Hermite-Hadamard type inequalities for the $m$- and $(\alpha, m)$-logarithmically convex functions

Rui-Fang Bai$^a$, Feng Qi$^b$, Bo-Yan Xi$^c$

$^a$College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China; The Seventh School of Neijiang, Neijiang City, Sichuan Province, 641000, China

$^b$Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300160, China; School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

$^c$College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China

Abstract. In the paper, the authors introduce concepts of $m$- and $(\alpha, m)$-logarithmically convex functions and establish some Hermite-Hadamard type inequalities of these classes of functions.

1. Introduction

For convex functions, the following Hermite-Hadamard type inequalities were given in [8].

**Theorem A** ([8]). Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I$ and $a, b \in I$ with $a < b$. If $|f'(x)|^q$ for $q \geq 1$ is convex on $[a, b]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q} \quad (1.1)$$

and

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}. \quad (1.2)$$

The $m$-convex function was defined in [12] as follows.

**Definition 1.1.** A function $f : [0, b] \to \mathbb{R}$ is said to be $m$-convex if

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) \quad (1.3)$$

holds for all $x, y \in [0, b]$, $t \in [0, 1]$, and $m \in (0, 1]$.
In [1, p. 48, Theorem 2] and [2], the following Hermite-Hadamard type inequality for \( m \)-convex functions was proved.

**Theorem B.** Let \( f : [0, \infty) \to \mathbb{R} \) be \( m \)-convex and \( m \in (0, 1) \). If \( f \in L([a, b]) \) for \( 0 \leq a < b < \infty \), then

\[
\frac{1}{b - a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + mf(b)/m + f(b)}{2}, \frac{mf(a) + f(b)}{2} \right\}.
\]

(1.4)

The \((a, m)\)-convex function was defined in [7] as follows.

**Definition 1.2.** A function \( f : [0, b] \to \mathbb{R} \) is said to be \((a, m)\)-convex if

\[
f(tx + m(1 - t)y) \leq t^a f(x) + m(1 - t)^a f(y)
\]

is valid for all \( x, y \in [0, b], t \in [0, 1], \) and \((a, m) \in (0, 1) \times (0, 1)\).

For \((a, m)\)-convex functions, the following Hermite-Hadamard type inequalities appeared in [5].

**Theorem C ([5, Theorem 2.2]).** Let \( I \supset [0, \infty) \) be an open real interval and let \( f : I \to \mathbb{R} \) be a differentiable function on \( I \) such that \( f' \in L([a, b]) \) for \( 0 \leq a < b < \infty \). If \( |f'(x)|^q \) is \( m \)-convex on \([a, b]\) for some given numbers \( m \in (0, 1) \) and \( q \in [1, \infty) \), then

\[
\left| \frac{f(a + b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4} \min \left\{ \left( \frac{|f'(x)|^q + m|f'(b)/m|^q}{2} \right)^{1/q}, \left( \frac{m|f'(a/m)|^q + |f'(b)|^q}{2} \right)^{1/q} \right\}.
\]

(1.6)

**Theorem D ([5, Theorem 3.1]).** Let \( I \supset [0, \infty) \) be an open real interval and let \( f : I \to \mathbb{R} \) be a differentiable function on \( I \) such that \( f' \in L([a, b]) \) for \( 0 \leq a < b < \infty \). If \( |f'(x)|^q \) is \((a, m)\)-convex on \([a, b]\) for \((a, m) \in (0, 1) \times (0, 1) \) and \( q \in [1, \infty) \), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2} \left( \frac{1}{2} \right)^{1/q} \min \left\{ v_1 |f'(a)|^q + v_2 m |f'(b)|^q, v_2 m |f'(a/m)|^q + v_1 |f'(b)|^q \right\}
\]

(1.7)

where

\[
v_1 = \frac{1}{(a + 1)(a + 2)} \left( a + \frac{1}{2} \right) \quad \text{and} \quad v_2 = \frac{1}{(a + 1)(a + 2)} \left( a^2 + a + \frac{1}{2} \right).
\]

(1.8)

The aim of this paper is to introduce concepts of \( m \)- and \((a, m)\)-logarithmically convex functions, and then to present some Hermite-Hadamard type inequalities for them.

