Fixed and periodic point theorems for $T$-contractions on cone metric spaces

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Abstract. Recently, Filipović et al. [M. Filipović, L. Paunović, S. Radenović, M. Rajović, Remarks on “Cone metric spaces and fixed point theorems of $T$-Kannan and $T$-Chatterjea contractive mappings”, Math. Comput. Modelling. 54 (2011) 1467-1472] proved several fixed and periodic point theorems for solid cones on cone metric spaces. In this paper several fixed and periodic point theorems for $T$-contraction of two maps on cone metric spaces with solid cone are proved. The results of this paper extend and generalize well-known comparable results in the literature.

1. Introduction and preliminaries

In 1922, Banach proved the following famous fixed point theorem [3]. Suppose that $(X,d)$ is a complete metric space and a self-map $T$ of $X$ satisfies $d(Tx,Ty) \leq \lambda d(x,y)$ for all $x,y \in X$ where $\lambda \in [0,1)$; that is, $T$ is a contractive mapping. Then $T$ has a unique fixed point. Afterward, other people considered various definitions of contractive mappings and proved several fixed point theorems [4, 8, 11, 12, 17]. In 2007, Huang and Zhang [9] introduced cone metric space and proved some fixed point theorems. Several fixed and common fixed point results on cone metric spaces were introduced in [1, 15, 16, 18, 19].

Recently, Morales and Rajes [14] introduced $T$-Kannan and $T$-Chatterjea contractive mappings in cone metric spaces and proved some fixed point theorems. Later, Filipović et al. [6] defined $T$-Hardy-Rogers contraction in cone metric space and proved some fixed and periodic point theorems. In this work we prove several fixed and periodic point theorems for a $T$-contraction of two maps on cone metric spaces. Our results extend various comparable results of Abbas and Rhoades [2], Filipović et al. [6] and, Morales and Rajes [14].

We begin with some important definitions.

Definition 1.1. (See [7, 9]). Let $E$ be a real Banach space and $P$ a subset of $E$. Then $P$ is called a cone if and only if

(a) $P$ is closed, non-empty and $P \neq \{0\}$;

(b) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;

(c) if $x \in P$ and $-x \in P$, then $x = \theta$.


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Given a cone $P \subseteq E$, a partial ordering $\preceq$ with respect to $P$ is defined by

$$x \preceq y \iff y - x \in P.$$  

We shall write $x < y$ to mean $x \preceq y$ and $x \not= y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is the interior of $P$). If $intP \not= \emptyset$, the cone $P$ is called solid. A cone $P$ is called normal if there exists a number $K > 0$ such that, for all $x, y \in E$,

$$\theta \preceq x \preceq y \implies \|x\| \leq K\|y\|.$$  

The least positive number satisfying the above inequality is called the normal constant of $P$.

**Example 1.2.** (See [16]).

(i) Let $E = C_R[0,1]$ with the supremum norm and $P = \{f \in E : f \geq 0\}$, then $P$ is a normal cone with normal constant $K = 1$.

(ii) Let $E = C^2_R[0,1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and consider the cone $P = \{f \in E : f \geq 0\}$ for every $K \geq 1$. Then $P$ is a non-normal cone.

**Definition 1.3.** (See [9]). Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

1. $(d1)$ $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
2. $(d2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $(d3)$ $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then, $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.

**Example 1.4.** (See [9]). Let $E = R^2$, $P = \{(x, y) \in E, x, y \geq 0\} \subseteq R^2$, $R = R$ and $d : X \times X \to E$ is such that $d(x, y) = (|x - y|, |x - y|)$, where $\alpha \geq 0$ is a constant. Then $(X, d)$ is a cone metric space.

**Definition 1.5.** (See [6]). Let $(X, d)$ be a cone metric space, $\{x_n\}$ a sequence in $X$ and $x \in X$. Then

(i) $\{x_n\}$ converges to $x$ if, for every $c \in E$ with $\theta \ll c$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$. We denote this by $\lim_{n \to \infty} d(x_n, x) = \theta$.
(ii) $\{x_n\}$ is called a Cauchy sequence if, for every $c \in E$ with $\theta \ll c$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$. We denote this by $\lim_{n, m \to \infty} d(x_n, x_m) = \theta$.

