Definition 3.4.

Definition 4.1. ([19]) In CTC, for four spaces \(X_{n,k_0} := X\) and \(Y_{n,k_1} := Y\), a subspace \(A_{n,k_0} := A \subseteq X_{n,k_0}\) and a Khalimsky interval \([0, m]_Z\), let \(f, g : X \rightarrow Y\) be \((k_0, k_1)\)-continuous functions. Suppose that there exist \(m \in \mathbb{N}\) and a function \(F : X \times [0, m]_Z \rightarrow Y\) such that

- for all \(x \in X, F(x, 0) = f(x)\) and \(F(x, m) = g(x)\);
- for all \(x \in X\), the induced function \(F_x : [0, m]_Z \rightarrow Y\) defined by \(F_x(t) = F(x, t)\) for all \(t \in [0, m]_Z\) is \((2, k_1)\)-continuous;
- for all \(t \in [0, m]_Z\), the induced function \(F_t : X \rightarrow Y\) defined by \(F_t(x) = F(x, t)\) for all \(x \in X\) is \((k_0, k_1)\)-continuous.

Then we say that \(F\) is a \((k_0, k_1)\)-homotopy between \(f\) and \(g\), and \(f\) and \(g\) are \((k_0, k_1)\)-homotopic in \(Y\). And we use the notation \(f \simeq_{(k_0, k_1)} g\).

If, further, for all \(t \in [0, m]_Z\), then the induced map \(F_t\) on \(A\) is a constant which is the prescribed function from \(A\) to \(Y\). In other words, \(F_t(x) = f(x) = g(x)\) for all \(x \in A\) and for all \(t \in [0, m]_Z\). Then, we say that the homotopy is a \((k_0, k_1)\)-homotopy relative to \((briefly, rel.)A\) and denote it \(f \simeq_{(k_0, k_1), rel.\ A} g\). In particular, if \(A = \{x_0\} \subseteq X\), then we say that \(F\) is a pointed \((k_0, k_1)\)-homotopy.

If \(X = [0, m]_Z\) for all \(t \in [0, m]_Z\), we have \(F(0, t) = F(0, 0)\) and \(F(m, t) = F(m, 0)\), then we say that \(F\) holds the endpoints fixed.

In CTC, as an analogous version of the notion of digital k-contractibility of [3], we say that a space \(X_{n,k} := X\) is pointed k-contractible if the identity map \(1_X\) is pointed k-homotopic relative to \(\{x_0\}\) in \(X\) to a constant map with the space consisting of some point \(x_0 \in X\).

Example 4.2. Both of two spaces \(X_{2,8}\) and \(Y_{2,8}\) in Figure 5 are pointed 8-contractive.

Proof. First of all, let us prove the 8-contractibility of \(X_{2,8}\) in Figure 5(a). Let us consider the map \(H : X_{2,8} \times [0, 3]_Z \rightarrow X_{2,8}\) given by

\[
H(x_i, 0) = x_i, \quad i \in \{0, 9\}_Z;
\]

\[
H(x_4, x_3, 1) = \{x_8\}, \quad H(x_2, x_3, 1) = \{x_1\}, \quad \text{and} \quad H(x_i, 1) = x_i, \quad i \in \{0, 1, 6, 7, 8, 9\};
\]

\[
H(x_2, x_3, x_4, x_5, x_6, 2) = \{x_0\}, \quad H(x_7, x_8, 2) = \{x_0\}, \quad \text{and} \quad H(x_i, 2) = x_i, \quad i \in \{0, 1, 9\};\]

\[
H(x_i, 3) = x_0, \quad i \in \{0, 9\}_Z.
\]

Then we can observe that this map \(H\) is an 8-homotopy which makes \(X_{2,8}\) 8-contractive in CTC.

By the similar method used for proving the 8-contractibility of \(X_{2,8}\), we can prove the 8-contractibility of \(Y_{2,8}\).

For a space pair \((X, A)_{n,k}\), consider the inclusion map \(i : A_{n,k} \rightarrow X_{n,k}\). The paper [18] established the notion of k-retract in CTC.

Definition 4.3. ([18]) In CTC \(A_{n,k} := A\) is called a k-retract of \(X_{n,k}\) if there is a k-continuous map \(r : X_{n,k} \rightarrow A_{n,k}\) such that \(r(a) = a\) for all \(a \in A_{n,k}\).

