The maximal rank of a kind of partial banded block matrix subject to linear equations

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Abstract. We establish the formulas of the maximal rank of a 3×3 partial banded block matrix
\[
\begin{pmatrix}
M_{11} & M_{12} & X \\
M_{21} & Y & M_{32} \\
Z & M_{32} & M_{33}
\end{pmatrix}
\]
where X, Y, and Z are three variant quaternion matrices subject to linear matrix equations
\[
A_1X = C_1, \quad XB_1 = C_2, \quad A_2Y = D_1, \quad YB_2 = D_2, \quad A_3Z = E_1, \quad ZB_3 = E_2.
\]
In order to demonstrate the feasibility of the result obtained, we present a necessary and sufficient condition for the solvability to the cubic system
\[
A_1X = C_1, \quad XB_1 = C_2, \quad A_2Y = D_1, \quad YB_2 = D_2, \quad A_3Z = E_1, \quad ZB_3 = E_2, \quad XYZ = J
\]
over the quaternion algebra.

1. Introduction

Throughout this paper, we denote the real number field by \( \mathbb{R} \), the set of all \( m \times n \) matrices over the quaternion algebra
\[
\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = ij = k = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}
\]
by \( \mathbb{H}^{\text{max}} \), the identity matrix with the appropriate dimension by \( I \), the rank of matrix \( A \) by \( r(A) \), and a reflexive inverse of matrix \( A \) over \( \mathbb{H} \) by \( A^+ \) which satisfies simultaneously \( AA^+A = A, A^+AA^+ = A^+ \). Moreover, \( R_A \) and \( L_A \) stand for the two projectors \( R_A = I - AA^+ \), \( L_A = I - A^+A \) induced by \( A \), where \( A^+ \) is any but fixed reflexive inverse of \( A \). Clearly, \( R_A \) and \( L_A \) are idempotent and one of its reflexive inverses is itself.

In matrix theory, the solvability of matrix equations is one of the important topics. In recent years, some authors ([2]-[9]) investigate the extremal ranks of the general solutions of systems subject to consistent
systems and provide applications over the quaternion algebra $\mathbb{H}$. In this paper, we formulate the maximal rank of a kind of $3 \times 3$ partial banded block matrix

$$M(X, Y, Z) = \begin{bmatrix} M_{11} & M_{12} & X \\ M_{21} & Y & M_{23} \\ Z & M_{32} & M_{33} \end{bmatrix},$$

where $M_{ij}[i, j] = (1, 1); (1, 2); (2, 1); (2, 3); (3, 2); (3, 3)] \in \mathbb{H}^{m \times n_i}$ are known, $X \in \mathbb{H}^{m \times n_3}$, $Y \in \mathbb{H}^{m \times n_2}$ and $Z \in \mathbb{H}^{m \times n_1}$ are three independent variant matrices subject to consistent system

$$A_1X = C_1, \; XB_1 = C_2, \; A_2Y = D_1, \; YB_2 = D_2, \; A_3Z = E_1, \; ZB_3 = E_2$$

(1.2)

over $\mathbb{H}$. As an application of the result derived, in Section 3, we present a necessary and sufficient condition for the solvability to the cubic system

$$A_1X = C_1, \; XB_1 = C_2, \; A_2Y = D_1, \; YB_2 = D_2, \; A_3Z = E_1, \; ZB_3 = E_2, \; XYZ = J$$

(1.3)

over $\mathbb{H}$ by the rank equalities. It demonstrates that the simple non-linear matrix equations (1.3) are solvable.

2. Maximal rank of (1.1) subject to system (1.2)

By Lemma 2.1 in [5], we can obtain the following lemma.

**Lemma 2.1.** Let $A_1 \in \mathbb{H}^{p \times m_1}, A_2 \in \mathbb{H}^{p \times m_2}, A_3 \in \mathbb{H}^{p \times m_3}, B_1 \in \mathbb{H}^{m \times q_1}, B_2 \in \mathbb{H}^{m \times q_2}, B_3 \in \mathbb{H}^{m \times q_3}, C_1 \in \mathbb{H}^{m_1 \times r_1}, C_2 \in \mathbb{H}^{m_2 \times r_2}, D_1 \in \mathbb{H}^{m_3 \times r_3}, D_2 \in \mathbb{H}^{m_3 \times r_3}, E_1 \in \mathbb{H}^{m_3 \times r_3}, E_2 \in \mathbb{H}^{m_3 \times r_3}$ be known, and $X \in \mathbb{H}^{m_1 \times n_3}, Y \in \mathbb{H}^{m_2 \times n_2}, Z \in \mathbb{H}^{m_3 \times n_1}$ be unknown. Then the following statements are equivalent: (1) The system (1.2) is consistent. (2)