The notation $\theta \ll c$ for $c \in intP$ of a positive cone is used by Krein and Rutman [13]. Also, a cone metric space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$. In the sequel we shall always suppose that $E$ is a real Banach space, $P$ is a solid cone in $E$, and $\preceq$ is a partial ordering with respect to $P$.

**Lemma 1.6.** (See [6]). Let $(X, d)$ be a cone metric space over an ordered real Banach space $E$. Then the following properties are often used, particularly when dealing with cone metric spaces in which the cone need not be normal.

$$(P_1)$$ If $x \preceq y \preceq z$, then $x \ll z$.

$$(P_2)$$ If $\theta \preceq x \preceq c$ for each $c \in intP$, then $x = \theta$.

$$(P_3)$$ If $x \preceq \lambda x$ where $x \in P$ and $0 \leq \lambda < 1$, then $x = \theta$.

$$(P_4)$$ Let $x_n \to \theta$ in $E$ and $\theta \ll c$. Then there exists a positive integer $n_0$ such that $x_n \ll c$ for each $n > n_0$.

**Definition 1.7.** (See [6]). Let $(X, d)$ be a cone metric space, $P$ a solid cone and $S : X \to X$. Then

(i) $S$ is said to be sequentially convergent if we have, for every sequence $\{x_n\}$, if $\{Sx_n\}$ is convergent, then $\{x_n\}$ also is convergent.

(ii) $S$ is said to be subsequentially convergent if, for every sequence $\{x_n\}$ that $\{Sx_n\}$ is convergent, $\{x_n\}$ has a convergent subsequence.

(iii) $S$ is said to be continuous if $\lim_{n \to \infty} x_n = x$ implies that $\lim_{n \to \infty} Sx_n = Sx$, for all $\{x_n\}$ in $X$.

**Definition 1.8.** (See [6]). Let $(X, d)$ be a cone metric space and $T, f : X \to X$ be two mappings. A mapping $f$ is said to be a $T$-Hardy-Rogers contraction, if there exist $\alpha_i \geq 0, i = 1, \ldots, 5$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ such that for all $x, y \in X$,

$$d(Tfx, Tfy) \leq \alpha_1 d(Tx, Ty) + \alpha_2 d(Tx, Tfx) + \alpha_3 d(Ty, Tfy) + \alpha_4 d(Tx, Tfy) + \alpha_5 d(Ty, Tfx).$$  

(1)

In Definition 1.8 if one assumes that $\alpha_1 = \alpha_4 = \alpha_5 = 0$ and $\alpha_2 = \alpha_3 \neq 0$ (resp. $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4 = \alpha_5 \neq 0$), then one obtains a $T$-Kannan (resp. $T$-Chatterjea) contraction. (See [14].)
2. Fixed point results

The following is the cone metric space version of a contractive condition of Ćirić for an ordinary metric space.

**Definition 2.1.** Let $(X, d)$ be a cone metric space. A mapping $f : X \to X$ is said to be a $\lambda$-generalized contraction if and only if for every $x, y \in X$, there exist nonnegative functions $q(x, y), r(x, y), s(x, y)$ and $t(x, y)$ such that

$$\sup_{x,y\in X} [q(x, y) + r(x, y) + s(x, y) + 2t(x, y)] \leq \lambda < 1$$

and

$$d(fx, fy) \leq q(x, y)d(fx, fy) + r(x, y)d(fx, fx) + s(x, y)d(fy, fy) + 2t(x, y)[d(fx, fy) + d(fy, fx)]$$

holds for all $x, y \in X$.

**Theorem 2.2.** Suppose that $(X, d)$ is a complete cone metric space, $P$ is a solid cone, and $T : X \to X$ is a continuous and one to one mapping. Moreover, let $f$ and $g$ be two mappings of $X$ satisfying

$$d(Tfx, Tgy) \leq q(x, y)d(Tx, Ty) + r(x, y)d(Tx, Tf x) + s(x, y)d(Ty, Tgy) + t(x, y)[d(Tx, Tgy) + d(Ty, Tf x)]$$

for all $x, y \in X$, where $q, r, s, t$ are nonnegative functions satisfying

$$\sup_{x,y\in X} [q(x, y) + r(x, y) + s(x, y) + 2t(x, y)] \leq \lambda < 1; \quad (3)$$

that is, $f$ and $g$ are $T$-contractions. Then

1. There exists a $z_r \in X$ such that $\lim_{n \to \infty} Tfx_n = \lim_{n \to \infty} Tgx_{2n+1} = z_r$.
2. If $T$ is subsequentially convergent, then $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ have a convergent subsequence.
3. There exists a unique $w_r \in X$ such that $fw_r = gw_r = w_r$; that is, $f$ and $g$ have a unique common fixed point.
4. If $T$ is sequentially convergent, then the sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to $w_r$.

