Vertex-removal in $\alpha$-domination

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Abstract. Let $G = (V,E)$ be any graph without isolated vertices. For some $\alpha$ with $0 < \alpha < 1$ and a dominating set $S$ of $G$, we say that $S$ is an $\alpha$-dominating set if for any $x \in V - S$, $|N(x) \cap S| \geq \alpha|N(x)|$. The cardinality of a smallest $\alpha$-dominating set of $G$ is called the $\alpha$-domination number of $G$ and is denoted by $\gamma_\alpha(G)$. In this paper, we study the effect of vertex removal on $\alpha$-domination.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order $n$. We denote the open neighborhood of a vertex $v$ of $G$ by $N_G(v)$, or just $N(v)$, and its closed neighborhood by $N_C(v) = N[v]$. For a vertex set $S \subseteq V(G)$, $N[S] = \cup_{v \in S} N[v]$. The degree $\deg(x)$ of a vertex $x$ denotes the number of neighbors of $x$ in $G$. The maximum degree and minimum degree of vertices of a graph $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A leaf is a vertex of degree one and a support vertex is one that is adjacent to a leaf. We denote by $S(G)$ the set of all support vertices of $G$. A set of vertices $S$ in $G$ is a dominating set if $N[S] = V(G)$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. If $S$ is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of $G$ induced by $S$. A subdivided star is obtained from a star with at least two edges by subdividing every edge exactly once. The corona $cor(H)$ of a graph $H$ is that graph obtained from $H$ by adding a pendant edge to each vertex of $H$. For notation and graph theory terminology in general we follow [7].

Let $G$ be a graph with no isolated vertex. For $0 < \alpha < 1$, a set $S \subseteq V$ is said to $\alpha$-dominate a graph $G$, if for any vertex $v \in V - S$, $|N(v) \cap S| \geq \alpha|N(v)|$. The minimum cardinality of an $\alpha$-dominating set is the $\alpha$-domination number, denoted $\gamma_\alpha(G)$. We refer an $\alpha$-dominating set of cardinality $\gamma_\alpha(G)$ as a $\gamma_\alpha(G)$-set. For references on $\alpha$-domination in graphs see, for example, [2–4, 6]. Dunbar et al. in [4] suggested the study of graphs in which removing of any edge changes the $\alpha$-domination number.

For a $\gamma_\alpha(G)$-set $S$ in a graph $G$ and a vertex $x \in S$, if $S - \{x\}$ is an $\alpha$-dominating set for $G - x$, then we denote $pn(x, S) = \{x\}$.

We remark that $\alpha$-domination could be defined for any graph $G$. However in the first introductory paper [4], Dunbar et al. defined it only for graphs with no isolated vertex. So we adopt this definition in this paper.

For many graph parameters, criticality is a fundamental question. Much has been written about graphs where a parameter increases or decreases whenever an edge or vertex is removed or added. For the

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obtained from \(K_1\). Proposition 2.4. Lemma 2.3. It is obvious that 2.1. Proposition 2.2. Brigham, Chinn, and Dutton [1] began the study of those graphs where the domination number decreases on the removal of any vertex. They defined a graph \(G\) to be domination vertex critical, or just \(\gamma\)-vertex critical, if removal of any vertex decreases the domination number. This concept is now well studied in domination theory.

In this paper we study the same concept for \(\alpha\)-domination. We call a graph \(G\), \(\alpha\)-domination vertex critical if removal of any vertex decreases the \(\alpha\)-domination number.

**Observation 1.1.** For any graph \(G\) of order \(n\), \(\gamma_\alpha(G) < n\).

**Observation 1.2.** In any graph \(G \neq P_2\), there is a \(\gamma_\alpha(G)\)-set containing all support vertices of \(G\).