$$R_{A_1}C_1 = 0, \; C_2L_{B_1} = 0, \; A_1C_2 = C_1B_1.$$  

$$R_{A_2}D_1 = 0, \; D_2L_{B_2} = 0, \; A_2D_2 = D_1B_2.$$  

$$R_{A_3}E_1 = 0, \; E_2L_{B_3} = 0, \; A_3E_2 = E_1B_3.$$  

(3)

$$A_1C_2 = C_1B_1, \; r[A_1, C_1] = r(A_1), \; r \begin{bmatrix} B_1 \\ C_2 \end{bmatrix} = r(B_1).$$  

$$A_2D_2 = D_1B_2, \; r[A_2, D_1] = r(A_2), \; r \begin{bmatrix} B_2 \\ D_2 \end{bmatrix} = r(B_2).$$  

$$A_3E_2 = E_1B_3, \; r[A_3, E_1] = r(A_3), \; r \begin{bmatrix} B_3 \\ E_2 \end{bmatrix} = r(B_3).$$

In that case, the general solution of (1.2) can be expressed as

$$X = A_1^*C_1 + L_{A_1}C_2B_1^* + L_{A_1}UR_{B_1},$$

(2.4)

$$Y = A_2^*D_1 + L_{A_2}D_2B_2^* + L_{A_2}VR_{B_2},$$

(2.5)

$$Z = A_3^*E_1 + L_{A_3}E_2B_3^* + L_{A_3}WR_{B_3},$$

(2.6)

where $U, V$, and $W$ are arbitrary matrices over $\mathbb{H}$ with compatible dimensions.
Lemma 2.2. (Lemma 2.4 in [6]) Let $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{p \times k}$, $C \in \mathbb{H}^{k \times q}$, $D \in \mathbb{H}^{k \times r}$, $E \in \mathbb{H}^{r \times j}$, $U \in \mathbb{H}^{r \times q}$, and $V \in \mathbb{H}^{q \times j}$.

Then the following rank equalities are true:

(a) \[
\text{rank} (CL_A) = r \begin{bmatrix} A \\ C \end{bmatrix} - r(A), \quad \text{rank} (RB_A) = r \begin{bmatrix} B \\ A \end{bmatrix} - r(B),
\]

(b) \[
\text{rank} (B, AL_C) = r \begin{bmatrix} B & A \\ 0 & C \end{bmatrix} - r(C), \quad \text{rank} (RC_B) = r \begin{bmatrix} C & 0 \\ A & B \end{bmatrix} - r(B),
\]

(c) \[
\text{rank} (RB_{AL_C}) = r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C),
\]
\[
\text{rank} \begin{bmatrix} A & B \\ R_C & BL_D \end{bmatrix} = r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(D) - r(E).
\]

Lemma 2.2 below plays an important role in simplifying ranks of various kinds of block matrices. Tian in [10] has given the following Lemma over a field. The result can be generalized to $\mathbb{H}$.

Lemma 2.3. Let $f(X_1, X_2, X_3) = A + B_1 X_1 C_1 + B_2 X_2 C_2 + B_3 X_3 C_3$ be a matrix expression over $\mathbb{H}$. Then the maximal rank of $f(X_1, X_2, X_3)$ can be shown as the following:

\[
\max_{X_1, X_2, X_3} \text{rank} (f(X_1, X_2, X_3)) = \min \left\{ \text{rank} \begin{bmatrix} A & B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}, \text{rank} \begin{bmatrix} A & B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}, \text{rank} \begin{bmatrix} A & B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}, \text{rank} \begin{bmatrix} A & B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \right\}.
\]

Now we consider the maximal rank of $M(X, Y, Z)$ subject to the consistent systems (1.2) and (1.3). For convenience of representation, the following notations are adopted:

\[
\begin{align*}
J_1 &= \{ X \in \mathbb{H}^{m \times n} | A_1 X = C_1, \quad XB_1 = C_2 \}, \\
J_2 &= \{ Y \in \mathbb{H}^{p \times k} | A_2 Y = D_1, \quad YB_2 = D_2 \}, \\
J_3 &= \{ Z \in \mathbb{H}^{m \times q} | A_3 Z = E_1, \quad ZB_3 = E_2 \}.
\end{align*}
\]

