Harmonic Bergman spaces on the complement of a lattice

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Abstract. We investigate harmonic Bergman spaces $b^p = b^p(\Omega)$, $0 < p < \infty$, where $\Omega = \mathbb{R}^n \setminus \mathbb{Z}^n$ and prove that $b^p \subset b^q$ for $n/(k+1) \leq q < p < n/k$. In the planar case we prove that $b^p$ is non empty for all $0 < p < \infty$. Further, for each $0 < p < \infty$ there is a non-trivial $f \in b^p$ tending to zero at infinity at any prescribed rate.

1. Introduction

We denote the space of all complex valued harmonic functions on a domain $V \subset \mathbb{R}^n$ by $h(V)$, with topology of locally uniform convergence. For $0 < p < \infty$ we set $b^p(V) = L^p(V) \cap h(V)$. With respect to $L^p$ (quasi)norm these spaces are Frechet spaces for $0 < p < 1$ and Banach spaces for $p \geq 1$. Let $\Gamma = \mathbb{Z}^n$, $\Omega = \mathbb{R}^n \setminus \Gamma$. In the planar case the analytic Bergman spaces $B^p(\Omega)$ were studied in [1], in this paper we investigate harmonic Bergman spaces $b^p(\Omega)$.

For $x \in \mathbb{R}^n$ and $r > 0$ $B(x,r)$ denotes the open ball of radius $r$ centered at $x$. We set, for $x \in \mathbb{R}^n$, $\|x\|_\infty = \max_{1 \leq j \leq n} |x_j|$. Also, $Q(z,a) = \{w : \|w-z\|_\infty < a/2\}$ denotes an open cube centered at $z \in \mathbb{R}^n$ of side length $a > 0$ and $Q(z,a) = Q(z,a) \setminus \{z\}$. In the planar case we also use notation $D(z,r) = \{w : |z-w| < r\}, D_r = D(0,r)$. The $n$ dimensional Lebesgue measure is denoted by $dm$. Letter $C$ denotes a constant, its value can vary from one occurrence to the next. For future reference we state some known facts.

Proposition 1.1. If $f : \mathbb{R}^n \to \mathbb{C}$ is a harmonic function, not identically equal to zero, then $f \notin b^p(\mathbb{R}^n), p > 0$. Moreover:

$$\left( \int_{B(x,R)} |f(y)|^p dy \right)^{1/p} \geq C_{p,n} R^{n/p} |f(x)|, \quad x \in \mathbb{R}^n. \quad (1)$$

Proof. Indeed, (1) follows from subharmonic behavior of $|f|^p$ for $0 < p < \infty$, see [3]. Therefore

$$\left( \int_{\mathbb{R}^n} |f(y)|^p dy \right)^{1/p} \geq \lim_{R \to +\infty} C_{p,n} |f(x)| R^{n/p} = +\infty$$

2010 Mathematics Subject Classification. Primary 30H20

Keywords. Bergman spaces, harmonic functions, integer lattice

Received: 10 December 2011; Accepted: 5 April 2012

Communicated by Prof. M. Mateljević

Research supported by Ministry of Science, Serbia, project OI174017

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whenever \( f(x) \neq 0 \) for some \( x \in \mathbb{R}^n \). \( \Box \)

It is a standard fact that for \( f \in b^p(V), V \subset \mathbb{R}^n, 0 < p < +\infty \) we have

\[
|f(x)| \leq C_{p,n} \frac{\|f\|_p}{p^n}, \quad \text{where} \quad r = d(x,V).
\]

In fact, using (1.1), we get

\[
|f(x)|^p \leq \frac{C_{n,p}}{m(B(x,r))} \int_{B(x,r)} |f|_p dm \leq C_{n,p} r^{-n} \|f\|_p^p,
\]

and (2) easily follows. Note that this allows one to conclude that convergence in \( b^p(V) \) implies locally uniform convergence on \( V \).

We need certain facts about expansions of harmonic functions near singularities, for details see [2].

