Paracompactness with respect to an ideal

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Abstract. In this paper, we study \(I\)-paracompact spaces and discuss their properties. Also, we characterize \(I\)-paracompact spaces. Some of the results in paracompact spaces have been generalized in terms of \(I\)-paracompact spaces.

1. Introduction

The subject of ideals in topological spaces has been studied by Kuratowski [10] and Vaidyanathaswamy [14]. An ideal \(I\) on a set \(X\) is a nonempty collection of subsets of \(X\) which satisfies (i) \(A \in I\) and \(B \subseteq A\) implies \(B \in I\) and (ii) \(A \in I\) and \(B \in I\) implies \(A \cup B \in I\). An ideal \(I\) is said to be a \(\sigma\)–ideal [9] if it is countably additive. Given a topological space \((X, \tau)\) with an ideal \(I\) on \(X\) and if \(\varphi(X)\) is the set of all subsets of \(X\), a set operator \((\mathcal{I}): \varphi(X) \to \varphi(X)\), called a local function [9] of \(A\) with respect to \(\tau\) and \(I\), is defined as follows: for \(A \subseteq X\), \(A^*(I, \tau) = \{x \in X \mid \lambda \cap A \not\in I\ \text{for every } \lambda \in \tau(x)\}\) where \(\tau(x) = \{\lambda \subseteq \tau \mid x \in \lambda\}\). A Kuratowski closure operator \(cl^*(I)\) for a topology \(\tau^*(I, \tau)\), called \(\star\)-topology, finer than \(\tau\) is defined by \(cl^*(A) = A^* \cup A^*(I, \tau)\) [9].

If \(I\) is an ideal on \(X\), then \((X, \tau, \star)\) is called an ideal space. A subset \(A\) of a topological space \((X, \tau)\) is said to be a generalized \(F_\sigma\)-subset [13] if for each open subset \(U\) of \(X\) containing \(A\), there exists an \(F_\sigma\)-subset \(B\) of \(X\) which is contained in \(U\) and contains \(A\). A space \(X\) is said to be totally normal [12] if it is normal and every open subset \(G\) of \(X\) is expressible as a union of a locally finite (in \(G\)) family of open \(F_\sigma\)-subsets of \(X\). A space \(X\) is said to be perfectly normal [6] if it is normal and in which each open set is an \(F_\sigma\)-set. A subset \(A\) of a space \((X, \tau)\) is said to be \(\gamma\)-closed [11] if \(cl(A) \subseteq U\), whenever \(A \subseteq U\) and \(U \in \tau\). By a space \((X, \tau)\), we always mean a topological space \((X, \tau)\) with no separation properties assumed. If \(A \subseteq X\), \(cl(A)\) and \(int(A)\) will, respectively, denote the closure and interior of \(A\) in \((X, \tau)\).

Lemma 1.1. [1] The union of a finite family of locally finite collection of sets in a space \((X, \tau)\) is again locally finite.

Lemma 1.2. [1] If \(\mathcal{V}\) is a locally finite family of sets in a space \((X, \tau)\), then \(\lambda = \{cl(Q) \mid Q \in \mathcal{V}\}\) is locally finite in \(X\).

Lemma 1.3. [3] If \(\{A_\alpha \mid \alpha \in \Delta\}\) is a locally finite family of subsets in a space \((X, \tau)\), and if \(B_\alpha \subseteq A_\alpha\) for each \(\alpha \in \Delta\), then the family \(\{B_\alpha \mid \alpha \in \Delta\}\) is locally finite in \(X\).
2. \(I\)-paracompact subsets

The concept of paracompactness with respect to an ideal was introduced by Zahid \[15\] and is further studied by T.R. Hamlett, D. Rose and D. Janković \[8\]. An ideal space \((X, \tau, I)\) is said to be paracompact modulo \(I\) or \(I\)-paracompact \[8\] if and only if every open cover \(\mathcal{U}\) of \(X\) has a locally finite open refinement \(\mathcal{V}\) (not necessarily a cover) such that \(X - \bigcup\{V \mid V \in \mathcal{V}\} \in I\). A subset \(A\) of an ideal space \((X, \tau, I)\) is said to be \(I\)-paracompact relative to \(X\) \((I\)-paracompact subset \[8\]) if for any open cover \(\mathcal{U}\) of \(A\), there exist \(I \in I\) and locally finite family \(\mathcal{V}\) of open sets such that \(\mathcal{V}\) refines \(\mathcal{U}\) and \(A \subset \bigcup\{V \mid V \in \mathcal{V}\} \cup I\). \(A\) is said to be \(I\)-paracompact \((I\)-paracompact subspace \[8\]) if \((A, \tau_A, I_A)\) is \(I_A\)-paracompact as a subspace, where \(\tau_A\) is the usual subspace topology. Theorem 2.1 below shows that a space \((X, \tau)\) is \(I\)-paracompact if and only if it is paracompact modulo \([0]\), the easy proof of which is omitted. A space \(X\) is said to be \(I\)-paracompact if every subset of \(X\) is \(I\)-paracompact. In this section, we characterize \(I\)-paracompact spaces.

