On new inequalities for $h$-convex functions via
Riemann-Liouville fractional integration

Mevlüt TUNÇ

University of Kilis 7 Aralık, Faculty of Science and Arts, Department of Mathematics, 79000, Kilis, Turkey

Abstract. In this paper, some new inequalities of the Hermite-Hadamard type for $h$-convex functions via Riemann-Liouville fractional integral are given.

1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and let $a, b \in I$, with $a < b$. The following inequality;

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}$$

is known in the literature as Hadamard’s inequality. Both inequalities hold in the reversed direction if $f$ is concave.

In [16], Varošanec introduced the following class of functions.

Definition 1.1. Let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a positive function. We say that $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is $h$-convex function or that $f$ belongs to the class $SX(h, I)$, if $f$ is nonnegative and for all $x, y \in I$ and $\lambda \in (0, 1)$ we have

$$f(\lambda x + (1 - \lambda)y) \leq h(\lambda)f(x) + h(1 - \lambda)f(y).$$

If the inequality in (2) is reversed, then $f$ is said to be $h$-concave, i.e., $f \in SV(h, I)$.

Obviously, if $h(\lambda) = \lambda$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(\lambda) = \frac{1}{\lambda}$, then $SX(h, I) = Q(I)$; if $h(\lambda) = 1$, then $SX(h, I) \supseteq P(I)$ and if $h(\lambda) = \lambda^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K_s^2$. For some recent results for $h$-convex functions we refer to the interested reader to the papers [3], [4], [12], [13].

Definition 1.2. ([1]) A function $h : J \rightarrow \mathbb{R}$ is said to be a superadditive function if

$$h(x + y) \geq h(x) + h(y)$$

for all $x, y \in J$.
In [12], Sarıkaya et al. proved the following Hadamard type inequalities for \( h \)-convex functions.

**Theorem 1.3.** Let \( f \in SX(h, I), a, b \in I \) with \( a < b \) and \( f \in L_1[a, b] \). Then

\[
\frac{1}{2h\left(\frac{1}{2}\right)} \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq [f(a) + f(b)] \int_0^1 h(x)dx.
\]

(4)

In [14], Sarıkaya et al. proved the following Hadamard type inequalities for fractional integrals as follows.

**Theorem 1.4.** Let \( f : [a, b] \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in L_1[a, b] \). If \( f \) is convex function on \([a, b]\), then the following inequalities for fractional integrals hold:

\[
f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(a+1)}{2(b-a)^{a}} \left[ J_a^b f'(b) + J_b^a f'(a) \right] \leq \frac{f(a) + f(b)}{2}
\]

with \( \alpha > 0 \).

Now we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

**Definition 1.5.** Let \( f \in L_1[a, b] \). The Riemann-Liouville integrals \( J_a^b f \) and \( J_b^a f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J_a^b f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a
\]

and

\[
J_b^a f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b
\]

respectively where \( \Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du \). Here is \( J_a^0 f(x) = J_b^0 f(x) = f(x) \).

In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities see [2, 5–8, 10, 14, 15].

In [14], Sarıkaya et al. proved a variant of the identity that established by Dragomir and Agarwal in [9, Lemma 2.1] for fractional integrals as the following.

**Lemma 1.6.** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \( a < b \). If \( f' \in L[a, b] \), then the following equality for fractional integrals holds:

\[
\frac{f(a) + f(b)}{2} = \frac{\Gamma(a+1)}{2(b-a)^{a}} \left[ J_a^b f'(b) + J_b^a f'(a) \right] = \frac{b-a}{2} \int_0^1 [(1-t)^{a} - t^a] f'(ta + (1-t)b)dt.
\]

The aim of this paper is to establish Hadamard type inequalities for \( h \)-convex functions via Riemann-Liouville fractional integral.
2. Main results

Theorem 2.1. Let \( f \in SX(h, l) \), \( a, b \in I \) with \( a < b \) and \( f \in L_1[a, b] \). Then one has inequality for \( h \)-convex functions via fractional integrals

\[
\frac{\Gamma(a)}{(b - a)^a} \left[ p_v^a(b) + f_v^a(a) \right] \leq [f(a) + f(b)] \int_0^1 t^{a-1} [h(t) + h(1-t)] dt \leq \frac{2[f(a) + f(b)]}{(ap + p + 1)^{\frac{1}{q}}} \left( \int_0^1 (h(t))^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \tag{6}
\]

where \( p^{-1} + q^{-1} = 1 \).