2. **Definitions and lemmas**

Firstly we introduce concepts of \( m \)- and \((a, m)\)-logarithmically convex functions.

**Definition 2.1.** A function \( f : [0, b] \to (0, \infty) \) is said to be \( m \)-logarithmically convex if the inequality

\[
f(tx + m(1 - t)y) \leq [f(x)]^t [f(y)]^{m(1 - t)}
\]

holds for all \( x, y \in [0, b], m \in (0, 1), \) and \( t \in (0, 1] \).

Obviously, if putting \( m = 1 \) in Definition 2.1, then \( f \) is just the ordinary logarithmically convex function on \([0, b]\).
**Theorem 3.1.** A function \( f : [0, b] \to (0, \infty) \) is said to be \((\alpha, \eta, m)-\logarithmically\ convex\) if

\[
\log f(tx + m(1-t)y) \leq (1-t)^\alpha [f(y)]^\eta [f(y)]^{\eta(1-t)}
\]

holds for all \( x, y \in [0, b] \) and \( (\alpha, \eta, m) \in (0, 1] \times (0, 1] \), for all \( t \in [0, 1] \).

Clearly, when taking \( \alpha = 1 \) in Definition 2.2, then \( f \) becomes the standard \( m \)-logarithmically convex function on \([0, b]\).

Secondly, we recite the following lemmas which will be used in proofs of our main results.

**Lemma 2.1 (3).** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I \) and \( a, b \in I \) with \( a < b \). If \( f' \in L([a, b]) \), then

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{b-a}{2} \int_0^1 (1-2t)f'(ta + (1-t)b) \, dt.
\]

**Lemma 2.2 (4).** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I \) and \( a, b \in I \) with \( a < b \). If \( f' \in L([a, b]) \), then

\[
\int_{\frac{a+b}{2}}^b f(x) \, dx = (b-a)\left[\int_0^{1/2} f'(ta + (1-t)b) \, dt + \int_{1/2}^1 (1-t)f'(ta + (1-t)b) \, dt \right].
\]

### 3. Hermite-Hadamard type inequalities

In this section, we will present several Hermite-Hadamard type inequalities for the \( m \)- and \((\alpha, \eta, m)\)-logarithmically convex functions.

**Theorem 3.1.** Let \( I \supset [0, \infty) \) be an open real interval and let \( f : I \to (0, \infty) \) be a differentiable function on \( I \) such that \( f' \in L([a, b]) \) for \( 0 \leq a < b < \infty \). If \( f'(x) \) is \((\alpha, \eta, m)\)-logarithmically convex on \([0, b]\) for \((\alpha, \eta, m) \in (0, 1] \times (0, 1] \), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2} \left( \frac{1}{2} \right)^{1-\frac{1}{q}} \left\| f' \left( \frac{b}{m} \right) \right\|^m \left| E_1(\alpha, \eta, m, q) \right|^{1/q}
\]

is valid for \( q \geq 1 \), where

\[
\mu = \frac{|f'(a)|}{|f'(b/m)|^m}, \quad E_1(\alpha, \eta, m, q) = \begin{cases} 
\frac{1}{2}, & \mu = 1, \\
F_1(\mu, \alpha, q), & \mu < 1, \\
\mu^{1-\alpha} F_1(\mu, \alpha, q), & \mu > 1,
\end{cases}
\]

and

\[
F_1(u, v) = \frac{1}{v^2 \ln^2 u} \left[ v(u^v - 1) \ln u - 2\left(u^{v/2} - 1\right)^2 \right]
\]

for \( u, v > 0 \) and \( u \neq 1 \).