**Theorem 2.1.** Let \((X, \tau)\) be a space with an ideal \(I = \{0\}\). Then \((X, \tau)\) is paracompact if and only if \((X, \tau)\) is paracompact modulo \(I\). The following Theorem 2.2 gives a property of subsets of \(X\) which are \(I\)-paracompact.

**Theorem 2.2.** If every open subset of \((X, \tau, I)\) is \(I\)-paracompact, then every subset of \(X\) is \(I\)-paracompact.

**Proof.** Let \(B\) be an open subset of \((X, \tau, I)\) such that \(B\) is \(I\)-paracompact. By Theorem 2.1, \(B\) is paracompact. The following Theorem 2.4 is a generalization of the above result. If \(I = \{0\}\) in the above Theorem 2.2, we have the following Corollary 2.3.

**Corollary 2.3.** [4, 7] If every open subset of a space \((X, \tau)\) is paracompact, then every subset of \(X\) is paracompact.

Hamlett, Rose and Janković \[8\] established that every closed subset of an \(I\)-paracompact space is \(I\)-paracompact. The following Theorem 2.4 is a generalization of the above result. If \(I = \{0\}\) in Theorem 2.4, we have Corollary 2.6.

**Theorem 2.4.** Every \(F_{\alpha}\)-set (countable union of closed sets) of an \(I\)-paracompact space \((X, \tau, I)\) is an \(I\)-paracompact subspace of \(X\).

**Proof.** Let \(A\) be a \(F_{\alpha}\)-subset of \(X\). Then \(A = \bigcup\{A_i \mid i \in N\}\) where each \(A_i\) is closed. Let \(\mathcal{U} = \{U_a \mid a \in A\}\) be a \(\tau_A\)-open cover of \(A\) where \(U_a = V_a \cap A\) such that \(V_a\) is open in \(X\). Then \(\mathcal{U} = \{U_a \mid a \in A\}\) is an open cover of \(X\). By hypothesis, there exist \(I \in I\) and \(\tau\)-locally finite family \(\mathcal{V} = \{V_\beta \mid \beta \in \mathcal{V}\}\) which refines \(\mathcal{U}\) such that \(\mathcal{V} = \bigcup\{V_\beta \mid \beta \in \mathcal{V}\} \cup I\). Then \(V \cap B = \bigcup\{V_\beta \mid \beta \in \mathcal{V}\} \cup I\) which implies that \(B = \bigcup\{V_\beta \mid \beta \in \mathcal{V}\} \cup (I \cap B)\) which implies that \(B = \bigcup\{V_\beta \mid \beta \in \mathcal{V}\} \cup I\) where \(I = I \cap B\) in \(\mathcal{I}\). Let \(x \in B\). Since \(\mathcal{V}\) is \(\tau\)-locally finite, there exists \(U \in \tau(x)\) such that \(V_\beta \cap U = \emptyset\) for all \(\beta \neq \beta_1, \beta_2, ..., \beta_n\) and \((V_\beta \cap U) \cap B = \emptyset\) for all \(\beta \neq \beta_1, \beta_2, ..., \beta_n\). Therefore, \(\mathcal{V} = \{V_\beta \mid \beta \in \mathcal{V}\}\) is \(\tau\)-locally finite. Let \(V_\beta \cap B \in \mathcal{V}\). Then \(V_\beta \in \mathcal{V}\). Hence every subset of \(X\) is an \(I\)-paracompact subspace. \(\square\)

If \(I = \{0\}\) in the above Theorem 2.2, we have the following Corollary 2.3.

**Corollary 2.3.** [4, 7] If every open subset of a space \((X, \tau)\) is paracompact, then every subset of \(X\) is paracompact.
Corollary 2.5. [8] Let \((X, \tau, I)\) be an \(I\)–paracompact space. If \(A \subseteq X\) is closed, then \(A\) is \(I\)–paracompact.

Corollary 2.6. [7, P.218, Theorem 8] Every \(F_o\)–set of a paracompact space \((X, \tau)\) is paracompact.

Theorem 2.7. Let \((X, \tau, I)\) be a space and let \(A\) be a subset of \(X\) such that for each open set \(U \supseteq A\), there is an \(I\)–paracompact set \(B\) with \(A \subset B \subset U\). Then \(A\) is \(I\)–paracompact.