Suppose \( n \geq 3, a \in V \subset \mathbb{R}^n \), and \( f \in h(V \setminus \{a\}) \). Then there are homogeneous harmonic polynomials \( p_m \) and \( q_m \) of degree \( m \) such that

\[
f(x) = \sum_{m=0}^{\infty} p_m(x-a) + \sum_{m=0}^{\infty} \frac{q_m(x-a)}{|x-a|^{m+n-2}}.
\]

A classification of singularities follows from this expansion: \( f \) has a removable singularity at \( a \) if and only if \( \lim_{z \to a} |x-a|^{n-2} |f(x)| = 0 \), \( f \) has a pole at \( a \) of order \( M + n - 2 \) if and only if \( 0 < \limsup_{z \to a} |x-a|^{M+n-2} |f(x)| < \infty \), and finally point \( a \) is an essential singularity if and only if \( \limsup_{z \to a} |x-a|^N |f(x)| = \infty \) for every positive integer \( N \).

When \( n = 2 \) the situation is slightly different, in that case there are homogeneous harmonic polynomials \( p_m \) and \( q_m \) of degree \( m \) on \( \mathbb{R}^2 \) such that

\[
f(z) = \sum_{m=0}^{\infty} p_m(z-a) + q_0 \log |z-a| + \sum_{m=1}^{\infty} \frac{q_m(z-a)}{|z-a|^{2m}}.
\]

The presence of the logarithmic factor makes a difference between analytic and harmonic case, see for example Proposition 2.3 below.

In the above situation \( f \) has a removable singularity at \( a \) iff \( \lim_{z \to a} \frac{f(z)}{\log |z-a|} = 0 \), it has a fundamental pole at \( a \) if and only if \( 0 < \lim_{z \to a} \frac{f(z)}{\log |z-a|} < \infty \), it has a pole at \( a \) of order \( M \) if and only if \( 0 < \limsup_{z \to a} |z-a|^M |f(z)| < \infty \), and finally \( f \) has an essential singularity at \( a \) if and only if \( \limsup_{z \to a} |z-a|^N |f(z)| = \infty \) for every positive integer \( N \).

There is an alternative, but equivalent way to expand \( u \in h(V \setminus \{a\}), V \subset \mathbb{C} \), namely to use analytic and conjugate analytic functions. We assume, for simplicity, that \( a = 0 \). Then we have

\[
u(z) = a_0 + b_0 \log |z| + \sum_{n \neq 0} (c_n z^n + d_n \bar{z}^n), \quad 0 < |z| < r.
\]

Note that \( a_0 = a_0(u), b_0 = b_0(u), c_n = c_n(u) \) and \( d_n = d_n(u) \).

**Proposition 1.2.** The functionals \( a_0, b_0, c_n \) and \( d_n, n \neq 0, \) are continuous on the Frechet space \( h(V'), V' = V \setminus \{0\} \).

**Proof.** Using

\[
b_0(u) = \frac{1}{2\pi} \int_{C_r} \frac{\partial u}{\partial n} ds, \quad 0 < \rho < \text{dist}(0, \partial V),
\]

where \( C_r \) is the circle of radius \( r \) centered at \( 0 \), and \( n \) is the normal to \( \partial V \).
where \( C_\rho \) is the circle centered at 0 of radius \( \rho \), we conclude, using continuity of derivatives on the space \( h(V') \) that \( b_0 \) is continuous on \( h(V') \). Now we fix \( 0 < \rho_1 < \rho_2 < \text{dist}(0, \partial V) \). For any \( k \neq 0 \) we have
\[
\phi_k(u) = \frac{1}{2\pi i} \int_{C_\rho} u(z)z^{-k}ds = c_k(u) + \rho_1^{-2k}d_k(u) \tag{7}
\]
and
\[
\psi_k(u) = \frac{1}{2\pi i} \int_{C_\rho} u(z)z^{-k}ds = \rho_2^{-2k}c_k(u) + d_k(u). \tag{8}
\]
Both \( \phi_k \) and \( \psi_k \) are continuous on \( h(V') \), since (7) and (8) represent a system of linear equations with determinant \( 1 - (\rho_2/\rho_1)^k \neq 0 \) it follows immediately that \( c_k \) and \( d_k \) are continuous. The case of \( a_0 \) is left to the reader. \( \square \)

2. Inclusions between \( b^p \) spaces

We start with an auxiliary proposition.

**Proposition 2.1.** Assume \( f \in b^p(V') \), where \( V' = V \setminus \{a\} \) for some \( a \in V \subset \mathbb{R}^n \). Then
\[
|f(x)| = o(|x - a|^{-n/p}), \quad x \to a. \tag{9}
\]
In particular, \( a \) is either a removable singularity of \( f \) or a pole of order \( k < n/p \). If \( n \geq 3 \) and \( p \geq \frac{n}{n-2} \), then \( a \) is a removable singularity.