**Proof.** When \( q > 1 \), by Definition 2.2, Lemma 2.1, and Hölder inequality, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2} \left( \int_0^1 (1-2t)f'(ta + (1-t)b) \, dt \right)^{1/q} \leq \frac{b-a}{2} \left( \frac{1}{2} \right)^{1-\frac{1}{q}} \left\| f' \left( \frac{b}{m} \right) \right\|^m \left( \int_0^1 (1-2t)f'(ta + (1-t)b) \, dt \right)^{1/q}.
\]
For \( \mu = 1 \), we have
\[
\int_0^1 |1 - 2|^{\mu^{aq}} \, dt = \int_0^1 |1 - 2| \, dt = \frac{1}{2}.
\]
For \( \mu < 1 \), we have \( \mu^{aq} \leq \mu^{aq} \), thereby
\[
\int_0^1 |1 - 2|^{\mu^{aq}} \, dt \leq \int_0^1 |1 - 2|^{\mu^{aq}} \, dt = \frac{\alpha q \mu^{aq} \ln \mu - \alpha q \ln \mu - 2 \mu^{aq} + 4 \mu^{aq/2} - 2}{\alpha^2 q^2 \ln^3 \mu}.
\]
For \( \mu > 1 \), we have \( \mu^{aq} \leq \mu^{(a+1-a)} \), thereby
\[
\int_0^1 |1 - 2|^{\mu^{aq}} \, dt \leq \mu^{a(1-a)} \int_0^1 |1 - 2|^{\mu^{aq}} \, dt = \mu^{a(1-a)} \frac{\alpha q \mu^{aq} \ln \mu - \alpha q \ln \mu - 2 \mu^{aq} + 4 \mu^{aq/2} - 2}{\alpha^2 q^2 \ln^3 \mu}.
\]
Thus, the inequality (3.1) follows.

When \( q = 1 \), we have
\[
\frac{|f(a) + f(b)|}{2} - \frac{b - a}{b - a} \int_a^b f(x) \, dx \leq \frac{b - a}{2} \left( \int_0^1 (1 - 2t) f'(ta + (1 - t)b) \, dt \right) \leq \frac{b - a}{2} \int_0^1 |1 - 2t| |f'(b/2)|^{m(1-e^t)} \, dt \leq \frac{b - a}{2} |f'(b/2)|^m E_1(a, m, q).
\]

The proof of Theorem 3.1 is complete. \( \square \)

**Corollary 3.2.** Let \( I \supseteq [0, \infty) \) be an open real interval and let \( f : I \to (0, \infty) \) be a differentiable function on \( I \) such that \( f' \in L([a, b]) \) for \( 0 \leq a < b < \infty \). If \( |f'(x)|^q \) is \( m \)-logarithmically convex on \( \left[ \frac{b}{m} \right] \) for some given numbers \( m \in (0, 1] \), then
\[
\frac{|f(a) + f(b)|}{2} - \frac{b - a}{b - a} \int_a^b f(x) \, dx \leq \frac{b - a}{2} \left( \frac{1}{2} \right)^{1-1/q} \left| f' \left( \frac{b}{m} \right) \right|^m E_1(1, m, q)^{1/q} \tag{3.4}
\]
holds for \( q \geq 1 \), where \( E_1 \) is defined as in Theorem 3.1.