Proof. Let \(\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}\) be a \(\tau_A\)–open cover of \(A\) where \(U_\alpha = A \cap V_\alpha\) such that \(V_\alpha\) is open in \(X\). By the given condition, there exists an \(I\)–paracompact subset \(B\) of \(X\) such that \(A \subset B \subset \bigcup V_\alpha\). Then \(\mathcal{U}_B = \{V_\alpha \cap B \mid \alpha \in \Delta\}\) is a \(\tau_B\)–open cover of \(B\). By hypothesis, there exist \(I \cap B = I_B \in I_B\) and \(\tau_B\)–locally finite family \(\mathcal{V}_B = \{V_\beta \cap B \mid \beta \in \mathcal{V}\}\) which refines \(\mathcal{U}_B\) such that \(B \subset \bigcup V_\beta \cap B \mid \beta \in \mathcal{V}\) \(\cup (I \cap B)\). Then \(A = B \cap A \subset (\bigcup V_\beta \cap B \mid \beta \in \mathcal{V}) \cup (I \cap B)\) which implies that \(A\) is \(I\)–paracompact in \(X\). Since \(\mathcal{V} = \{V_\beta \cap B \mid \beta \in \mathcal{V}\}\) is \(\tau_B\)–locally finite, there exists \(W \in \tau(X)\) such that \((V_\beta \cap B) \cap W = \emptyset\) for all \(\beta \neq \beta_1, \beta_2, ..., \beta_n\) which implies that \((V_\beta \cap B) \cap (\bigcup \cap B))) = \emptyset\) for all \(\beta \neq \beta_1, \beta_2, ..., \beta_n\). Therefore, \(\mathcal{V} = \{V_\beta \cap A \mid \beta \in \mathcal{V}\}\) is \(\tau_A\)–locally finite. Let \(V_\beta \cap A \in \mathcal{V}\). Then \(V_\beta \cap B \in \mathcal{V}\). Since \(\mathcal{V}_B\) refines \(\mathcal{U}_B\), there is some \(V_\alpha \cap B \in \mathcal{U}_B\) such that \(V_\alpha \cap B \subset V_\beta \cap B\). Also, \(A \subset B\) implies that \(V_\alpha \cap A \subset V_\alpha \cap B\). Thus, \(V_\alpha \cap A \subset V_\alpha \cap B = U_\alpha\) so that \(\mathcal{V}\) refines \(\mathcal{U}\). Hence \(A\) is \(I\)–paracompact.

Corollary 2.8. Every generalized \(F_o\)–subset of an \(I\)–paracompact space \((X, \tau, I)\) is \(I\)–paracompact.

Proof. Let \(X\) be an \(I\)–paracompact space. Let \(A\) be a generalized \(F_o\)–subset of \(X\). Then for every open subset \(U\) of \(X\) containing \(A\), there exists an \(F_o\)–subset \(B\) of \(X\) which is contained in \(U\) and contains \(A\). By Theorem 2.4, \(B\) is \(I\)–paracompact. Therefore, by Theorem 2.7, \(A\) is \(I\)–paracompact.

If \(I = \emptyset\) in the above Theorem 2.7, we have the following Corollary 2.9.

Corollary 2.9. [6] Let \((X, \tau)\) be a space and let \(A\) be a subset of \(X\) such that for each open set \(U \supseteq A\), there is a paracompact set \(B\) with \(A \subset B \subset U\). Then \(A\) is paracompact.

If \(I = \emptyset\) in the above Corollary 2.8, we have Corollary 2.10.

Corollary 2.10. Every generalized \(F_o\)–subset of a paracompact space \((X, \tau)\) is paracompact.

Theorem 2.11. Every subset of a perfectly normal \(I\)–paracompact space \((X, \tau, I)\) is \(I\)–paracompact.

Proof. Suppose that \((X, \tau, I)\) is a perfectly normal \(I\)–paracompact space. Since \(X\) is perfectly normal, every open set is an \(F_o\) set and so every open set is \(I\)–paracompact, by Theorem 2.4. Therefore, by Theorem 2.2, every subset of \(X\) is \(I\)–paracompact.

If \(I = \emptyset\) in the above Theorem 2.11, we have Corollary 2.12.

Corollary 2.12. [5, 7] Every subset of a perfectly normal, paracompact space \((X, \tau)\) is paracompact.

Corollary 2.13. Every perfectly normal \(I\)–paracompact space \((X, \tau, I)\) is hereditarily \(I\)–paracompact.

If \(I = \emptyset\) in the above Corollary 2.13, we have Corollary 2.14.

Corollary 2.14. Every perfectly normal paracompact space \((X, \tau)\) is hereditarily paracompact.

Theorem 2.15. Let \(\{V_\alpha \mid \alpha \in \Delta\}\) be a locally finite open covering of a space \((X, \tau, I)\) such that each \(cI(V_\alpha)\) is \(I\)–paracompact relative to \(X\). Then \(X\) is \(I\)–paracompact.
Proof. Let \( U = \{ U_\alpha \mid \gamma \in \Delta_0 \} \) be an open cover of \( X \). Then for each \( \alpha \), \( U \) is a cover of \( cl(V_\alpha) \) by \( \tau \)-open sets. By hypothesis, there exist \( I \in I \) and locally finite family \( V_1 = \{ V_\beta \mid \beta \in \Delta_1 \} \) of open sets which refines \( U \) such that \( cl(V_\alpha) \subset \cup \{ V_\beta \mid \beta \in \Delta_1 \} \cup I \). Now \( V_\alpha = cl(V_\alpha) \cap V_\alpha \subset \cup \{ V_\beta \mid \beta \in \Delta_1 \} \cup I \cap V_\alpha = \cup \{ V_\beta \cap V_\alpha \mid \beta \in \Delta_1 \} \cup (I \cap V_\alpha) \) which implies that \( V_\alpha \subset \cup \{ V_\beta \cap V_\alpha \mid \beta \in \Delta_1 \} \cup I \). Since \( \{ V_\alpha \mid \alpha \in \Delta_0 \} \) is an open covering of \( X \), \( X = \cup \{ V_\beta \cap V_\alpha \mid \alpha \in \Delta_0 , \beta \in \Delta_1 \} \cup I \). Since \( \{ V_\alpha \mid \alpha \in \Delta_0 \} \) and \( V_1 = \{ V_\beta \mid \beta \in \Delta_1 \} \) are locally finite, \( \mathcal{V} = \{ V_\beta \cap V_\alpha \mid \alpha \in \Delta_0 , \beta \in \Delta_1 \} \) is locally finite. If \( V_\beta \cap V_\alpha \subset \mathcal{V} \), then \( V_\beta \in \mathcal{V} \) and since \( V_1 \) refines \( U \), there is some \( U_\beta \in U \) such that \( V_\beta \subset U_\beta \). Also, \( V_\beta \cap V_\alpha \subset \mathcal{V} \subset U_\beta \), implies that \( V_\beta \cap V_\alpha \subset U_\beta \). Therefore, \( \mathcal{V} \) refines \( U \). Hence \( X \) is \( I \)-paracompact. \( \square \)