Theorem 2.4. The maximal rank of the matrix expression (1.1) subject to the consistent system (1.2) is given in the following:

\[
\max_{(X \in J_1, Y \in J_2, Z \in J_3)} \text{rank} (M(X, Y, Z)) = \min \{ s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 \},
\]

where

\[
s_1 = r \begin{bmatrix} A_1 M_{11} & A_1 M_{12} & C_1 \\ A_2 M_{21} & D_1 & A_2 M_{23} \\ E_1 & A_3 M_{32} & A_3 M_{33} \end{bmatrix} - r(A_1) - r(A_2) - r(A_3) + m_1 + m_2 + m_3,
\]
\[
s_2 = r \begin{bmatrix} M_{11} B_3 & M_{12} B_2 & C_2 \\ M_{21} B_3 & D_2 & M_{23} B_1 \\ E_2 & M_{32} B_2 & M_{33} B_1 \end{bmatrix} - r(B_1) - r(B_2) - r(B_3) + n_1 + n_2 + n_3,
\]
$s_3 = r \begin{bmatrix} M_{11} & M_{12} & C_2 \\ A_{2M_{21}} & D_1 & A_{2M_{23}}B_1 \\ E_1 & A_{3M_{32}} & A_{3M_{33}}B_1 \end{bmatrix} - r(B_1) - r(A_2) - r(A_3) + m_2 + m_3 + n_3,$

$s_4 = r \begin{bmatrix} A_1M_{11} & A_1M_{12}B_2 & C_1 \\ M_{21} & D_2 & M_{23} \\ E_1 & A_{3M_{32}}B_2 & A_{3M_{33}} \end{bmatrix} - r(A_1) - r(B_2) - r(A_3) + m_1 + n_2 + m_3,$

$s_5 = r \begin{bmatrix} A_1M_{11}B_3 & A_1M_{12} & C_1 \\ A_{2M_{21}}B_3 & D_1 & A_{2M_{23}}B_1 \\ E_2 & M_{32} & M_{33} \end{bmatrix} - r(A_1) - r(A_2) - r(B_3) + m_1 + m_2 + n_1,$

$s_6 = r \begin{bmatrix} A_1M_{11}B_3 & A_1M_{12}B_2 & C_1 \\ M_{21}B_3 & D_2 & M_{23}B_1 \\ E_2 & M_{32}B_2 & M_{33} \end{bmatrix} - r(A_1) - r(B_2) - r(B_3) + m_1 + n_1 + n_2,$

$s_7 = r \begin{bmatrix} M_{11}B_3 & M_{12} & C_2 \\ A_{2M_{21}}B_3 & D_1 & A_{3M_{32}}B_1 \\ E_2 & M_{32}B_2 & M_{33}B_1 \end{bmatrix} - r(B_1) - r(A_2) - r(B_3) + m_2 + m_1 + n_3,$

and $s_8 = r \begin{bmatrix} M_{11} & M_{12}B_2 & C_2 \\ M_{21} & D_2 & M_{23}B_1 \\ E_1 & A_{3M_{32}}B_2 & A_{3M_{33}}B_1 \end{bmatrix} - r(B_1) - r(B_2) - r(A_3) + m_3 + n_2 + n_3.$

Proof. Substituting (2.4), (2.5) and (2.6) into (1.1)

\[
M(X, Y, Z) = \begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{21} & 0 & M_{23} \\ 0 & M_{32} & M_{33} \end{bmatrix} + \begin{bmatrix} l & 0 & l \\ 0 & 0 & 0 \\ 0 & l & 0 \end{bmatrix} X + \begin{bmatrix} 0 & l & 0 \\ l & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Y + \begin{bmatrix} 0 & 0 & l \\ 0 & 0 & 0 \\ l & 0 & 0 \end{bmatrix} Z
\]


\[
= \begin{bmatrix} M_{11} & M_{12} & A_1^*C_1 + L_{A_1}C_2B_1^* \\ M_{21} & A_2^*D_1 + L_{A_2}D_2B_2^* & M_{23} \\ A_{3}^*E_1 + L_{A_3}E_2B_3^* & M_{32} & M_{33} \end{bmatrix} + \begin{bmatrix} L_{A_1} & 0 & 0 \\ 0 & L_{A_2} & 0 \\ 0 & 0 & L_{A_3} \end{bmatrix} U + \begin{bmatrix} 0 & 0 & R_{B_1} \\ 0 & R_{B_2} & 0 \\ R_{B_3} & 0 & 0 \end{bmatrix} W.
\]