**Proof.** Applying (2) to \( V = B(x, |x - a|) \) one gets (9) and that suffices in view of the above classification of isolated singularities. \( \square \)

Combining the last proposition and Proposition 1.1 we obtain the following:

**Corollary 2.2.** If \( f \in b^p(\Omega) \), \( p \geq \frac{n}{n-2} \) and \( n \geq 3 \), then \( f \) is identically zero.

Our first result demonstrates a basic difference between harmonic and analytic Bergman spaces on \( \Omega \) in the planar case, namely \( B^p(\Omega) = \{0\} \) for \( p \geq 2 \), see [1]. However we have:

**Proposition 2.3.** If \( n = 2 \), then \( b^p(\Omega) \neq \{0\} \) for \( 0 < p < \infty \).

**Proof.** The function \( f(z) = \log |z - 1| - 2 \log |z| + \log |z + 1| \) is harmonic in \( \Omega \) and, by Lagrange’s theorem, \(|f(z)| = O(|z|^{-2})\) as \( z \to 0 \). Therefore \( f \in b^2(\Omega) \).

Similarly, \( f(z) = \log |z + 1| - \log |z| \) is harmonic in \( \Omega \) and, by Lagrange’s theorem, \(|f(z)| = O(|z|^{-1})\). Therefore \( f \in b^p(\Omega) \) for \( 2 < p < \infty \).

Finally, for \( 0 < p < 2 \) the analytic Bergman spaces \( \mathcal{B}^p(\Omega) \) are non-empty, in fact they contain nontrivial rational functions, see [1]. \( \square \)

**Lemma 2.4.** Let \( k \in \mathbb{N} \) and \( n/(k+1) \leq q < p < n/k \). Then there is a constant \( C = C_{p,q,n} \) such that
\[
\|u\|_{b^p(Q(a,1))} \leq C\|u\|_{b^p(Q(a,3/2))} \quad \text{for every} \quad u \in b^p(Q(a,3/2)), \quad a \in \Gamma.
\]

**Proof.** This lemma states that the restriction operator \( R : b^p(Q(a,3/2)) \to b^p(Q(a,1)) \) given by \( Ru = u_{|Q(a,1)} \) is continuous. Since both spaces \( b^p(Q(a,3/2)) \) and \( b^p(Q(a,1)) \) are complete it suffices, by the closed graph theorem, to prove that \( R \) maps \( b^p(Q(a,3/2)) \) into \( b^p(Q(a,1)) \). Let \( u \in b^p(Q(a,3/2)) \). Since \( q \geq n/(k+1) \) Proposition 3 implies that the order of pole of \( u \) at \( a \) is at most \( k \). Therefore, \( |u(z)|^p = O(|a - z|^\text{-kp}) \) where \( kp < n \). Hence \( |u|^p \) is integrable in a neighborhood of \( a \) and that implies \( u \in b^p(Q(a,1)) \). \( \square \)

The main result of this section is the following result.
4. Some generalizations and open problems

Theorem 2.5. If \( n/(k+1) \leq q < n/k \) for \( k = 1, 2, \ldots \), then \( b^q(\Omega) \subseteq b^p(\Omega) \).

Proof. Set \( Q_\omega = Q(\omega, 1) \) for \( \omega \in \Gamma \). Let \( u \in b^q(\Omega) \). The poles of \( u \) have orders at most \( k \) hence \( u(z) = O(|z-\omega|^{-\alpha}) \) as \( z \to \omega \). Therefore \( u|_{Q_\omega} \in L^p(Q_\omega) \). Using Lemma 1 we get

\[
\|u\|_p = \int_\Omega |u|^p dm = \sum_{\omega \in \Gamma} \int_{Q_\omega} |u|^p dm \leq C \sum_{\omega \in \Gamma} \left( \int_{Q(\omega, 3/2)} |u|^p dm \right)^{p/q}
\]

\[
\leq C \left( \sum_{\omega \in \Gamma} \int_{Q(\omega, 3/2)} |u|^p dm \right)^{p/q} \leq 4^{p/q} C \left( \sum_{\omega \in \Gamma} \int_{Q_\omega} |u|^p dm \right)^{p/q} = 4^{p/q} C \|u\|_p^p
\]

because \( p/q \geq 1 \) and almost every point in \( \mathbb{C} \) lies in precisely 4 squares \( Q(\omega, 3/2) \). \( \square \)

We note that the above proof can be used to prove Theorem 1 from [1], in fact it presents a simplification of the proof given in [1].