If \( I = \{ \emptyset \} \) in the above Theorem 2.15, we have the following Corollary 2.16.

**Corollary 2.16.** Let \( \{ V_\alpha \mid \alpha \in \Delta \} \) be a locally finite open covering of a space \((X, \tau)\) such that each \( cl(V_\alpha) \) is paracompact relative to \( X \). Then \( X \) is paracompact.

**Theorem 2.17.** Every subset of a totally normal \( I \)-paracompact space \((X, \tau, I)\) is \( I \)-paracompact.

**Proof.** Let \( X \) be a totally normal \( I \)-paracompact space. Let \( G \) be an open subset of \( X \). Since \( X \) is totally normal, \( \mathcal{G} = U_G \), where \( C_G \)’s are open \( F_\sigma \)-subset of \( X \) and locally finite in \( G \). Therefore, \( |C_G| \) is a locally finite open covering of \( G \). Also, for each \( i \), \( cl(C_G) \) is a closed subsets of \( X \) and so by Theorem IV.3[8], \( cl(G) \) is \( I \)-paracompact relative to \( X \) for each \( i \). Then \( cl(G) \) is \( I \)-paracompact relative to \( G \) for each \( i \). Therefore, \( G \) is \( I \)-paracompact, by Theorem 2.15. Since \( G \) is an open subset of \( X \), by Theorem 2.2, every subset of \( X \) is \( I \)-paracompact. \( \square \)

**Corollary 2.18.** Every totally normal \( I \)-paracompact space is hereditarily \( I \)-paracompact.

If \( I = \{ \emptyset \} \) in the above Theorem 2.17, we have the following Corollary 2.19.

**Corollary 2.19.** Every subset of a totally normal paracompact space is paracompact.

A collection \( \mathcal{V} \) of subsets of \( X \) is said to be an \( I \)-cover [15] of \( X \) if \( X = \cup \{ V_\alpha \mid V_\alpha \in \mathcal{V} \} \in I \). A collection \( \mathcal{A} \) of subsets of a space \((X, \tau)\) is said to be \( \sigma \)-locally finite [8] if \( \mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n \) where each collection \( \mathcal{A}_n \) is a locally finite. The following Theorem 2.20 gives a property of \( I \)-paracompact spaces.

**Theorem 2.20.** Let \((X, \tau, I)\) be a regular ideal space. If \( X \) is \( I \)-paracompact, then every open cover of \( X \) has a closed locally finite \( I \)-cover refinement.

**Proof.** Let \( \mathcal{U} \) be an open cover of \( X \). For each \( x \in X \), let \( U_x \in \mathcal{U} \) such that \( x \in U_x \). Since \((X, \tau)\) is regular, for each \( x \in X \), there exists a neighborhood \( V_x \) of \( x \) such that \( cl(V_x) \subset U_x \). Now \( \mathcal{U}_1 = \{ V_x \mid x \in X \} \) is an open cover of \( X \) and so there exist an \( I \) in \( I \) and a locally finite family \( \mathcal{W}_1 = \{ W_\beta \mid \beta \in \Delta \} \) of open sets which refines \( \mathcal{U}_1 \) such that \( X = \cup \{ W_\beta \mid \beta \in \Delta \} \cup I \) which implies that \( X = \cup \{ cl(W_\beta) \mid \beta \in \Delta \} \cup I \). Since the family \( \mathcal{W}_1 = \{ W_\beta \mid \beta \in \Delta \} \) is locally finite, the family \( \mathcal{W} = \{ cl(W_\beta) \mid W_\beta \in \mathcal{W}_1 \} \) is locally finite, by Lemma 1.2. Let \( cl(W_\beta) \in \mathcal{W} \). Then \( W_\beta \in \mathcal{W}_1 \). Since \( \mathcal{W}_1 \) refines \( \mathcal{U}_1 \), there is some \( V_x \in \mathcal{U}_1 \) such that \( W_\beta \subset V_x \) and so \( cl(W_\beta) \subset cl(V_x) \). Also, \( cl(V_x) \subset U_x \) implies that \( cl(W_\beta) \subset U_x \). Hence \( \mathcal{W} \) refines \( \mathcal{U} \). Thus, \( \mathcal{W} = \{ cl(W_\beta) \mid \beta \in \Delta \} \) is a closed locally finite family which refines \( \mathcal{U} \) which completes the proof. \( \square \)

**Corollary 2.21.** [7, P.210, Lemma 2] If every covering of a regular space \( X \) has a locally finite refinement, then every open covering of that space also has closed locally finite refinement.