**Proof.** Suppose that $x_0$ is an arbitrary point of $X$, and define $\{x_n\}$ by

$$x_1 = fx_0, \quad x_2 = gx_1, \quad \ldots, \quad x_{2n+1} = fx_{2n}, \quad x_{2n+2} = gx_{2n+1} \quad \text{for} \quad n = 0, 1, 2, \ldots.$$  

First we shall prove that $\{Tx_n\}$ is a Cauchy sequence. Applying the triangle inequality we get

$$d(Tx_{2n+1}, Tx_{2n+2}) = d(Tfx_{2n}, Tgx_{2n+1}) \leq q(x_{2n}, x_{2n+1})d(Tx_{2n}, Tx_{2n+1}) + r(x_{2n}, x_{2n+1})d(Tx_{2n}, Tfx_{2n}) + s(x_{2n}, x_{2n+1})d(Tx_{2n+1}, Tgx_{2n+1}) + t(x_{2n}, x_{2n+1})d(Tx_{2n+1}, Tfx_{2n+1})$$

Consequently

$$d(Tx_{2n+1}, Tx_{2n+2}) \leq \frac{q(x_{2n}, x_{2n+1}) + r(x_{2n}, x_{2n+1}) + t(x_{2n}, x_{2n+1})}{1 - s(x_{2n}, x_{2n+1}) - t(x_{2n}, x_{2n+1})}d(Tx_{2n}, Tx_{2n+1}). \quad (4)$$

Using (3), we have

$$\frac{q(x, y) + r(x, y) + t(x, y)}{1 - s(x, y) - t(x, y)} \leq \lambda$$
for all \( x, y \in X \). Thus, from (4), it follows that

\[
d(Tx_{2n+1}, Tx_{2n+2}) \leq \lambda d(Tx_{2n}, Tx_{2n+1}),
\]

which shows that a generalized contraction is a contraction for certain pairs of points. Following arguments similar to those given above, we obtain

\[
d(Tx_{2n+3}, Tx_{2n+2}) \leq \lambda d(Tx_{2n+2}, Tx_{2n+1}),
\]

where

\[
\frac{q(x, y) + s(x, y) + t(x, y)}{1 - r(x, y) - t(x, y)} \leq \lambda
\]

for all \( x, y \in X \). Therefore, for all \( n \),

\[
d(Tx_n, Tx_{n+1}) \leq \lambda d(Tx_{n-1}, Tx_n) \leq \lambda^2 d(Tx_{n-2}, Tx_{n-1}) \leq \cdots \leq \lambda^n d(Tx_0, Tx_1).
\]

(5)

Now, for any \( m > n \) and \( \lambda < 1 \),

\[
d(Tx_n, Tx_m) \leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{m-1}, Tx_m)
\]

\[
\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1})d(Tx_0, Tx_1)
\]

\[
\leq \frac{\lambda^n}{1 - \lambda} d(Tx_0, Tx_1) \to \theta \quad \text{as} \quad n \to \infty.
\]

From (P1) we have \( (\lambda^n/(1 - \lambda))d(Tx_0, Tx_1) \ll c \) for all \( n \) sufficiently large and \( \theta \ll c \). From (P1), we have \( d(Tx_n, Tx_m) \ll c \). It follows that \( \{Tx_n\} \) is a Cauchy sequence by Definition 1.5(ii). Since a cone metric space \( X \) is complete, there exists a \( x_\star \in X \) such that \( Tx_n \to x_\star \) as \( n \to \infty \). Thus,

\[
\lim_{n \to \infty} Tx_{2n} = x_\star, \quad \lim_{n \to \infty} Tx_{2n+1} = x_\star.
\]

(6)

Now, if \( T \) is subsequentially convergent, \( \{fx_{2n}\} \) (resp. \( \{gx_{2n+1}\} \)) has a convergent subsequence. Thus, there exist \( x_{\star 1} \in X \) and \( \{fx_{2n}\} \) (resp. \( x_{\star 2} \in X \) and \( \{gx_{2n+1}\} \)) such that

\[
\lim_{n \to \infty} fx_{2n} = x_{\star 1}, \quad \lim_{n \to \infty} gx_{2n+1} = x_{\star 2}.
\]