**Theorem 3.3.** Let \( I \supseteq [0, \infty) \) be an open real interval and let \( f : I \to (0, \infty) \) be a differentiable function on \( I \) such that \( f' \in L([a, b]) \) for \( 0 \leq a < b < \infty \). If \( |f'(x)|^q \) is \( (a, m) \)-logarithmically convex on \( \left[ \frac{b}{m} \right] \) for \( (a, m) \in (0, 1] \times (0, 1] \), then
\[
\frac{f \left( \frac{a+b}{2} \right)}{2} - \frac{b - a}{b - a} \int_a^b f(x) \, dx \leq \frac{b - a}{4} \left( \frac{1}{2} \right)^{1-3/q} \left| f' \left( \frac{b}{m} \right) \right|^m E_2(a, m, q) \tag{3.5}
\]
is valid for \( q \geq 1 \), where \( E_2 \) is defined by (3.2) and
\[
E_2(a, m, q) = \begin{cases} 2 \left( \frac{1}{8} \right)^{1/q}, & \mu = 1 \\ [F_2(a, \mu q)]^{1/q} + [F_3(\mu, \mu q)]^{1/q}, & 0 < \mu < 1 \\ \mu^{1-a} [F_2(a, \mu q)]^{1/q} + [F_3(\mu, \mu q)]^{1/q}, & \mu > 1 \end{cases} \tag{3.6}
\]
with
\[
F_2(a, \nu) = \frac{1}{v^2 \ln^2 u} \left( \frac{v^2}{2} u^{v/2} \ln u - u^{v/2} + 1 \right) \quad \text{and} \quad F_3(a, \nu) = \frac{1}{v^2 \ln^2 u} \left( u^{v} - \frac{v}{2} u^{v/2} \ln u - u^{v/2} \right) \tag{3.7}
\]
for \( u, v > 0 \) and \( u \neq 1 \).
Proof. When \( q > 1 \), by Definition 2.2, Lemma 2.2, and Hölder inequality yield

\[
\left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2} \left[ \int_0^{1/2} f'(ta + (1 - t)b) \, dt + \int_{1/2}^1 (1 - t)f'(ta + (1 - t)b) \, dt \right]
\]

\[
\leq \frac{b - a}{4} (1/2)^{1-3/q} \left\{ \left[ \int_0^{1/2} t |f'(a)|^{q_1} \left( \int_0^{1/2} \frac{t}{m} \right)^{\frac{q_{1q}}{m}} \, dt \right]^{1/q_1} + \left[ \int_{1/2}^1 (1 - t)f'(a) |f'(\frac{b}{m})|^{m(1 - t)} \, dt \right]^{1/q_1} \right\}
\]

\[
= \frac{b - a}{4} (1/2)^{1-3/q} \left\{ \left( \int_0^{1/2} t^{\frac{1}{1/q_1}} \, dt \right)^{1/q_1} + \left[ \int_{1/2}^1 (1 - t) \mu^{q_1 d} \, dt \right]^{1/q_1} \right\}.
\]

If \( \mu = 1 \), we have

\[
\left( \int_0^{1/2} t \mu^{q_1 d} \, dt \right)^{1/q_1} + \left[ \int_{1/2}^1 (1 - t) \mu^{q_1 d} \, dt \right]^{1/q_1} = \left( \frac{1}{s} \right)^{1/q_1}.
\]

If \( \mu < 1 \), we obtain

\[
\left( \int_0^{1/2} t \mu^{q_1 d} \, dt \right)^{1/q_1} + \left[ \int_{1/2}^1 (1 - t) \mu^{q_1 d} \, dt \right]^{1/q_1} \leq \left( \int_0^{1/2} t \mu^{q_1 d} \, dt \right)^{1/q_1} + \left[ \int_{1/2}^1 (1 - t) \mu^{q_1 d} \, dt \right]^{1/q_1}
\]

\[
= \left[ \frac{1}{\alpha^2 q_1 d} \left( \frac{\alpha q_1}{2} \mu^{q_1 d} \ln \mu - \mu^{q_1 d/2} + 1 \right) \right]^{1/q_1} + \left[ \frac{1}{\alpha^2 q_1 d} \ln \mu \left( \mu^{q_1 d} - \frac{\alpha q_1}{2} \mu^{q_1 d} \ln \mu - \mu^{q_1 d/2} \right) \right]^{1/q_1}.
\]