**Proposition 1.3.** (4) If \(0 < \alpha < \frac{1}{\Delta(G)}\), then \(\gamma_\alpha(G) = \gamma(G)\).

**Proposition 1.4.** (4) If \(1 \geq \alpha > 1 - \frac{1}{\Delta(G)}\), then \(\gamma_\alpha(G) = \alpha_0(G)\).

A set \(S \subseteq V(G)\) is a 2-packing of \(G\) if for every two different vertices \(x, y \in S\), \(N[x] \cap N[y] = \emptyset\).

**2. Results**

**Proposition 2.1.** Let \(G\) be a graph without isolated vertices. For any vertex \(v \in V(G) - S(G)\), and any \(0 < \alpha \leq 1\), \(\gamma_\alpha(G) - 1 \leq \gamma_\alpha(G - v) \leq \gamma_\alpha(G) + \deg(v) - 1\) and these bounds are sharp.

**Proof.** Let \(G\) be a graph without isolated vertices and \(v \in V(G) - S(G)\). Let \(S\) be a \(\gamma_\alpha(G)\)-set. If \(v \notin S\), then \(S\) is an \(\alpha\)-dominating set for \(G - v\), and so \(\gamma_\alpha(G - v) \leq \gamma_\alpha(G)\). Thus we assume that \(v \in S\). Then \(S \cup N_G(v) - \{v\}\) is an \(\alpha\)-dominating set for \(G - v\), and so \(\gamma_\alpha(G - v) \leq \gamma_\alpha(G) + \deg(v) - 1\). Thus the upper bound follows.

For the lower bound let \(D\) be a \(\gamma_\alpha(G - v)\)-set. Then \(D \cup \{v\}\) is an \(\alpha\)-dominating set for \(G\), and so the lower bound follows.

To see the sharpness of the upper bound, let \(x\) be the center of a star \(K_{1,k}\) for \(k \geq 2\), and \(\alpha \leq \frac{1}{2}\). Let \(G\) be obtained from \(K_{1,k}\) by subdividing each edge of \(K_{1,k}\) three times. Note the \(G\) has \(3k\) vertices of degree one, \(k\) vertices of degree two, and a vertex of degree \(k\) (the vertex \(x\)). Now it is easy to see that \(\gamma_\alpha(G) = k + 1\), and \(\gamma_\alpha(G - x) = 2k\). To see the sharpness of the lower bound consider a cycle \(C_4\).

We call a graph \(G\), \(\alpha\)-domination vertex critical, or just \(\gamma_\alpha\)-vertex critical if for any \(v \in V(G) - S(G)\), \(\gamma_\alpha(G - v) < \gamma_\alpha(G)\).

We note that if for a graph \(G\) with no isolated vertex, \(V(G) - S(G) = \emptyset\), then \(G\) is \(\alpha\)-domination vertex critical. Thus \(P_2\) is obviously \(\alpha\)-domination vertex critical, since \(V(P_2) - S(P_2) = \emptyset\).

**2.1. \(\gamma_\alpha\)-vertex critical graphs**

In this subsection we present our results on \(\gamma_\alpha\)-vertex critical graphs.

**Proposition 2.2.** A graph \(G\) is \(\gamma_\alpha\)-vertex critical if and only if for any non-support vertex \(x\), there is a \(\gamma_\alpha(G)\)-set \(S\) containing \(x\) such that \(\text{pn}(x, S) = \{x\}\).

**Proof.** \((\Rightarrow)\) Let \(G\) be a \(\gamma_\alpha\)-vertex critical graph and \(x \notin S(G)\). Then \(\gamma_\alpha(G - x) = \gamma_\alpha(G) - 1\). Let \(S\) be a \(\gamma_\alpha(G - x)\)-set. It is obvious that \(D = S \cup \{x\}\) is an \(\alpha\)-dominating set for \(G\) and \(\text{pn}(x, D) = \{x\}\).