(7)

Because of the continuity of \( T \), we have

\[
\lim_{n \to \infty} Tx_{2n} = Tw_{\star 1}, \quad \lim_{n \to \infty} Tx_{2n+1} = Tw_{\star 2}.
\]

(8)

From (6) and (8) and using the injectivity of \( T \), there exists a \( w_\star \in X \) (set \( w_\star = w_{\star 1} = w_{\star 2} \)) such that \( Tw_\star = x_\star \).

On the other hand, from (d3) and (2) we have

\[
d(Tw_{\star 1}, Tw_{\star 2}) \leq d(Tw_{\star 1}, Tw_{\star 2}) + d(Tw_{\star 1}, Tw_{\star 2}) + d(Tw_{\star 1}, Tw_{\star 2})
\]

\[
\leq \frac{1}{1 - \lambda} d(Tw_{\star 1}, Tw_{\star 2}) + \frac{1}{1 - \lambda} d(Tw_{\star 1}, Tw_{\star 2}) + \frac{1}{1 - \lambda} d(Tw_{\star 1}, Tw_{\star 2})
\]

\[
= B_1 d(Tw_{\star 1}, Tw_{\star 2}) + B_2 \lambda^{2n+1} + B_3 d(Tw_{\star 1}, Tw_{\star 2}) + B_4 d(Tw_{\star 1}, Tw_{\star 2}),
\]

(9)
where
\[ B_1 = \frac{1}{1 - \lambda}, \quad B_2 = \frac{1}{1 - \lambda} d(Tx_0, Tx_1), \quad B_3 = \frac{\lambda}{1 - \lambda}, \quad B_4 = \frac{\lambda}{1 - \lambda}. \]

Let \( \theta \ll c \). Since \( \lambda^{2n+1} \to \theta \) and \( Tx_i \to Tw \) as \( i \to \infty \), there exists a natural number \( n_0 \) such that, for each \( i \geq n_0 \), (by Definition 1.5.(i)) we have
\[
d(Tw, Tx_{2n+2}) \ll \frac{c}{4B_1}, \quad \lambda^{2n} \ll \frac{c}{4B_2}, \quad d(Tx_{2n+1}, Tw) \ll \frac{c}{4B_3}, \quad d(Tw, Tx_{2n+1}) \ll \frac{c}{4B_4}.
\]

By \((P_1)\), we obtain
\[
d(Tw, Tw_{n+1}) \ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c.
\]

Thus, \( d(Tw_{n+1}, Tw_{n+2}) \ll c \) for each \( c \in \text{int} P \). Using \((P_2)\), we obtain \( d(Tw_n, Tw_{n+1}) = 0 \); that is, \( Tw_n = Tw_{n+1} \). Since \( T \) is one to one, \( gTw_n = Tw_n \). Now we shall show that \( fTw_n = Tw_n \).

\[
d(TfTw_n, Tw_n) = d(TfTw_n, Tgw_n)
\]
\[
\leq q(w_n, w_n)d(Tw_n, Tw_n) + r(w_n, w_n)d(Tw_n, TfTw_n) + s(w_n, w_n)d(Tw_n, Tgw_n)
\]
\[
+ t(w_n, w_n)[d(Tw_n, Tgw_n) + d(Tw_n, TfTw_n)]
\]
\[
= (r + t)(w_n, w_n)d(Tw_n, TfTw_n) \leq \lambda d(Tw_n, TfTw_n).
\]

Using \((P_3)\), it follows that \( d(TfTw_n, Tw_n) = 0 \), which implies the equality \( TfTw_n = Tw_n \). Since \( T \) is one to one, then \( fTw_n = Tw_n \). Thus \( fTw_n = Tw_n \); that is, \( Tw_n \) is a common fixed point of \( f \) and \( g \). Now we shall show that \( Tw_n \) is the unique common fixed point. Suppose that \( Tw_n \) is another common fixed point of \( f \) and \( g \). Then
\[
d(Tw_n, Tw'_n) = d(TfTw_n, Tgw'_n)
\]
\[
\leq q(w_n, w_n')d(Tw_n, Tw'_n) + r(w_n, w_n')d(Tw_n, TfTw_n) + s(w_n, w_n')d(Tw_n, Tgw'_n)
\]
\[
+ t(w_n, w_n')[d(Tw_n, Tgw'_n) + d(Tw_n, TfTw_n)]
\]
\[
= (q + 2r)(w_n, w_n')d(Tw_n, TfTw_n) \leq \lambda d(Tw_n, TfTw_n).
\]