Proof. Since \( f \in SX(h, l) \), we have

\[
f(tx + (1 - t)y) \leq h(t) f(x) + h(1 - t) f(y)
\]

and

\[
f((1 - t)x + ty) \leq h(1 - t) f(x) + h(t) f(y).
\]

By adding these inequalities we get

\[
f(tx + (1 - t)y) + f((1 - t)x + ty) \leq [h(t) + h(1 - t)] [f(x) + f(y)]. \tag{7}
\]

By using (7) with \( x = a \) and \( y = b \) we have

\[
f(ta + (1 - t)b) + f((1 - t)a + tb) \leq [h(t) + h(1 - t)] [f(a) + f(b)]. \tag{8}
\]

Then multiplying both sides of (8) by \( t^{a-1} \) and integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we get

\[
\int_0^1 t^{a-1} [f(ta + (1 - t)b) + f((1 - t)a + tb)] dt \leq \int_0^1 t^{a-1} [h(t) + h(1 - t)] [f(a) + f(b)] dt, \tag{9}
\]

and

\[
\frac{\Gamma(a)}{(b - a)^a} \left[ p_v^a(b) + f_v^a(a) \right] \leq [f(a) + f(b)] \int_0^1 t^{a-1} [h(t) + h(1-t)] dt \tag{10}
\]

and thus the first inequality is proved.

To obtain the second inequality in (6), by using Holder inequality for the right hand side of (10), we obtain

\[
\int_0^1 t^{a-1} [h(t) + h(1 - t)] dt
\]

\[
\leq \left( \int_0^1 (t^{a-1})^p dt \right)^{\frac{1}{p}} \left( \int_0^1 (h(t) + h(1-t))^{\frac{q}{p}} dt \right)^{\frac{1}{q}}
\]

\[
= \left( \frac{1}{ap - p + 1} \right)^{\frac{1}{p}} \left( \int_0^1 (h(t) + h(1-t))^{\frac{q}{p}} dt \right)^{\frac{1}{q}}
\]

\[
= \left( \frac{1}{ap - p + 1} \right)^{\frac{1}{p}} \left( \int_0^1 (h(t) + h(1-t))^{\frac{q}{p}} dt \right)^{\frac{1}{q}}
\]
Then using Minkowski inequality
\[
\left(\frac{1}{ap^{p+1}}\right)^{\frac{1}{q}} \left(\int_0^1 (h(t) + h(1-t))^q \, dt\right)^{\frac{1}{q}}
\]
\[
\leq \frac{1}{ap^{p+1}} \left(\left(\int_0^1 (h(t))^q \, dt\right)^{\frac{1}{q}} + \left(\int_0^1 (h(1-t))^q \, dt\right)^{\frac{1}{q}}\right)
\]
\[
= \frac{2}{ap^{p+1}} \left(\int_0^1 (h(t))^q \, dt\right)^{\frac{1}{q}}
\]
where the proof is completed. 

Remark 2.2. If we choose \( \alpha = 1 \) in Theorem 1, we obtain
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \left[ f(a) + f(b) \right] \int_0^1 h(t) \, dt \leq \left[ f(a) + f(b) \right] \left(\int_0^1 (h(t))^q \, dt\right)^{\frac{1}{q}}.
\]

Corollary 2.3. (1) If we choose \( h(\lambda) = \lambda \) in Remark 2.2, we get
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2} \leq \frac{f(a) + f(b)}{(q + 1)^{\frac{1}{q}}}
\]
for ordinary convex functions.

(2) If we choose \( h(\lambda) = 1 \) in Remark 2.2, we get
\[
\frac{2}{b-a} \int_a^b f(x) \, dx \leq 2 \left( f(a) + f(b) \right)
\]
for \( P \)-functions. This inequality is a refinement of right hand side of (1) for \( P \)-functions.