In other words, the map \(r\) satisfies the identity \(r \circ i = 1_A\). As an analogy of a strong k-deformation retract in DTC [12], we can establish the following:

Definition 4.4. In CTC, for a space pair \((X, A)_{n,k_0} := (X_{n,k_0} := X, A_{n,k_0} := A)\), \(A_{n,k}\) is said to be a strong k-deformation retract of \(X_{n,k}\) if there is a k-retraction \(r\) of \(X_{n,k}\) to \(A_{n,k}\) such that \(F : i \circ r \simeq_{k, rel.\ A} 1_X\), i.e., the strong k-deformation satisfies the condition \(F(x, t) = x\) for \(x \in A, t \in [0, m]_Z\). Then the k-homotopy \(F\) is called a strong k-deformation of \(X_{n,k}\) to \(A_{n,k}\). Then the point \(x \in X_{n,k} \setminus A_{n,k}\) is called a strongly k-deformable point.
Furthermore, in relation to the compression of spaces $X_{n,k}$ in CTC, we introduce the notion of $k$-homotopic thinning which can be used in topology and the field of applied science.

**Definition 4.5.** In CTC, for a space $X_{n,k}$ we can delete all strongly $k$-deformable points from $X_{n,k}$ in terms of a strong $k$-deformation retract. Then this processing is called a $k$-homotopic thinning.

**Example 4.6.** In CTC, consider the space $X = \{x_i | i \in [0, 12]_\mathbb{Z} \} := X_{3,18}$ in Figure 6. Then we can observe that the subspaces $X_1 := (X_1)_{3,18}$ and $X_2 := (X_2)_{3,18}$ in Figure 6 are two types of strong 18-deformation retracts of $X$.

Based on the $k$-homotopic thinning method in CTC, we can write an algorithm for compressing 2D spaces $X_{2,k}$ in CTC. In this section we assume that every point $x \in X_{2,k}$ has an $N^*_k(x, \epsilon) \subset X_{2,k}$ for some element $\epsilon$ of $\mathbb{N}$. If there is no hypothesis of the existence of $N^*_k(x, \epsilon) \subset X_{2,k}$, then we cannot perform the compression of a space $X_{2,k}$.

**[An algorithm for compressing non-empty spaces $X_{2,k}$ in terms of the $k$-homotopic thinning in CTC]**

1. Consider a space $X_{2,k}$ in CTC.
2. Scan the given space $X_{2,k}$ row by row from the top to the bottom and from the left to the right. Finally, if there is no point in the row, then restart to scan from the next row from the left to the right.
(3) According to the step(2) above, we can meet the first point \( x \in X_{2,k} \) and label it as \( x_1 \). Next, keep going to the labeling on the next points to the right, e.g. \( x_2, x_3, \cdots \) and so on (see the spaces \( X_{2,4} \) and \( Y_{2,8} \) in Figure 7 (a) and (b)).

(4) According to the step(3) above, after labeling the points in \( X_{2,k} \), examine if the labeled point is a \( k \)-homotopic thinning point from the first labeled point \( x_1 \) to the next \( x_2, x_3, \cdots \) and so on (see the labeled points in Figure 7 (a) and (b)). If yes, then delete it. Next, according to the order of the labeled points, keep going the \( k \)-homotopic thinning of all labeled points.

(5) In (4), when examining if the point \( x \in X_{2,k} \) is a \( k \)-homotopic thinning point, if the labeled point is not a \( k \)-homotopic thinning point, then remain the point and keep going to the right and proceed the above work (4). Finally, if there is no point in the row then restart the work of (4) from the next row.

(6) After finishing the works of (1)-(5), we can finally obtain a \( k \)-homotopic thinned space from the given space \( X_{2,k} \) (see the \( k \)-homotopic thinned spaces in Figure 7 (a) and (b), \( k \in \{4, 8\} \)).

**Example 4.7.** Consider the two spaces \( X_{2,4} := X \) and \( Y_{2,8} := Y \) in Figure 7 (a) and (b). According to the above algorithm, we can proceed the compression of \( X \) (resp. \( Y \)) in terms of a 4-homotopic thinning of \( X \) (resp. an 8-homotopic thinning of \( Y \)).

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5. **Summary and further work**

In relation to the compression of Khalimsky topological spaces, we have developed a \( k \)-homotopic thinning of spaces \( X_{n,k} \) in CTC and further, have suggested an algorithm for it. Such a case might be made by demonstrating that the methods yield useful information about image processing algorithms, or provide useful compression, or that they in any way increase our understanding of the digital images in a concretely applicable way.

As a further work, we need to find a map between Khalimsky topological spaces expanding both a \( K \)-continuous map and a \( (k_0,k_1) \)-continuous map in CTC.
References

[25] T.Y. Kong, The Khalimsky topologies are precisely those simply connected topologies on $\mathbb{Z}^n$ whose connected sets include all 2n-connected sets, Theoretical Computer Science 305 (2003) 221–235.