Using \((P_3)\), it follows that \( d(Tw_n, Tw'_n) = 0 \), which implies the equality \( Tw_n = Tw'_n \). Since \( T \) is one to one, \( w_n = w'_n \). Thus \( f \) and \( g \) have a unique common fixed point.

Ultimately, if \( T \) is sequentially convergent, then we can replace \( n \) by \( n_1 \). Thus we have
\[
\lim_{n \to \infty} fTw_n = Tw_n, \quad \lim_{n \to \infty} gTw_{n+1} = Tw_n.
\]

Therefore if \( T \) is sequentially convergent, then the sequences \( \{fTw_n\} \) and \( \{gTw_{n+1}\} \) converge to \( Tw_n \). □

The following results is obtained from Theorem 2.2.

**Corollary 2.3.** Suppose that \((X, d)\) is a complete cone metric space, \( P \) is a solid cone, and \( T : X \to X \) is a continuous and one to one mapping. Moreover, let \( f \) and \( g \) be two maps of \( X \) satisfying
\[
d(Tfx, Tgy) \leq \alpha d(Tx, Ty) + \beta[d(Tx, Tfx) + d(Ty, Tgy)] + \gamma[d(Tx, Tgy) + d(Ty, Tfx)],
\]
for all \( x, y \in X \), where
\[
\alpha, \beta, \gamma \geq 0 \quad \text{and} \quad \alpha + 2\beta + 2\gamma < 1;
\]
that is, \( f \) and \( g \) are \( T \)-contractions. Then

(1) There exists a \( z_0 \in X \) such that \( \lim_{n \to \infty} fTw_{2n} = \lim_{n \to \infty} gTw_{2n+1} = z_0 \).

(2) If \( T \) is subsequentially convergent, then \( \{fTw_n\} \) and \( \{gTw_{n+1}\} \) have a convergent subsequence.

(3) There exists a unique \( w_0 \in X \) such that \( fTw_n = gTw_n = w_0 \); that is, \( f \) and \( g \) have a unique common fixed point.

(4) If \( T \) is sequentially convergent, then the sequences \( \{fTw_n\} \) and \( \{gTw_{n+1}\} \) converge to \( w_0 \).
Proof. Corollary 2.3 follows from Theorem 2.2 by setting \( q = \alpha, r = s = \beta \) and \( t = \gamma \). 

**Corollary 2.4.** Let \((X,d)\) be a complete cone metric space, \(P\) a solid cone and \(T: X \to X\) a continuous and one to one mapping. Moreover, let the mapping \(f\) be a map of \(X\) satisfying

\[
\begin{align*}
    d(Tfx, Tf y) &\leq q(x,y) d(Tx, Ty) + r(x,y) d(Tx, Tfx) + s(x,y) d(Ty, Tf y) \\
    &\quad + t(x,y) [d(Tx, Tfy) + d(Ty, Tfx)],
\end{align*}
\]

for all \(x, y \in X\), where \(q, r, s\) and \(t\) are nonnegative functions satisfying

\[
\sup_{x,y \in X} \{q(x,y) + r(x,y) + s(x,y) + 2t(x,y)\} \leq \lambda < 1;
\]

that is, \(f \) is a \(T\)-contraction. Then

(1) For each \(x_0 \in X\), \(\{f^n x_0\}\) is a Cauchy sequence, (Define the iterate sequence \(\{x_n\}\) by \(x_{n+1} = f^n x_0\)).

(2) There exists a \(z_{x_0} \in X\) such that \(\lim_{n \to \infty} T^n x_0 = z_{x_0}\).

(3) If \(T\) is subsequentially convergent, then \(\{f^n x_0\}\) has a convergent subsequence.

(4) There exists a unique \(w_{x_0} \in X\) such that \(f w_{x_0} = w_{x_0}\); that is, \(f\) has a unique fixed point.

(5) If \(T\) is subsequentially convergent, then, for each \(x_0 \in X\), the sequence \(\{f^n x_0\}\) converges to \(w_{x_0}\).