\((\Leftarrow)\) Let \(x \notin S(G)\) and let \(S\) be a \(\gamma_\alpha(G)\)-set containing \(x\) such that \(\text{pn}(x, S) = \{x\}\). Then \(S - \{x\}\) is an \(\alpha\)-dominating set for \(G - x\) implying that \(\gamma_\alpha(G - x) < \gamma_\alpha(G)\). Thus \(G\) is \(\gamma_\alpha\)-vertex critical.

Since \(\gamma_\alpha(K_{1,n}) = 1\), we obtain the following.

**Lemma 2.3.** \(K_{1,n}\) is \(\gamma_\alpha\)-vertex critical if and only if \(n = 1\).

**Proposition 2.4.** Every support vertex in a \(\gamma_\alpha\)-vertex critical graph is adjacent to exactly one leaf.
Proof. Let \( G \) be a \( \gamma_a \)-vertex critical. Assume that there is a support vertex \( x \) such \( x \) is adjacent to two leaves \( x_1 \) and \( x_2 \). By Lemma 2.3, we may assume that \( N(x) \) contains a vertex of degree at least two. Then \( x \) is a support vertex in \( G - x_1 \). By Observation 1.2, let \( S \) be a \( \gamma_a(G - x_1) \)-set such that \( x \in S \). Then \( S \) is an \( a \)-dominating set for \( G \), a contradiction. \( \square \)

Observation 2.5. A subdivided star is not \( \gamma_a \)-vertex critical.

Theorem 2.6. Let \( H \) be a connected graph of order at least two. Then \( G = cor(H) \) is \( \gamma_a \)-vertex critical.

Proof. Since \( G = cor(H) \), each vertex \( x \) of \( G \) is either a leaf a support vertex adjacent to exactly one leaf. We observe that \( \gamma_a(G) = |S(G)| \). Let \( x \) be a leaf of \( G \). We show that \( \gamma_a(G - x) < \gamma_a(G) \). Let \( y \) be the support vertex adjacent to \( x \). Since \( H \) is connected of order at least two, there is a vertex \( z \in N(y) \) such that \( deg(z) > 1 \). Then \( z \) is a support vertex. Now \( S(G) - \{y\} \) is an \( a \)-dominating set for \( G - x \), implying that \( \gamma_a(G - x) < \gamma_a(G) \). Thus \( G \) is \( \gamma_a \)-vertex critical. \( \square \)

Let \( T \) be the class of all trees \( T \) such that \( T \in T \) if and only if:

(1) \( T = P_2 \), or
(2) \( diam(T) \geq 3 \), and for any vertex \( x \) of \( T \) either \( x \) is a leaf or \( x \) is a support vertex adjacent to exactly one leaf.

Theorem 2.7. A tree \( T \) is \( \gamma_a \)-vertex critical for \( 0 < a \leq \frac{1}{\Delta(T)} \) if and only if \( T \in T \).

Proof. \((\Rightarrow) \) It is obvious that \( P_2 \) is \( \gamma_a \)-vertex critical. If \( T \neq P_2 \) is a tree in \( T \), then Theorem 2.6 implies that \( T \) is \( \gamma_a \)-vertex critical.

\((\Leftarrow) \) Let \( T \) be a \( \gamma_a \)-vertex critical tree. If \( diam(T) = 1 \), then \( T = P_2 \) and so \( T \in T \). If \( diam(T) = 2 \), then by Lemma 2.3, \( T \) is not \( \gamma_a \)-domination vertex critical. Thus we assume that \( diam(T) \geq 3 \). We show that any vertex of \( T \) is either a leaf or a support vertex.

Let \( y \) be vertex of \( T \) such that \( y \) is neither a leaf nor a support vertex. If each leaf of \( T \) is at distance two from \( y \), then by Proposition 2.4, \( y \) is the center of a subdivided star, a contradiction to Observation 2.5. Thus assume that there is a leaf \( x \) in \( T \) such that \( d(x, y) \geq 3 \). Let \( d(x, y) = t \) and \( P : x - x_1 - x_2 - \ldots - x_t = y \) be the shortest path between \( x \) and \( y \).