Let

\[
\begin{bmatrix} M_{11} & M_{12} & A_1^*C_1 + L_{A_1}C_2B_1^* \\ M_{21} & A_2^*D_1 + L_{A_2}D_2B_2^* & M_{23} \\ A_{3}^*E_1 + L_{A_3}E_2B_3^* & M_{32} & M_{33} \end{bmatrix} = \tilde{A},
\]

\[
\begin{bmatrix} L_{A_1} \\ 0 \\ 0 \end{bmatrix} = \tilde{B}_1, \quad \begin{bmatrix} 0 \\ L_{A_2} \\ 0 \end{bmatrix} = \tilde{B}_2, \quad \begin{bmatrix} 0 \\ 0 \\ L_{A_3} \end{bmatrix} = \tilde{B}_3,
\]

\[
\begin{bmatrix} 0 & 0 & R_{B_1} \\ 0 & R_{B_2} & 0 \\ R_{B_3} & 0 & 0 \end{bmatrix} = \tilde{C}_1, \quad \begin{bmatrix} 0 \\ 0 & 0 \\ 0 \end{bmatrix} = \tilde{C}_2, \quad \begin{bmatrix} 0 \\ 0 & 0 \\ 0 \end{bmatrix} = \tilde{C}_3.
\]

Then

\[
\max_{(X \in J_1, Y \in J_2, Z \in J_3)} r[M(X, Y, Z)] = \max_{U, V, W} [\tilde{A} + \tilde{B}_1U\tilde{C}_1 + \tilde{B}_2V\tilde{C}_2 + \tilde{B}_3W\tilde{C}_3]
\]
\[
= \min \left\{ \begin{bmatrix} r \left[ \begin{array}{c} \tilde{A} \\ \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \end{array} \right], \begin{bmatrix} A \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} \right], r \begin{bmatrix} \begin{array}{c} \tilde{A} \\ \tilde{B}_2 \\ \tilde{B}_3 \end{array} \end{bmatrix}, \begin{bmatrix} A \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} \right], r \begin{bmatrix} \begin{array}{c} \tilde{A} \\ \tilde{B}_1 \\ \tilde{B}_3 \end{array} \end{bmatrix}, \begin{bmatrix} A \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} \right) \right\},
\]

where

\[
\begin{bmatrix} M_{11} & M_{12} & A_1^T C_1 + L_{A_1} C_2 B_1^T & 0 \\
M_{21} & M_{22} & A_2^T D_1 + L_{A_2} D_2 B_2^T & M_{23} \\
0 & 0 & 0 & R_{B_1} \\
0 & 0 & R_{B_1} & 0 \\
\end{bmatrix} = r \begin{bmatrix} M_{11} & M_{12} & A_1^T C_1 + L_{A_1} C_2 B_1^T & 0 \\
M_{21} & M_{22} & A_2^T D_1 + L_{A_2} D_2 B_2^T & M_{23} \\
0 & 0 & 0 & R_{B_1} \\
0 & 0 & R_{B_1} & 0 \\
\end{bmatrix},
\]

\[
\begin{bmatrix} M_{11} & M_{12} & A_1^T C_1 + L_{A_1} C_2 B_1^T & 0 \\
M_{21} & M_{22} & A_2^T D_1 + L_{A_2} D_2 B_2^T & M_{23} \\
0 & 0 & 0 & R_{B_1} \\
0 & 0 & R_{B_1} & 0 \\
\end{bmatrix} = r \begin{bmatrix} M_{11} & M_{12} & A_1^T C_1 + L_{A_1} C_2 B_1^T & 0 \\
M_{21} & M_{22} & A_2^T D_1 + L_{A_2} D_2 B_2^T & M_{23} \\
0 & 0 & 0 & R_{B_1} \\
0 & 0 & R_{B_1} & 0 \\
\end{bmatrix} - r(B_1) - r(B_2) - r(A_3).
\]