**Corollary 2.5.** Let \((X,d)\) be a complete cone metric space, \(P\) a solid cone and \(T: X \to X\) a continuous and one to one mapping. Moreover, let the mapping \(f\) be a map of \(X\) satisfying

\[
\begin{align*}
    d(Tfx, Tf y) &\leq \alpha d(Tx, Ty) + \beta d(Tx, Tfx) + d(Ty, Tf y) + \gamma [d(Tx, Tfy) + d(Ty, Tfx)],
\end{align*}
\]

for all \(x, y \in X\), where

\[
\alpha, \beta, \gamma \geq 0 \quad \text{and} \quad \alpha + 2\beta + 2\gamma < 1;
\]

that is, \(f\) is a \(T\)-contraction. Then

(1) For each \(x_0 \in X\), \(\{f^n x_0\}\) is a Cauchy sequence, (Define the iterate sequence \(\{x_n\}\) by \(x_{n+1} = f^n x_0\)).

(2) There exists a \(z_{x_0} \in X\) such that \(\lim_{n \to \infty} T^n x_0 = z_{x_0}\).

(3) If \(T\) is subsequentially convergent, then \(\{f^n x_0\}\) has a convergent subsequence.

(4) There exists a unique \(w_{x_0} \in X\) such that \(f w_{x_0} = w_{x_0}\); that is, \(f\) has a unique fixed point.

(5) If \(T\) is subsequentially convergent, then, for each \(x_0 \in X\), the sequence \(\{f^n x_0\}\) converges to \(w_{x_0}\).

**Example 2.6.** (See [14]). Let \(X = [0, 1], E = C_2^* [0, 1]\) with the norm \(\|f\| = \|f\|_\infty + \|f\|_\infty\), \(P = \{f \in E|f \geq 0\}\) and \(d(x, y) = |x - y|/2\) where \(2' \in P \subset E\). Moreover, suppose that \(Tx = x^2\) and \(fx = x/2\), which map the set \(X\) into \(X\). \((X,d)\) is a cone metric space with non-normal solid cone [9, 16]. Also, \(T\) is a one to one, continuous mapping, and \(f\) is not a Kannan contraction [14]. All of the conditions of Corollary 2.5 are satisfied with \(\alpha = \gamma = 0\) and \(\beta = \frac{1}{2}\). Therefore, \(x = 0\) is the unique fixed point of \(f\).

**Corollary 2.7.** Let \((X,d)\) be a complete cone metric space, \(P\) a solid cone and \(T : X \to X\) a continuous and one to one mapping. Moreover, let the mapping \(f\) be a \(T\)-Hardy-Rogers contraction. Then, the results of the previous Corollary hold.

Proof. See [6].

3. Periodic point results

Obviously, if \(f\) is a map which has a fixed point \(z\), then \(z\) is also a fixed point of \(f^n\) for each \(n \in \mathbb{N}\). However the converse need not be true [2]. If a map \(f : X \to X\) satisfies \(\text{Fix}(f) = \text{Fix}(f^n)\) for each \(n \in \mathbb{N}\), where \(\text{Fix}(f)\) stands for the set of fixed points of \(f\) [10], then \(f\) is said to have property \(P\). Recall also that two mappings \(f, g : X \to X\) are said to have property \(Q\) if \(\text{Fix}(f) \cap \text{Fix}(g) = \text{Fix}(f^n) \cap \text{Fix}(g^n)\) for each \(n \in \mathbb{N}\). The following results extend some theorems of [2, 6].
Theorem 3.1. Let \((X,d)\) be a cone metric space, \(P\) be a solid cone and \(T : X \to X\) be a one to one mapping. Moreover, let the mapping \(f\) be a map of \(X\) satisfying

\((i)\) \(d(fx_T, f^2x) \leq \lambda d(x, fx)\) for all \(x \in X\), where \(\lambda \in [0, 1)\), or \((ii)\) with strict inequality, \(\lambda = 1\) for all \(x \in X\) with \(x \neq fx\). If \(\text{Fix}(f) \neq \emptyset\), then \(f\) has property \(P\).

Proof. See [6]. \(\Box\)

Theorem 3.2. Let \((X, d)\) be a complete cone metric space, and \(P\) a solid cone. Suppose that mappings \(f, g : X \to X\) satisfy all of the conditions of Corollary 2.3. Then \(f\) and \(g\) have property \(P\).