If \( x_2 \) is not a support vertex, then by Proposition 2.2, there is a \( \gamma_a(T) \)-set \( S \) containing \( x_2 \) such that \( pm(x_2, S) = \{x_2\} \). But then \( \{x_1, x\} \neq S \neq \emptyset \). Since \( a \Delta(T) \leq 1 \), we see that \( (S - \{x, x_2\}) \cup \{x_1\} \) is an \( a \)-dominating set for \( T \), a contradiction. Thus \( x_2 \) is a support vertex. Let \( y_2 \) be a leaf adjacent to \( x_2 \). If \( x_3 \neq 2 \) is a vertex of degree at least two, then by Proposition 2.2, there is a \( \gamma_a(S) \)-set \( S \) containing \( x_3 \) such that \( pm(x_3, S) = \{x_3\} \). Since \( S \neq \emptyset \), \( S \neq \{x_2, y_2\} \neq \emptyset \). Then \( S_1 = (S - \{y_2\}) \cup \{x_2\} \) is a \( \gamma_a(T) \)-set such that \( pm(x_3, S_1) = \{x_3\} \) and \( x_2 \in S_1 \). So \( S_1 \) is an \( a \)-dominating set for \( T \), a contradiction. Thus \( x_3 \) is a support vertex. By continuing this process we obtain that \( x_{t-1} \in S(T) \) for \( t = 1, 2, \ldots, t-1 \). By Proposition 2.2, there is a \( \gamma_a(T) \)-set \( D \) containing \( y \) such that \( P_n(y, D) = \{y\} \). We may assume that \( x_{t-1} \in D \), since \( x_{t-1} \in S(T) \). Then \( D - \{y\} \) is an \( a \)-dominating set for \( T \), a contradiction. \( \square \)

Problem 2.8. Characterize \( \gamma_a \)-vertex critical trees for \( a > \frac{1}{\Delta(T)} \).

Proposition 2.9. \((\Leftrightarrow) \) If \( \frac{1}{2} < a \leq 1 \), then:

(1) \( \gamma_a(P_n) = \left\lceil \frac{n}{2} \right\rceil \).
(2) \( \gamma_a(C_n) = \left\lceil \frac{n}{2} \right\rceil \).

Proposition 2.10. \((\Leftrightarrow) \) If \( 0 < a \leq \frac{1}{2} \), then \( \gamma_a(P_n) = \gamma_a(C_n) = \left\lceil \frac{n}{2} \right\rceil \).

Proposition 2.11. \((\Rightarrow) \) For \( 0 < a \leq \frac{1}{2} \), the path \( P_n \) is \( \gamma_a \)-vertex critical if and only if \( n \in [2, 4] \).

(2) For \( \frac{1}{2} < a \leq 1 \), the path \( P_n \) is \( \gamma_a \)-vertex critical if and only if \( n = 2k \).

Proof. If \( 0 < a \leq \frac{1}{2} \), then the result follows from Theorem 2.7.

Assume next that \( \frac{1}{2} < a \leq 1 \). By Proposition 2.9, \( \gamma_a(P_n) = \left\lceil \frac{n}{2} \right\rceil \). Let \( n = 2k \) for some integer \( k \geq 1 \). It is easy to see that \( P_n \) is \( \gamma_a \)-vertex critical. Thus we assume now that \( n \geq 6 \). Let \( x \) be a vertex such that \( x \) is not a support vertex. If \( x \) is a leaf then by Proposition 2.9,

\[ \gamma_a(P_n - x) = \gamma_a(P_{n-1}) = \left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil - 1 < \left\lceil \frac{n}{2} \right\rceil. \]
Thus assume now that \( x \) is not a leaf. Let \( G = P_n - x \). Then \( G \) has two components \( P_{n_1} \) and \( P_{n_2} \). Clearly we may assume that \( n_1 \) is even and \( n_2 \) is odd. Then

\[
\gamma_a(G) = \gamma_a(P_{n_1}) + \gamma_a(P_{n_2}) = \left\lceil \frac{n_1}{2} \right\rceil + \left\lfloor \frac{n_2}{2} \right\rfloor.
\]

A simple calculation shows that \( \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \). Thus \( P_n \) is \( \gamma_a \)-vertex critical.