**Theorem 2.22.** Let \((X, \tau, I)\) be a regular ideal space. Then \( X \) is \( I \)-paracompact if and only if every open cover of \( X \) has an open \( \sigma \)-locally finite \( I \)-cover refinement.

**Proof.** Since every locally finite refinement is \( \sigma \)-locally finite refinement, it is enough to prove the sufficiency. Let \( \mathcal{U} \) be an open cover of \( X \). Then there exists \( I \in I \) and open \( \sigma \)-locally finite refinement \( \mathcal{V} \) of \( \mathcal{U} \) such that \( X \subset \cup \{ V \mid V \in \mathcal{V} \} \cup I \). Also, \( \mathcal{V} = \bigcup_{\alpha \in \mathcal{N}} V_\alpha \) where each \( V_\alpha \) is locally finite. For each \( n \in \mathcal{N} \), let
which implies $V \in \mathcal{V}_n$. Then $W_n$ refines $W_n$. Let $x \in X$ and $n$ be the smallest member of $\{ n \in N \mid x \in W_n \}$. Then $x \in W_n$. Hence $X \subset \bigcup \{ W_n \mid n \in N \} \cup I$. Also, $W_n$ is a neighborhood of $x$ that intersect only finite number of members of $W_n$ so that $\{ W_n \mid n \in N \}$ is locally finite. Let $\mathcal{W} = \{ W_n \mid n \in N \}$ and $V \in \mathcal{V}_n$. Let $x \in X$. Since $W_n \mid n \in N \}$ is locally finite, there exists a neighborhood $P$ containing $x$ that intersects only a finite number of members of $W_n \mid n \in N \}$. Also, for each $i = 1, 2, ... , k$, there exists a neighborhood $O_{x_i}$ containing $x$ that intersects only a finite number of members of $\mathcal{V}_n$. Then $P \cap O_{x_i}$ is a neighborhood of $x$ that intersects only a finite number of members of $W_i$. Hence $\mathcal{W}$ is locally finite. Let $W \in \mathcal{W}$. Then $V \in \mathcal{V}_n$. Since $\mathcal{V}_n$ refines $\mathcal{U}$, there is some $U \in \mathcal{U}$ such that $V \subset U$. Then $W \subset \bigcup \{ V \mid n \in N \}$. Thus, $\mathcal{W}$ refines $\mathcal{U}$. Since $X \subset \bigcup \{ V \mid V \in \mathcal{V}_n \} \cup I$ and $X \subset \bigcup \{ W_n \mid n \in N \} \cup I$, $X \subset \bigcup \{ (W_n \cap V) \mid n \in N \} \cup V \in \mathcal{V}_n \cup I$. Therefore, $(X, \tau, I)$ is $I$–paracompact. 

Corollary 2.23. [7, P.210, Theorem 4] Let $(X, \tau)$ be a regular space. Then $(X, \tau)$ is paracompact if and only if every open cover of $X$ has an open $\alpha$–locally finite refinement.

3. Relative $I$–paracompact subsets

In this section, we discuss some of the properties of subsets of $I$–paracompact spaces.

Theorem 3.1. Let $(X, \tau, I)$ be an ideal space. If $B$ is an open subset of $X$, $A \subset B$ and $A$ is $I$–paracompact relative to $X$, then $A$ is $I$–paracompact subset of $B$.

Proof. Let $\mathcal{U} = \{ U_a \mid a \in \Delta \}$ be a cover of $A$ by sets open in $A$. Then $\mathcal{U} = \{ U_a \mid a \in \Delta \}$ is an open cover of $A$, since $B$ is open in $X$. By hypothesis, there exist $I \in \mathcal{I}$ and locally finite family $\mathcal{V} = \{ V_{\beta} \mid \beta \in \Delta \}$ by sets open in $X$ which refines $\mathcal{U}$ such that $A \subset \bigcup \{ V_{\beta} \mid \beta \in \Delta \} \cup I$ which implies $A \subset \bigcup \{ V_{\beta} \cap B \mid \beta \in \Delta \} \cup I$. Let $x \in B$. Since $\mathcal{V} = \{ V_{\beta} \mid \beta \in \Delta \}$ is locally finite in $X$, there exists $W \in \tau(x)$ such that $W \cap V_{\beta} = \emptyset$ for $\beta \neq \beta_1, \beta_2, ... , \beta_n$ which implies $(W \cap V_{\beta}) \cap B = \emptyset$ for $\beta \neq \beta_1, \beta_2, ... , \beta_n$. Therefore, the family $\mathcal{V}_1 = \{ V_{\beta} \cap B \mid \beta \in \Delta \}$ is $B$–locally finite. Let $V_{\beta} \cap B \in \mathcal{V}_1$. Then $V_{\beta} \cap B \in \mathcal{V}$. Since $\mathcal{V}$ refines $\mathcal{U}$, there is some $U_{\alpha} \in \mathcal{U}$ such that $V_{\beta} \subset U_{\alpha}$ which implies $V_{\beta} \cap B \subset U_{\alpha} \cap B \subset U_{\alpha}$. Hence $\mathcal{V}_1$ refines $\mathcal{U}$. Therefore, $A$ is $I$–paracompact relative to $B$.