Note that

\[
\begin{align*}
(A_1^T C_1 + L_{A_1} C_2 B_1^T) B_1 &= C_2, \\
(A_2^T D_1 + L_{A_2} D_2 B_2^T) B_2 &= C_2, \\
A_3 (A_2^T E_1 + L_{A_2} E_2 B_2^T) &= E_1,
\end{align*}
\]

so

\[
r \begin{bmatrix} \tilde{A} \\ \tilde{B}_3 \end{bmatrix}, \begin{bmatrix} \tilde{A} \\ \tilde{C}_1 \\ \tilde{C}_2 \\ \tilde{C}_3 \end{bmatrix} = s_8.
\]

Similarly,

\[
r \begin{bmatrix} \tilde{A} \\ \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \end{bmatrix} = s_1, r \begin{bmatrix} \tilde{A} \\ \tilde{C}_1 \\ \tilde{C}_2 \\ \tilde{C}_3 \end{bmatrix} = s_2, r \begin{bmatrix} \tilde{A} \\ \tilde{B}_2 \\ \tilde{B}_3 \end{bmatrix}, \begin{bmatrix} \tilde{A} \\ \tilde{C}_1 \\ \tilde{C}_2 \\ \tilde{C}_3 \end{bmatrix} = s_3, r \begin{bmatrix} \tilde{A} \\ \tilde{B}_1 \\ \tilde{B}_3 \end{bmatrix}, \begin{bmatrix} \tilde{A} \\ \tilde{C}_1 \\ \tilde{C}_2 \\ \tilde{C}_3 \end{bmatrix} = s_4.
\]
is true, the cubic matrix equations (1.3) are consistent too, and the solution set of (1.3) is equal to that of (1.2).

This completes the proof of Theorem 2.4. □

3. Some solvability conditions of system (1.3)

In order to demonstrate the power of Theorem 2.4, in this section, we apply it to the solvability conditions to cubic matrix equations (1.3).

**Theorem 3.1.** Let the linear matrix equations (1.2) be consistent and the solution set of (1.2) be (2.7). Then only one of following conclusions

\[
\begin{align*}
&\begin{cases}
 r \begin{bmatrix} A_1 & 0 & C_1 \\ 0 & D_1 & -A_2 \\ E_1 & A_3 & 0 \end{bmatrix} = r[A_2] + r[A_3], \\
r[A_1] = m_1 
\end{cases}, \\
&\begin{cases}
 r \begin{bmatrix} JB_3 & 0 & C_2 \\ 0 & D_2 & -B_1 \\ E_2 & B_2 & 0 \end{bmatrix} = r[B_1] + r[B_2], \\
r[B_3] = n_1 
\end{cases}, \\
&\begin{cases}
 r \begin{bmatrix} 0 & 0 & C_2 \\ J & 0 & -A_2B_1 \\ E_1 & A_3 & 0 \end{bmatrix} = r[A_2] + r[A_3], \\
r[B_1] = n_3 
\end{cases}, \\
&\begin{cases}
 r \begin{bmatrix} A_1 & C_1D_2 \\ E_1 & A_3B_2 \end{bmatrix} = r[B_2], \\
r[A_1] = m_1 \\
r[A_3] = m_3 
\end{cases}, \\
&\begin{cases}
 r \begin{bmatrix} A_1B_2 & C_1 \\ D_1E_2 & A_2 \end{bmatrix} = r[A_2], \\
r[A_1] = m_1 \\
r[B_3] = n_1 
\end{cases}, \\
&\begin{cases}
 r \begin{bmatrix} A_1B_3 & C_1D_2 \\ E_2 & B_2 \end{bmatrix} = r[B_2], \\
r[A_1] = m_1 \\
r[B_3] = n_1 
\end{cases}, \\
&\begin{cases}
 r \begin{bmatrix} JB_3 & C_2 \\ D_1E_2 & A_2B_1 \end{bmatrix} = r[B_1], \\
r[A_2] = m_2 \\
r[B_3] = n_1 
\end{cases}, \\
\text{or} \quad &\begin{cases}
 r \begin{bmatrix} J & 0 & C_2 \\ 0 & D_2 & -B_1 \\ E_1 & A_3B_2 & 0 \end{bmatrix} = r[B_1] + r[B_2], \\
r[A_3] = m_3 
\end{cases}
\end{align*}
\]

is true, the cubic matrix equations (1.3) are consistent too, and the solution set of (1.3) is equal to that of (1.2).
Proof. Use the following rank formula

\[
\begin{align*}
    r(J - XYZ) &= r \begin{bmatrix}
        J & 0 & X \\
        0 & Y & -I_{m_1} \\
        Z & I_{m_3} & 0
    \end{bmatrix} - m_3 - n_3.
\end{align*}
\]