Finally, we show that \( P_n \) is not \( \gamma_a \)-vertex critical if \( n \) is odd. Let \( n \) be odd and let \( x \) be a leaf. Then \( \gamma_a(P_n) = \gamma_a(P_{n-1}) \), since \( \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor \), as desired. \( \blacksquare \)

Using Propositions 2.9 and 2.10, we obtain the following proposition similarly.

**Proposition 2.12.** (1) For \( 0 < \alpha \leq \frac{1}{2} \), the cycle \( C_n \) is \( \gamma_a \)-vertex critical if and only if \( n \equiv 1 \pmod{3} \).

(2) For \( \frac{1}{2} < \alpha \leq 1 \), the cycle \( C_n \) is always \( \gamma_a \)-vertex critical.

**Observation 2.13.** (4) If \( K_n \) is the complete graph of order \( n \), then \( \gamma_a(K_n) = [\alpha(n-1)] \).

**Proposition 2.14.** A complete graph \( K_n \) of order \( n \geq 2 \) is \( \gamma_a \)-vertex critical if and only if

\[
\alpha > \frac{[\alpha(n-2)]}{n-1}.
\]

**Proof.** By Observation 2.13, the complete graph \( K_n \) is \( \gamma_a \)-vertex critical if and only if \( [\alpha(n-2)] < [\alpha(n-1)] \). This is equivalent with \( [\alpha(n-2)] < \alpha(n-1) \), and this is equivalent with \( \alpha > [\alpha(n-2)]/(n-1) \). \( \blacksquare \)

**Proposition 2.15.** (4) If \( K_{m,n} \) is a complete bipartite graph with \( 1 \leq m \leq n \), then \( \gamma_a(K_{m,n}) = \min[m, [am] + [an]] \).

**Proposition 2.16.** If \( 2 \leq m < n \), then \( K_{m,n} \) is \( \gamma_a \)-vertex critical if and only if \( m \geq [am] + [an] \),

\[
\alpha > \frac{[am]}{m} \quad \text{and} \quad \alpha > \frac{[an]}{n}.
\]

**Proof.** Let \( X \) and \( Y \) be the partite sets of \( G = K_{m,n} \) with \( |X| = m \) and \( |Y| = n \). First assume that \( m < [am] + [an] \).

By Proposition 2.15, \( \gamma_a(G) = m \). Let \( S \) be a \( \gamma_a(G - y) \)-set, where \( y \in Y \). Then Proposition 2.15 implies

\[
|S| = \gamma_a(G - y) = \gamma_a(K_{m,n-1}) = \min[m, [am] + [a(n-1)]],
\]

and therefore \( G \) is not \( \gamma_a \)-vertex critical in that case.

Next assume that \( m \geq [am] + [an] \). Then \( \gamma_a(G) = [am] + [an] \). Let \( S_1 \) be a \( \gamma_a(G - y) \)-set, where \( y \in Y \), and let \( S_2 \) be a \( \gamma_a(G - x) \)-set, where \( x \in X \). Then similar to the proof of Proposition 2.14, we observe that \( G \) is \( \gamma_a \)-vertex critical if and only if \( \alpha > [a(m-1)]/m \) and \( \alpha > [a(n-1)]/n \). \( \blacksquare \)

**Proposition 2.17.** If \( 2 \leq m \), then \( K_{m,n} \) is \( \gamma_a \)-vertex critical if and only if \( m \leq 2[am] \) or \( m > 2[am] \) and

\[
\alpha > \frac{[am]}{m}.
\]