(3) If we choose \( h(\lambda) = \lambda^s \) in Remark 2.2, we get
\[
\frac{1}{b-a} \int_0^1 f(x) \, dx \leq \frac{f(a) + f(b)}{s + 1} \leq \frac{f(a) + f(b)}{(sq + 1)^{\frac{1}{q}}}
\]
for \( s \)-convex functions in the second sense with \( s \in (0, 1] \).

Theorem 2.4. Let \( f \in SX(h, I) \), \( a, b \in I \) with \( a < b \), \( h \) be superadditive on \( I \) and \( f \in L_1[a, b], \ h \in L_1[0, 1] \). Then one has inequality for \( h \)-convex functions via fractional integrals
\[
\frac{\Gamma(a)}{(b-a)^a} \left[ p_a(b) + p_a(a) \right] \leq \frac{h(1)}{a} \left[ f(a) + f(b) \right].
\]
(11)

Proof. Since \( f \in SX(h, I) \) and \( h \) is superadditive, by using (8), we have
\[
f(ta + (1-t)b) + f((1-t)a + tb) \leq h(t) + h(1-t) \left[ f(a) + f(b) \right] \leq h(1) \left[ f(a) + f(b) \right].
\]
(12)

Then multiplying both sides of (12) by \( t^{a-1} \) and integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we get
\[
\int_0^1 t^{a-1} \left[ f(ta + (1-t)b) + f((1-t)a + tb) \right] dt \leq \int_0^1 t^{a-1} h(1) \left[ f(a) + f(b) \right] dt,
\]
and
\[
\frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[ f_+^a (b) + f_-^a (a) \right] \leq h(1) \left[ f(a) + f(b) \right] \int_0^1 t^{p-1} dt.
\]

This completes the proof. \( \square \)

**Remark 2.5.** If we choose \( \alpha = 1 \) in Theorem 2.4, then (11) reduce to special version of right hand side of (4).

**Theorem 2.6.** Let \( h : \mathbb{R} \to \mathbb{R} \) and \( f : [a, b] \to \mathbb{R} \) be positive functions with \( 0 \leq a < b \) and \( h \) \( \in \) \( L_1 \) \( [0, 1] \), \( f \in L_1 \) \( [a, b] \). If \( \left| f' \right| \) is an \( h \)-convex mapping on \([a, b] \), then the following inequality for fractional integrals holds,

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \right[ f_+^a (b) + f_-^a (a) \left| \right. \right] \leq \frac{b-a}{2} \int_0^1 \left[ (1-t)^\alpha - t^\alpha \right] \left| f'(ta + (1-t)b) \right| dt.
\]

where \( \alpha > 0 \), \( p > 1 \) and \( p^{-1} + q^{-1} = 1 \).

**Proof.** From Lemma 1.6 and using the properties of modulus, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \right[ f_+^a (b) + f_-^a (a) \left| \right. \right] \leq \frac{b-a}{2} \int_0^1 \left[ (1-t)^\alpha - t^\alpha \right] \left| f'(ta + (1-t)b) \right| dt.
\]

Since \( \left| f' \right| \) is \( h \)-convex on \([a, b] \), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \right[ f_+^a (b) + f_-^a (a) \left| \right. \right] \leq \frac{b-a}{2} \left\{ \int_0^{\frac{1}{2}} \left[ (1-t)^\alpha - t^\alpha \right] \left[ h(t) \right| f'(a) \right] dt + \int_0^{\frac{1}{2}} \left[ (1-t)^\alpha - t^\alpha \right] \left[ h(t) \right| f'(b) \right] dt \right\}
\]

\[
= \frac{b-a}{2} \left\{ \left[ f'(a) \right] \int_0^{\frac{1}{2}} (1-t)^\alpha h(t) dt - \left[ f'(a) \right] \int_0^{\frac{1}{2}} t^\alpha h(t) dt \right\}
\]

\[
+ \left[ f'(b) \right] \int_0^{\frac{1}{2}} (1-t)^\alpha h(t) dt - \left[ f'(b) \right] \int_0^{\frac{1}{2}} t^\alpha h(t) dt \right\}
\]

\[
+ \left[ f'(a) \right] \int_0^{\frac{1}{2}} t^\alpha h(t) dt - \left[ f'(a) \right] \int_0^{\frac{1}{2}} (1-t)^\alpha h(t) dt \right\}
\]

\[
+ \left[ f'(b) \right] \int_0^{\frac{1}{2}} (1-t)^\alpha h(t) dt - \left[ f'(b) \right] \int_0^{\frac{1}{2}} (1-t)^\alpha h(t) dt \right\}
\]