If \( \mu > 1 \), then

\[
\left( \int_0^{1/2} t \mu^{q_1 d} \, dt \right)^{1/q_1} + \left[ \int_{1/2}^1 (1 - t) \mu^{q_1 d} \, dt \right]^{1/q_1} \leq \left( \int_0^{1/2} t \mu^{q_1 d (a + 1 - a)} \, dt \right)^{1/q_1} + \left[ \int_{1/2}^1 (1 - t) \mu^{q_1 d (a + 1 - a)} \, dt \right]^{1/q_1}
\]

\[
= \mu^{1-\alpha} \left( \left[ \frac{1}{\alpha^2 q_1 d} \ln \mu \left( \frac{\alpha q_1}{2} \mu^{q_1 d} \ln \mu - \mu^{q_1 d/2} + 1 \right) \right]^{1/q_1} + \left[ \frac{1}{\alpha^2 q_1 d} \ln \mu \left( \mu^{q_1 d} - \frac{\alpha q_1}{2} \mu^{q_1 d} \ln \mu - \mu^{q_1 d/2} \right) \right]^{1/q_1} \right).
\]

Thus, the inequality (3.5) follows.

When \( q = 1 \), we have

\[
\left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2} \left[ \int_0^{1/2} f'(ta + (1 - t)b) \, dt + \int_{1/2}^1 (1 - t)f'(ta + (1 - t)b) \, dt \right]
\]

\[
\leq (b - a) \left[ \int_0^{1/2} |f'(a)| \left( \int_0^{1/2} \frac{t}{m} \right)^{\frac{1}{m(1 - t)}} \, dt + \int_{1/2}^1 (1 - t)|f'(a)| \left( \int_0^{1/2} \frac{t}{m} \right)^{\frac{1}{m(1 - t)}} \, dt \right]
\]

\[
\leq (b - a) \left| f'\left(\frac{b}{m}\right)\right|^m E_2(a, m, 1).
\]

This completes the proof of Theorem 3.3. \( \square \)

**Corollary 3.4.** Let \( I \supset [0, \infty) \) be an open real interval and let \( f : I \to (0, \infty) \) be a differentiable function on \( I \) such that \( f' \in L([a, b]) \) for \( 0 \leq a < b < \infty \). If \( |f'(x)|^q \) is \( m \)-logarithmically convex on \( [0, \frac{b}{m}] \) for some given numbers \( m \in (0, 1) \), then

\[
\left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4} \left( \frac{1}{2} \right)^{1-3/q} \left| f'\left(\frac{b}{m}\right)\right|^m E_2(1, m, q)
\]

(3.8)

holds for \( q \geq 1 \), where \( E_2 \) is defined as in Theorem 3.3.
**Theorem 3.5.** Let $f, g : [0, \infty) \to (0, \infty)$ such that $f, g \in \mathcal{L}([a, b])$ for $0 \leq a < b < \infty$. If $f(x)$ is $(\alpha, m_1)$-logarithmically convex and $g(x)$ is $(\alpha, m_2)$-logarithmically convex on $[0, \frac{b}{m_1}]$ for $(\alpha, m_i) \in (0, 1] \times (0, 1)$ and $i = 1, 2$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x) \, dx \leq (b-a) \left[ f \left( \frac{b}{m_1} \right) \right]^{m_1} \left[ g \left( \frac{b}{m_2} \right) \right]^{m_2} E_3(\alpha),$$

where

$$\eta = f(a)g(a) \left[ f \left( \frac{b}{m_1} \right) \right]^{m_1} \left[ g \left( \frac{b}{m_2} \right) \right]^{m_2} \text{ and } E_3(\alpha) = \begin{cases} \frac{\eta^{\alpha} - 1}{\alpha \ln \eta}, & 0 < \eta < 1, \\ \eta^{1-\alpha}(\eta^{\alpha} - 1), & \eta > 1. \end{cases}$$