**Proof.** Let \( G = K_{m,n} \). First assume that \( m \leq 2[am] \). By Proposition 2.15, \( \gamma_a(G) = m \). Let \( S \) be a \( \gamma_a(G - x) \)-set, where \( x \in V(G) \). Then Proposition 2.15 implies

\[
|S| = \gamma_a(G - x) = \gamma_a(K_{m-1,n}) = \min[m - 1, [am] + [a(m-1)]] \leq m - 1.
\]

and therefore \( G \) is \( \gamma_a \)-vertex critical in that case.

Next assume that \( m > 2[am] \). Then \( \gamma_a(G) = 2[am] \). Let \( S \) be a \( \gamma_a(G - x) \)-set, where \( x \in V(G) \). Then similar to the proof of Proposition 2.14, we observe that \( G \) is \( \gamma_a \)-vertex critical if and only if \( \alpha > [a(m-1)]/m \). \( \blacksquare \)
Proposition 2.18. There is no induced-subgraph characterization for $\gamma_n$-vertex critical graphs.

Proof. Let $G$ be an arbitrary graph, and $H = \text{cor}(G)$. Clearly, $G$ is an induced subgraph of $H$. Then $\gamma_n(H) = |V(G)|$, and $V(G)$ is a $\gamma_n(H)$-set. Let $v$ be a leaf of $H$ and $u \in N(v)$. Then $V(G) - \{u\}$ is an $\alpha$-dominating set for $H - v$, implying that $\gamma_n(H - v) < \gamma_n(H)$. Thus $H$ is $\gamma_n$-vertex critical. \qed

Theorem 2.19. ([5]) If $G$ is a $\gamma$-vertex critical graph of order $n = (\gamma(G) - 1)(\Delta(G) + 1) + 1$, then $G$ is regular.

Theorem 2.20. If $G$ is a $\gamma_n$-vertex critical graph of order $n$, then $n \leq (\gamma_n(G) - 1)(\Delta(G) + 1) + 1$. Furthermore, if $\delta(G) > 1$ and equality holds, then $G$ is regular.

Proof. Let $G$ be a $\gamma_n$-vertex critical graph of order $n$, and let $S$ be a $\gamma_n(G - v)$-set, where $v \in V(G) - S(G)$. Any vertex of $S$ dominates at most $1 + \Delta(G)$ vertices of $G$ including itself. Thus $S$ dominates at most $(\gamma_n(G) - 1)(\Delta(G) + 1) \geq n - 1$ vertices of $G$, as desired.

Now assume that $n = (\gamma_n(G) - 1)(\Delta(G) + 1) + 1$. Thus any vertex of $S$ dominates exactly $1 + \Delta(G)$ vertices of $G$, and so has degree $\Delta(G)$. Furthermore $S$ is a $2$-packing. Let $u \in N(v) - S$. Since $\delta(G) > 1$, the vertex $u \not\in V(G) - S(G)$. Let $D$ be a $\gamma_n(G - u)$-set. Then $|D| = \gamma_n(G) - 1$, and, as before, we obtain that any vertex of $D$ is of degree $\Delta(G)$, and $D$ is a $2$-packing. Since $S$ is a $\gamma_n(G - v)$-set, $u$ is adjacent to a vertex $a \in S$, and now $\text{deg}_{G-u}(a) < \Delta(G)$, and so $a \not\in D$. We deduce that $D - S \neq \emptyset$. Also clearly $v \not\in D$. Let $w \in D - S$. Since $S$ is a $\gamma_n(G - v)$-set, we obtain that $1 = |N(w) \cap S| \geq \text{deg}(w) = a\Delta(G)$, and so $a \leq \frac{1}{\Delta(G)}$. By Proposition 1.3, $\gamma_n(G) = \gamma(G)$, and also $\gamma_n(G - a) = \gamma(G - a)$ for any vertex $a$. Thus $G$ is $\gamma$-vertex critical. By Theorem 2.19, $G$ is regular. \qed

Fulman et al. [5] proved that if $G$ is a $\gamma$-vertex critical graph, then $\text{diam}(G) \leq 2(\gamma(G) - 1)$. However with a similar proof we obtain the following.