Let \( M_{11} \) in Theorem 2.4 be replaced by \( J, M_{23} \) by \(-I_{m_1}, M_{12}, M_{21}, M_{33} \) by zero matrices with dimensions, \( m_2 = n_3, n_2 = m_3 \) respectively. According to Theorem 2.4, we can obtain the following equality:

\[
\max_{(x \in J, \ y \in J, \ z \in J)} \ r(J - XYZ) = \min \begin{cases}
    r \begin{bmatrix}
        A_1 & 0 & C_1 \\
        0 & D_1 & -A_2 \\
        E_1 & A_3 & 0
    \end{bmatrix} - r[A_1] - r[A_2] - r[A_3] + m_1, \\
    r \begin{bmatrix}
        J & B_3 & 0 & C_2 \\
        0 & D_2 & -B_1 \\
        E_2 & B_2 & 0
    \end{bmatrix} - r[B_1] - r[B_2] - r[B_3] + n_1, \\
    r \begin{bmatrix}
        J & 0 & C_2 \\
        0 & D_1 & -A_2 B_1 \\
        E_1 & A_3 & 0
    \end{bmatrix} - r[B_1] - r[A_2] - r[A_3] + m_2, \\
    r \begin{bmatrix}
        A_1 & B_3 & C_1 \\
        E_1 & A_2 B_2 & 0
    \end{bmatrix} - r[A_1] - r[A_2] - r[A_3] + m_1 + m_3, \\
    r \begin{bmatrix}
        A_1 & B_3 & C_1 D_2 \\
        E_1 & A_3 B_2 & 0
    \end{bmatrix} - r[A_1] - r[A_2] - r[A_3] + m_1 + m_3, \\
    r \begin{bmatrix}
        A_1 & J & B_3 & C_1 D_2 \\
        E_1 & A_2 B_2 & 0
    \end{bmatrix} - r[A_1] - r[A_2] - r[A_3] + m_1 + m_3
\end{cases}.
\]

It is obvious that

\[
\max_{(x \in J, \ y \in J, \ z \in J)} \ r(J - XYZ) \geq 0.
\]

If one of the conditions from (3.8) to (3.15) is satisfied, then

\[
\max_{(x \in J, \ y \in J, \ z \in J)} \ r(J - XYZ) = 0.
\]

Conversely, assume that

\[
\begin{align*}
    r \begin{bmatrix}
        A_1 & 0 & C_1 \\
        0 & D_1 & -A_2 \\
        E_1 & A_3 & 0
    \end{bmatrix} - r[A_1] - r[A_2] - r[A_3] + m_1 &= 0.
\end{align*}
\]

Note that

\[
\begin{align*}
    r \begin{bmatrix}
        A_1 & 0 & C_1 \\
        0 & D_1 & -A_2 \\
        E_1 & A_3 & 0
    \end{bmatrix} &\geq r[A_2] + r[A_3]
\end{align*}
\]

and

\[
\begin{align*}
    r[A_1] &\leq m_1.
\end{align*}
\]
Therefore, the condition (3.8) is satisfied. Similarly, assume that one of conditions from (3.9) to (3.15) is satisfied, then
\[ \max_{(X \in \mathbb{J}, Y \in \mathbb{J}, Z \in \mathbb{J})} r \left(J - XYZ\right) = 0. \]
This corresponds to
\[ \text{XYZ} \equiv J. \]
It demonstrates that the cubic matrix equations (1.3) are consistent, and the solution set of (1.3) covers that of (2.7). On the other hand, the cubic matrix equations (1.3) contain the linear matrix equations (1.2), and the solution set of (1.3) covers that of (2.7). Therefore, the solution set of (1.3) is equal to that of (2.7).

4. Conclusion

In this paper, we derive the maximal rank of a kind of 3×3 partial banded block variant matrix (1.1) subject to the consistent system (1.2). Moreover, in order to demonstrate the feasibility of the result obtained, Theorem 2.4 is applied to derive the necessary and sufficient conditions for the solvability to the cubic system (1.3) by the rank equalities. It means the non-linear matrix equations (1.3) are solvable.

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