**Proof.** The $(\alpha, m)$-logarithmic convexity of $f(x)$ and $g(x)$ yields

$$f \left( ta + m_1(1-t) \left( \frac{b}{m_1} \right) \right) \leq [f(a)]^{\alpha} \left[ f \left( \frac{b}{m_1} \right) \right]^{m_1(1-\alpha)} \text{ and } g \left( ta + m_2(1-t) \left( \frac{b}{m_2} \right) \right) \leq [g(a)]^{\alpha} \left[ g \left( \frac{b}{m_2} \right) \right]^{m_2(1-\alpha)},$$

from which it follows that

$$\frac{1}{b-a} \int_a^b f(x)g(x) \, dx = (b-a) \int_0^1 f(ta + 1-tb)g(ta + 1-tb) \, dt$$

$$\leq (b-a) \int_0^1 [f(a)]^{\alpha} [g(a)]^{\alpha} \left[ f \left( \frac{b}{m_1} \right) \right]^{m_1(1-\alpha)} \left[ g \left( \frac{b}{m_2} \right) \right]^{m_2(1-\alpha)} \, dt$$

$$= (b-a) \left[ f \left( \frac{b}{m_1} \right) \right]^{m_1} \left[ g \left( \frac{b}{m_2} \right) \right]^{m_2} \int_0^1 \{f(a)g(a)\}^{\alpha} \left[ f \left( \frac{b}{m_1} \right) \right]^{m_1} \left[ g \left( \frac{b}{m_2} \right) \right]^{m_2} \, dt.$$

When $\eta = 1$, we have $\int_0^1 \eta^{\alpha} \, dt = 1$. When $\eta < 1$, we have

$$\int_0^1 \eta^{\alpha} \, dt \leq \frac{\eta^{\alpha} - 1}{\alpha \ln \eta},$$

When $\eta > 1$, we have

$$\int_0^1 \eta^{\alpha} \, dt \leq \frac{\eta^{1-%alpha}(\eta^{\alpha} - 1)}{\alpha \ln \eta}. $$

Theorem 3.5 is thus proved. \qed

**Corollary 3.6.** Let $f, g : [0, \infty) \to (0, \infty)$ such that $f : g \in \mathcal{L}([a, b])$ for $0 \leq a < b < \infty$. If $f(x)$ is $m_1$-logarithmically convex and $g(x)$ is $m_2$-logarithmically convex on $[0, \frac{b}{m_1}]$ for $i = 1, 2$ and some given numbers $m_1, m_2 \in (0, 1]$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x) \, dx \leq (b-a) \left[ f \left( \frac{b}{m_1} \right) \right]^{m_1} \left[ g \left( \frac{b}{m_2} \right) \right]^{m_2} E_3(1),$$

where $E_3$ is defined as in Theorem 3.5.

**Corollary 3.7.** Let $f, g : [0, \infty) \to (0, \infty)$ such that $f : g \in \mathcal{L}([a, b])$ for $0 \leq a < b < \infty$. If $f(x)$ and $g(x)$ are $(\alpha, m)$-logarithmically convex on $[0, \frac{b}{m}]$ for $(\alpha, m) \in (0, 1] \times (0, 1)$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x) \, dx \leq (b-a) \left[ f \left( \frac{b}{m} \right) \right]^{m} \left[ g \left( \frac{b}{m} \right) \right]^{m} E_3(\alpha),$$

where $E_3$ is defined as in Theorem 3.5.
Corollary 3.8. Let \( f, g : [0, \infty) \to (0, \infty) \) such that \( f \cdot g \in L([a, b]) \) for \( 0 \leq a < b < \infty \). If \( f(x) \) and \( g(x) \) are \( m \)-logarithmically convex on \( [0, \frac{b}{m}] \) for some given number \( m \in (0, 1] \), then

\[
\frac{1}{b-a} \int_a^b f(x)g(x) \, dx \leq (b-a) \left[ f\left( \frac{b}{m} \right) g\left( \frac{b}{m} \right) \right]^m E_3(1),
\]

(3.13)

where \( E_3 \) is defined as in Theorem 3.5.

Remark 3.1. In [6, 9–11, 13–16] the authors and their coauthors obtained some results on Hermite-Hadamard type inequalities for convex functions and for \( m \)- and \((\alpha, m)\)-geometrically convex functions.

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References