Theorem 3.2. Let $S$ be a closed subset of an ideal space $(X, \tau, I)$. If $F \subset S$ is $I$–paracompact relative to $S$ and if there exists an open set $G$ in $X$ such that $F \subset G \subset S$, then $F$ is $I$–paracompact relative to $X$.

Proof. Let $\mathcal{U} = \{ U_a \mid a \in \Delta \}$ be an open cover of $F$ by sets open in $X$. Then $\mathcal{U} = \{ U_a \mid a \in \Delta \}$ is an open cover of $F$ by sets open in $G$ so that $\mathcal{U}_1 = \{ U_a \cap G \mid a \in \Delta \}$ is an open cover of $F$ by sets open in $S$. By hypothesis, there exist $I \in \mathcal{I}$ and locally finite family $\mathcal{V}_1 = \{ V_{\beta} \mid \beta \in \Delta \}$ in $S$ where each $V_{\beta} = V_{\beta} \cap S$ is open in $S$ which refines $\mathcal{U}_1$, such that $F \subset \bigcup \{ V_{\beta} \cap S \mid \beta \in \Delta \} \cup I$ and $F \subset \bigcup \{ V_{\beta} \cap S \mid \beta \in \Delta \} \cup I \cap G \cap \mathcal{V}_1$ implies $F \subset \bigcup \{ V_{\beta} \cap S \mid \beta \in \Delta \} \cap (I \cap G)$ implies $F \subset \bigcup \{ V_{\beta} \cap S \mid \beta \in \Delta \} \cup \mathcal{V}_1$. Let $x \in X$. If $x \in S$, there exists $W \in \tau(x)$ such that $V_{\beta} \cap S \cap W = \emptyset$ for $\beta \neq \beta_1, \beta_2, ... , \beta_n$ which implies $(V_{\beta} \cap S \cap W) \cup G = \emptyset$ for $\beta \neq \beta_1, \beta_2, ... , \beta_n$. If $x \in X - S$, then $X - S$ is an open set containing $x$ such that $(V_{\beta} \cap G \cap (X - S)) = \emptyset$. Thus, the family $\mathcal{V}_1 = \{ V_{\beta} \cap G \mid \beta \in \Delta \}$ is locally finite in $X$. Let $V_{\beta} \cap G \in \mathcal{V}$. Then $V_{\beta} \cap S \in \mathcal{V}_1$. Since $\mathcal{V}_1$ refines $\mathcal{U}_1$, there is some $U_{\alpha} \in \mathcal{U}$, such that $V_{\beta} \cap S \subset U_{\alpha} \cap G \subset U_{\alpha}$. Hence $\mathcal{V}_1$ refines $\mathcal{U}$. Therefore, $F$ is $I$–paracompact relative to $X$.

Theorem 3.3. If $A$ is $I$–paracompact relative to $X$ and $B$ is a closed subset of $X$, then $A \cap B$ is $I$–paracompact relative to $X$.

Proof. Let $\mathcal{U} = \{ U_a \mid a \in \Delta \}$ be an open cover of $A \cup B$. Then $\mathcal{U}_A = \{ U_a \mid a \in \Delta \} \cup (X - B)$ is an open cover of $A$. By hypothesis, there exist $I \in \mathcal{I}$ and locally finite family $\mathcal{V}_A = \{ V_{\alpha} \cap (X - B) \mid a \in \Delta \}$ which refines $\mathcal{U}_A$ such that $A \subset \bigcup (V_{\alpha} \cap (X - B) \mid a \in \Delta \} \cup I$. Then $A \subset \bigcup (V_{\alpha} \cap (X - B) \mid a \in \Delta \} \cup (I \cap B)$ which implies that $A \cap B \subset \bigcup (V_{\alpha} \cap B \mid a \in \Delta \} \cup I$. Let $x \in X$. Since $\mathcal{V}_A = \{ V_{\alpha} \cap (X - B) \mid a \in \Delta \}$ is locally finite, there exists $W \in \tau(x)$ such that $(V_{\alpha} \cap (X - B)) \cap W = \emptyset$ for $\alpha \neq a_1, a_2, ... , a_n$ which implies $(V_{\alpha} \cap W) \cup ((X - B) \cap W) = \emptyset$.
for \( \alpha \neq \alpha_1, \alpha_2, ..., \alpha_n \) which implies \((V_{\alpha} \cap W) \cup ((X - B) \cap W)) \cap B = \emptyset \) for \( \alpha \neq \alpha_1, \alpha_2, ..., \alpha_n \). Therefore, the family \( \mathcal{V} = \{V_{\alpha} \cap B \mid \alpha \in \Delta \} \) is locally finite. Let \( V_{\alpha} \cap B \in \mathcal{V} \). Then \( V_{\alpha} \cup (X - B) \in \mathcal{V} \). Since \( \mathcal{V}_A \) refines \( \mathcal{U}_A \), there is some \( U_{\gamma} \subseteq (X - B) \subseteq \mathcal{U}_A \) such that \( V_{\alpha} \cup (X - B) \subseteq U_{\gamma} \subseteq (X - B) \) which implies \( (V_{\alpha} \cup (X - B)) \cap B \subseteq (U_{\gamma} \cup (X - B)) \cap B \) which implies \( V_{\alpha} \cap B \subseteq U_{\gamma} \cap B \subseteq U_{\gamma} \). Hence \( \mathcal{V} \) refines \( \mathcal{U} \). Therefore, \( A \cap B \) is \( I \)-paracompact relative to \( X \). □