Proposition 2.21. If $G$ is a $\gamma_n$-vertex critical graph, then $\text{diam}(G) \leq 2(\gamma_n(G) - 1)$.

Theorem 2.22. If $G$ is a $\gamma_n$-vertex critical graph of order $n$, then for any vertex $v \in V(G) - S(G)$,

$$\gamma_n(G) \geq \left\lceil \frac{a\delta(G - v)n + \Delta(G)}{a\delta(G - v) + \Delta(G)} \right\rceil.$$ 

Proof. Let $G$ be a $\gamma_n$-vertex critical graph of order $n$, and let $v \in V(G) - S(G)$. Let $H = G - v$ and let $S$ be a $\gamma_n(H)$-set. Then $|S| = \gamma_n(G) - 1$. Let $M$ be the set of edges between $S$ and $V(H) - S$. By counting the edges from $S$ to $V(H) - S$, we obtain that

$$|M| \leq \sum_{v \in S} \text{deg}(v) \leq |S|\Delta(G).$$

On the other hand, since $S$ is an $\alpha$-dominating set for $H$, we find that

$$|M| \geq \sum_{v \in V(H) - S} \text{adev}(v) \geq a\delta(H)(|V(H)| - |S|).$$

Now we obtain

$$|S|\Delta(G) \geq a\delta(H)(n - 1 - |S|).$$

Since $|S| = \gamma_n(G) - 1$ and $H = G - v$, a simple calculation imply that

$$\gamma_n(G) \geq \frac{a\delta(G - v)n + \Delta(G)}{a\delta(G - v) + \Delta(G)}.$$

\qed

Proposition 2.23. ([4]) If $0 < \alpha < 1$, then for any graph $G$, $\gamma_n(G) + \gamma_{1-\alpha}(G) \leq n$. 

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Theorem 2.24. If $G$ is a $\gamma_a$-vertex critical graph of order $n$ and size $m$, then
\[
\gamma_a(G) \geq \left\lceil \frac{2am - a\Delta(G) + \Delta(G)}{\Delta(G)(a + 1)} \right\rceil.
\]

Proof. Let $G$ be a $\gamma_a$-vertex critical graph of order $n$ and size $m$. Let $v \in V(G) - S(G)$ and $H = G - v$. Let $S$ be a $\gamma_a(H)$-set. Then $\sum_{v \in S} \deg_H(v) \geq \sum_{v \in V(H) - S} \deg_H(v)$. Now
\[
(a + 1)|S|\Delta(G) \geq a \sum_{v \in S} \deg_H(v) + \sum_{v \in S} \deg_H(v) \\
\geq a \sum_{v \in S} \deg_H(v) + \sum_{v \in V(H) - S} \deg_H(v) \\
\geq a \sum_{v \in V(H)} \deg_H(v) \\
= a(2m - 2\deg_C(v)) \geq a(2m - 2\Delta(G)).
\]
Since $|S| = \gamma_a(G) - 1$, a simple calculation completes the proof. \(\Box\)

By Proposition 2.23, we have the following.

Corollary 2.25. Let $0 < \alpha < 1$. If $G$ is a $\gamma_\alpha$-vertex critical graph of order $n$ and size $m$, then
\[
\gamma_{1-\alpha}(G) \leq \left\lceil \frac{(1 + \alpha)\Delta(G)n + \alpha\Delta(G) - 2am - \Delta(G)}{\Delta(G)(\alpha + 1)} \right\rceil.
\]

References