Proof. From Corollary 2.3, \(\text{Fix}(f) \cap \text{Fix}(g) = \{w\}\), where \(w\) is the unique common fixed point of \(f\) and \(g\). Suppose that \(z \in \text{Fix}(f^n) \cap \text{Fix}(g^n)\), where \(n > 1\) is arbitrary. Then we have

\[
d(Tw, Tz) = d(Tf^n w, Tg^n z) = d(T(f^{n-1} w), T(g^{n-1} z))
\]

\[
\leq ad(T(f^{n-1} w), T^g z^n - 1 z) + \beta[d(Tf^{n-1} w, T^f w) + d(Tg^{n-1} z, Tg^z z)]
\]

\[
+ \gamma[d(Tf^{n-1} w, Tg^z z) + d(Tg^{n-1} z, T^f w)]
\]

\[
= ad(Tw, Tg^z z) + \beta[\theta + d(Tg^{n-1} z, Tz)]
\]

\[
+ \gamma[d(Tw, Tz) + d(Tg^{n-1} z, Tw)]
\]

\[
\leq ad(Tw, Tg^z z) + \beta[d(Tg^{n-1} z, Tw) + d(Tw, Tz)]
\]

\[
+ \gamma[d(Tw, Tz) + d(Tg^{n-1} z, Tw)],
\]

which implies that

\[
d(Tw, Tz) = d(Tw, Tg^z z) \leq \lambda d(Tw, Tg^z z),
\]

where \(\lambda = (\alpha + \beta + \gamma)/(1 - \beta - \gamma) < 1\) (by relation (11)). Now, we have

\[
d(Tw, Tz) = d(Tw, Tg^z z) \leq \lambda d(Tw, Tg^z z) \leq \lambda^2 d(Tw, Tg^z z) \cdots \leq \lambda^n d(Tw, Tz).
\]

Since \(\lambda^n \in [0, 1)\), according to \((P_1)\), we have \(d(Tw, Tz) = \theta\); that is, \(Tw = Tz\). Since \(T\) is one to one, then \(w = z\), which implies that \(f\) and \(g\) have property \(P\). \(\Box\)

Theorem 3.3. Let \((X, d)\) be a complete cone metric space, and \(P\) a solid cone. Suppose that the mapping \(f : X \to X\) satisfies all of the conditions of Corollary 2.5. Then \(f\) has property \(P\).

Proof. From Corollary 2.5, \(f\) has a unique fixed point in \(X\). Suppose that \(z \in \text{Fix}(f^n)\). Then we have

\[
d(Tz, T^f z) = d(Tf^{n-1} z, T^f z)
\]

\[
\leq ad(Tf^{n-1} z, T^f z) + \beta[d(Tf^{n-1} z, T^f z) + d(Tf^{n-1} z, T^f z)]
\]

\[
+ \gamma[d(Tf^{n-1} z, T^f z) + d(Tf^{n-1} z, T^f z)]
\]

\[
\leq ad(Tf^{n-1} z, Tz) + \beta[d(Tf^{n-1} z, Tz) + d(Tz, T^f z)] + \gamma[d(Tf^{n-1} z, Tz) + d(Tz, T^f z)]
\]

\[
= (\alpha + \beta + \gamma) d(Tf^{n-1} z, Tz) + (\beta + \gamma) d(Tz, T^f z),
\]

which implies that

\[
d(Tz, T^f z) \leq \lambda d(Tf^{n-1} z, Tz)
\]

where \(\lambda = (\alpha + \beta + \gamma)/(1 - \beta - \gamma) < 1\), (by relation (15)). Hence,

\[
d(Tz, T^f z) = d(Tf^n z, T^f z) \leq \lambda d(Tf^{n-1} z, Tz) \leq \cdots \leq \lambda^n d(Tz, Tz).
\]

Therefore we have \(d(Tz, Tz) = \theta\); that is, \(Tz = Tz\). Since \(Tz \) is one to one, \(Tz = z\). \(\Box\)

Corollary 3.4. Let \((X, d)\) be a complete cone metric space, and \(P\) be a solid cone. Suppose that the mapping \(f : X \to X\) satisfies all of the conditions of Corollary 2.7. Then \(f\) has property \(P\).

Proof. See [6]. \(\Box\)
References