**Corollary 3.4.** If \( A \) is \( I \)-paracompact relative to \( X \) and \( B \subset A \) is a closed subset of \( A \), then \( B \) is \( I \)-paracompact relative to \( X \).

**Theorem 3.5.** Let \( A \) be \( I \)-paracompact relative to \( X \) and \( B \) an open set contained in \( A \). Then \( A - B \) is \( I \)-paracompact relative to \( X \).

**Proof.** Let \( \mathcal{U} = \{U_\alpha \mid \alpha \in \Delta \} \) be a cover of \( A - B \) by sets open in \( X \). Then \( \mathcal{U}_A = \{U_\alpha \mid \alpha \in \Delta \} \cup B \) is a cover of \( A \) by sets open in \( X \). By hypothesis, there exist \( I \in I \) and locally finite family \( \mathcal{V}_I = \{V_\beta \mid \beta \in \Delta_0 \} \cup B \) which refines \( \mathcal{U}_A \) such that \( A \subseteq \cup \{(V_\beta \mid \beta \in \Delta_0) \cup B) \cup I \} \). Then \( A - B \subseteq \cup \{(V_\beta \mid \beta \in \Delta_0) \cup B) \cup I \} - B \) which implies that \( A - B \subseteq \cup \{(V_\beta \mid \beta \in \Delta_0) \cup B) \cup I \} \). Since the family \( \mathcal{V}_I = \{V_\beta \mid \beta \in \Delta_0 \} \) is locally finite, the family \( \mathcal{V} = \{V_\beta - B \mid \beta \in \Delta_0 \} \) is locally finite, by Lemma 1.3. Let \( V_\beta - B \in \mathcal{V} \). Then \( V_\beta \cup B \subseteq \mathcal{V}_I \). Since \( \mathcal{V}_I \) refines \( \mathcal{U}_I \), there is some \( U_{\delta} \subseteq B \subseteq U_{\delta} \), such that \( V_\delta \cup B \subseteq U_{\delta} \cup B \) which implies \( (V_\delta \cup B) \cup B \subseteq (U_{\delta} \cup B) - B \subseteq (U_{\delta} \cup B) - B \) and so \( V_\delta - B \subseteq U_{\delta} - B \subseteq U_{\delta} \). Therefore, \( \mathcal{V} \) refines \( \mathcal{U} \). Hence \( A - B \) is \( I \)-paracompact relative to \( X \). □

**Theorem 3.6.** In a space \((X, \tau, I)\), if \( A \) and \( B \) are \( I \)-paracompact relative to \( X \), then \( A \cup B \) is \( I \)-paracompact relative to \( X \).

**Proof.** Let \( \mathcal{U} = \{U_\gamma \mid \gamma \in \Delta \} \) be a cover of \( A \cup B \) by sets open in \( X \). Then \( \mathcal{U}_A = \{U_\gamma \mid \gamma \in \Delta \} \cup B \) is an open cover of \( A \) and \( B \) by hypothesis. There exist \( I_\alpha, I_\beta \in I \) and locally finite families \( \mathcal{V}_A = \{V_\alpha \mid \alpha \in \Delta_0 \} \) of \( A \) and \( \mathcal{V}_B = \{V_\beta \mid \beta \in \Delta_1 \} \) of \( B \) which refines \( \mathcal{U} \) such that \( A \subseteq \cup \{(V_\alpha \mid \alpha \in \Delta_0) \cup I_\alpha \} \cup B \subseteq \cup \{(V_\beta \mid \beta \in \Delta_1) \cup I_\beta \} \). Then \( A \cup B \subseteq \cup \{(V_\alpha \mid \alpha \in \Delta_0) \cup I_\alpha \} \cup \{V_\beta \mid \beta \in \Delta_1 \} \cup I_\beta \cup B \). Then \( A \cup B \subseteq \cup \{(V_\alpha \mid \alpha \in \Delta_0) \cup I_\alpha \} \cup \{V_\beta \mid \beta \in \Delta_1 \} \cup I_\beta \cup B \). Since the family \( \mathcal{V} = \{V_\alpha \cup V_\beta \mid \alpha \in \Delta_0, \beta \in \Delta_1 \} \) is locally finite, by Lemma 1.1, which refines \( \mathcal{U} \). Therefore, \( A \cup B \) is \( I \)-paracompact relative to \( X \). □