3. Asymptotics at infinity of functions in \( b^p(\Omega) \)

One might conjecture that on the set \( \Omega_\epsilon = \{ z \in \mathbb{C} : d(z, \Gamma) > \epsilon \} \) we can control the size of functions \( f \in b^p(\Omega) \), for example that we can prove \( f(z) = O(|z|^{-2/p}) \), \( |z| \to \infty, z \in \Omega_\epsilon \). However, this is never true in general. The following theorem was proved in the case \( 0 < p < 2 \) for analytic Bergman spaces \( B^p(\Omega) \) in [1], and the same method of proof works in the present situation. We present this proof for reader’s convenience.

Theorem 3.1. Implication \( f \in b^p(\Omega) \Rightarrow f(z) = O(|z|^{-\alpha}) \) as \( |z| \to \infty, z \in \Omega_\epsilon \) does not hold for any \( 0 < p < \infty, \alpha > 0, 0 < \epsilon < 1/\sqrt{2} \).

Proof. Assume this implication holds for some \( 0 < p < \infty, \alpha > 0 \) and \( 0 < \epsilon < 1/\sqrt{2} \). One easily proves that

\[
h_{\epsilon, \alpha} = \{ f \in h(\Omega_\epsilon) : \|f\|_{\epsilon, \alpha} = \sup_{z \in \Omega_\epsilon} |z|^\alpha |f(z)| < +\infty \}
\]

is a Banach space. The restriction operator \( R : b^p(\Omega) \to h_{\epsilon, \alpha} \) has closed graph because convergence in both (quasi)-norms \( \| \cdot \|_p \) and \( \| \cdot \|_{\epsilon, \alpha} \) implies pointwise convergence. Hence \( R \) is bounded, that is \( \|f\|_{\epsilon, \alpha} \leq C\|f\|_p \) for all \( f \in b^p(\Omega) \). Let us pick a non-trivial \( f \in b^p(\Omega) \). Then

\[
\|f(z_0)\| = \|f_n(z_0 - n)\| \leq |z_0 - n|^{-\alpha}\|f_n\|_{\epsilon, \alpha} \leq C|z_0 - n|^{-\alpha}\|f_n\|_p
\]

for all \( n \in \mathbb{N}, z_0 \in \Omega_\epsilon \) (\( f_n \) denotes a function \( f_n(z) = f(z + n) \)). This gives, as \( n \to \infty \), \( f(z_0) = 0 \), hence \( f(z) = 0 \) on \( \Omega_\epsilon \) and therefore on \( \Omega \) as well. Contradiction. \( \square \)

Remark 3.2. The same proof works for a function \( \phi(|z|) \) instead of \( |z|^{-\alpha} \), where \( \phi(r) \) is strictly positive and \( \lim_{r \to +\infty} \phi(r) = 0 \).

4. Some generalizations and open problems

We alert reader to possible generalizations and open problems, these are parallel to those mentioned in [1]. One can define mixed norm spaces \( b^{p,q}(\Omega) \) using (quasi)-norms.
∥f∥_{p,q} = \left\{ \sum_{\omega \in \Gamma} \left( \int_{Q(\omega,1)} |f(z)|^p \, dx \, dy \right)^{q/p} \right\}^{1/q}, \quad 0 < p, q < \infty.

Note that \( b^{p,q}(\Omega) = b^r(\Omega) \). Some of our results generalize to the \( b^{p,q} \) spaces, without any substantial changes in the proofs. For example:

\[ b^{q,r} \subset b^{p,r}, \quad \frac{2}{n + 1} \leq q < p < \frac{2}{r}, \quad 0 < r < +\infty \quad (10) \]

Finally, we mention some natural questions on \( b^p(\Omega) \) spaces.
1. Is there a bounded projection from \( L^p(\Omega) \) onto \( b^p(\Omega) \)? This problem is related to the problem of finding the dual space of \( b^p(\Omega) \), see [5] for the problem in the context of analytic functions.
2. Describe the dual of \( b^p(\Omega) \).
3. Is \( b^p(\Omega) \) isomorphic to \( l^p \)? We note that there is a vast amount of literature related to classical Banach spaces, see [4].
4. Are there sequences \( z_n \) in \( \Omega \) such that \( \|f\|_p^p \sim \sum_{n=1}^{\infty} d(z_n, \Gamma)^2 |f(z_n)|^p \)?

References