In the right hand side of above inequality by using Hölder inequality for \( p^{-1} + q^{-1} = 1 \) and \( p > 1 \), we get

\[
\int_0^{\frac{1}{2}} (1-t)^\alpha h(t) dt = \int_0^{\frac{1}{2}} t^\alpha h(1-t) dt \leq \left[ \frac{2^{ap+1} - 1}{2^{ap+1}(ap + 1)} \right] ^\frac{1}{p} \left( \int_0^{\frac{1}{2}} \left[ h(t) \right]^p dt \right) ^\frac{1}{p},
\]
\[
\int_0^1 (1-t)^{a} h(t) dt = \int_0^1 t^a h(t) dt \leq \left[ \frac{2^{ap+1} - 1}{2^{ap+1} (ap+1)} \right]^{\frac{1}{p}} \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}},
\]
\[
\int_0^1 t^a h(t) dt = \int_0^1 (1-t)^a h(t) dt \leq \left[ \frac{1}{2^{ap+1} (ap+1)} \right]^{\frac{1}{p}} \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}}.
\]

and

\[
\int_0^1 t^a h(t) dt = \int_0^1 (1-t)^a h(t) dt \leq \left[ \frac{1}{2^{ap+1} (ap+1)} \right]^{\frac{1}{p}} \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}}.
\]

Then using the above inequalities in the right hand side of (14), we get
\[
\frac{1}{2} \left[ f(a) + f(b) \right] - \frac{\Gamma(a+1)}{2(b-a)^2} \left[ f_{a+}^n (b) + f_{b-}^n (a) \right] \leq
\]
\[
\frac{b-a}{2} \left\{ f'(a) \left[ \left( \frac{2^{ap+1} - 1}{2^{ap+1} (ap+1)} \right)^{\frac{1}{p}} - \left( \frac{1}{2^{ap+1} (ap+1)} \right)^{\frac{1}{p}} \right] \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}} \right\}
\]
\[
+ \left\{ \left( \frac{2^{ap+1} - 1}{2^{ap+1} (ap+1)} \right)^{\frac{1}{p}} - \left( \frac{1}{2^{ap+1} (ap+1)} \right)^{\frac{1}{p}} \right\} \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}} \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}}
\]
\[
+ f'(b) \left[ \left( \frac{2^{ap+1} - 1}{2^{ap+1} (ap+1)} \right)^{\frac{1}{p}} - \left( \frac{1}{2^{ap+1} (ap+1)} \right)^{\frac{1}{p}} \right] \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}} \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}}
\]
\[
= \frac{b-a}{2} \left\{ f'(a) \left[ \left( \frac{2^{ap+1} - 1}{2^{ap+1} (ap+1)} \right)^{\frac{1}{p}} - \left( \frac{1}{2^{ap+1} (ap+1)} \right)^{\frac{1}{p}} \right] \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}} \right\}
\]
\[
+ f'(b) \left[ \left( \frac{2^{ap+1} - 1}{2^{ap+1} (ap+1)} \right)^{\frac{1}{p}} - \left( \frac{1}{2^{ap+1} (ap+1)} \right)^{\frac{1}{p}} \right] \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}} \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}}
\]
\[
= \frac{(b-a)}{2} \left[ f'(a) + f'(b) \right] \left[ \left( \frac{2^{ap+1} - 1}{2^{ap+1} (ap+1)} \right)^{\frac{1}{p}} - \left( \frac{1}{2^{ap+1} (ap+1)} \right)^{\frac{1}{p}} \right] \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}} \left( \int_0^1 [h(t)]^p dt \right)^{\frac{1}{p}}
\]
\
\]

which is the desired result. The proof is completed. \(\square\)

References