**Theorem 3.7.** Every \( g \)-closed subset of an \( I \)-paracompact space is \( I \)-paracompact relative to \( X \).

**Proof.** Let \( A \) be a \( g \)-closed subset of \((X, \tau, I)\). Let \( \mathcal{U} = \{U_\alpha \mid \alpha \in \Delta \} \) be an open cover of \( A \). Then \( A \subseteq \cup U_\alpha \). Since \( A \) is \( g \)-closed, \( cI(A) \subseteq \cup U_\alpha \). Then \( \mathcal{U}_A = \{U_\alpha \mid \alpha \in \Delta \} \cup (X - cI(A)) \) is an open cover of \( X \). By hypothesis, there exist \( I \in I \) and locally finite family \( \mathcal{V}_I = \{V_\beta \mid \beta \in \Delta_0 \} \cup (X - cI(A)) \) which refines \( \mathcal{U}_A \) such that \( X \subseteq \cup \{(V_\beta \mid \beta \in \Delta_0) \cup I \} \). Then \( cI(A) - \cup V_\beta \in I \). Since \( A \subseteq \cup V_\beta \subseteq cI(A) - \cup V_\beta \subseteq I \), by heredity. Since \( \mathcal{V}_I = \{V_\beta \mid \beta \in \Delta_0 \} \) is locally finite, the family \( \mathcal{V} = \{V_\beta \mid \beta \in \Delta_0 \} \) is locally finite, by Lemma 1.3. Thus, the family \( \mathcal{V} \) is locally finite which refines \( \mathcal{U} \). Therefore, \( A \) is \( I \)-paracompact relative to \( X \). □

**Theorem 3.8.** Let \((X, \tau, I)\) be a perfectly normal ideal space with a \( \sigma \)-ideal \( I \) and \( G \) be a subset of \( X \) such that \( G \) is the union of countable number of open subsets \( G_n \) of \( X \). Then each \( G_n \in \mathbb{N} \) is \( I \)-paracompact relative to \( X \) if and only if \( G \) is \( I \)-paracompact relative to \( X \).

**Proof.** Suppose each \( G_n, n \in \mathbb{N} \) is \( I \)-paracompact relative to \( X \). Let \( \mathcal{U} = \{U_\alpha \mid \alpha \in \Delta \} \) be a cover of \( G \) by sets open in \( X \). Then \( \mathcal{U}_A = \{U_\alpha \mid \alpha \in \Delta \} \cup \{U_\beta \mid \beta \in \Delta_1 \} \cup I \) is an open cover of \( G \) for each \( n \in \mathbb{N} \). By hypothesis, there exist \( I_\beta \in I \) and locally finite family \( \mathcal{V}_n = \{V_{\beta, n} \mid \beta \in \Delta_1 \} \) which refines \( \mathcal{U}_A \) such that \( \mathcal{V}_n \subseteq \cup \{(V_{\beta, n} \mid \beta \in \Delta_1) \cup I \} \). Then \( \cup \{V_{\beta, n} \mid \beta \in \Delta_1 \} \subseteq \cup \{V_{\beta, n} \mid \beta \in \Delta_1 \} \cup I \) which implies that \( G \subseteq \cup \{W_n \mid n \in \mathbb{N} \} \cup I \) where \( W_n = \cup \{V_{\beta, n} \mid \beta \in \Delta_1 \} \) and \( I = \cup \{I_n \mid n \in \mathbb{N} \} \). Let \( x \in X \). Since \( \mathcal{V}_n = \{V_{\beta, n} \mid \beta \in \Delta_1 \} \) is locally finite, there exists a neighborhood \( U \) containing \( x \) such that \( U \cap V_{\beta, n} \neq \emptyset \) for every \( \beta \in \Delta_0 \) where \( \Delta_0 \) is a finite subset of \( \Delta_1 \). Suppose \( \mathcal{V} = \{W_n \mid n \in \mathbb{N} \} \)
is not locally finite. Then there exists an element $x \in X$ such that for all neighborhood $U$ of $x$, we have $U \cap W_i = \emptyset$ for all $i = 1, 2, ..., k$ which implies that $U \cap (\bigcup V_{\beta,i}) = \emptyset$ for all $i = 1, 2, ..., k$ which in turn implies that $U \cap V_{\beta,i} = \emptyset$ for all $i = 1, 2, ..., k$, which is a contradiction to the fact that $V_n$ is locally finite. Therefore, $V = \{W_n \mid n \in \mathbb{N}\}$ is locally finite. Let $W_n \in V$ where $W_n = \bigcup V_{\beta,n}$. Then $V_{\beta,n} \in V_n$. Since $V_n$ refines $U_n$, there is some $V_{a} \cap G_n \in U_n$ such that $V_{\beta,n} \subset V_{a} \cap G_n$ which implies $V_{\beta,n} \subset V_{a}$. Thus, $\bigcup V_{\beta,n} \subset V_{a}$ and so $W_n \subset V_{a}$ for some $V_{a} \in U$. Therefore, $V$ refines $U$. Hence $G$ is $I$–paracompact.

Conversely, suppose $G$ is $I$–paracompact. Since the subset of a perfectly normal space is perfectly normal, $G$ is perfectly normal. Then each $G_n$ is an $F_\sigma$-set. Therefore, by Theorem 2.4, each $G_n$ is $I$–paracompact. □

References