Property \((Bb)\) and Tensor product

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Abstract. In this paper, we find necessary and sufficient conditions for Banach Space operator to satisfy the property \((Bb)\). Then we obtain, if Banach Space operators \(A \in B(X)\) and \(B \in B(Y)\) satisfy property \((Bb)\) implies \(A \otimes B\) satisfies property \((Bb)\) if and only if the \(B\)-Weyl spectrum identity \(\sigma_{BW}(A \otimes B) = \sigma_{BW}(A) \sigma(B) \cup \sigma_{BW}(B) \sigma(A)\) holds. Perturbations by Riesz operators are considered.

1. Introduction

Throughout this paper we denote by \(B(X)\) the algebra of all bounded linear operators acting on a Banach space \(X\). For \(T \in B(X)\), let \(T^*\), \(\ker(T) = T^{-1}(0)\), \(\Re(T) = T(X), \sigma(T)\) and \(\sigma_p(T)\) denote respectively the adjoint, the null space, the range, the spectrum and the approximate point spectrum of \(T\). Let \(\alpha(T)\) and \(\beta(T)\) be the nullity and the deficiency of \(T\) defined by \(\alpha(T) = \dim \ker(T)\) and \(\beta(T) = \text{codim} \Re(T)\). If the range \(\Re(T)\) of \(T \in B(X)\) is closed and \(\alpha(T) < \infty\) (resp., \(\beta(T) < \infty\)) then \(T\) is upper semi-Fredholm (resp., lower semi-Fredholm) operator. Let \(\text{SF}_+(X)\) (resp., \(\text{SF}_-(X)\)) denote the semigroup of upper semi-Fredholm (resp., lower semi-Fredholm) operator on \(X\). An operator \(T \in B(X)\) is said to be semi-Fredholm if \(T \in \text{SF}_+(X) \cup \text{SF}_-(X)\) and Fredholm if \(T \in \text{SF}_+(X) \cap \text{SF}_-(X)\). If \(T\) is semi-Fredholm then the index of \(T\) is defined by \(\text{ind}(T) = \alpha(T) - \beta(T)\). Recall that the ascent of an operator \(T \in B(X)\) is the smallest non negative integer \(p = p(T)\) such that \(T^p(0) = T^{p+1}(0)\). If there is no such integer, i.e., \(T^p(0) \neq T^{p+1}(0)\) for all \(p\), then set \(p(T) = \infty\). The descent of \(T\) is defined as the smallest non negative integer \(q = q(T)\) such that \(T^q(X) = T^{q+1}(X)\). If there is no such integer, i.e., \(T^q(X) \neq T^{q+1}(X)\) for all \(q\), then set \(q(T) = \infty\). It is well known that if \(p(T)\) and \(q(T)\) are both finite then they are equal [13, Proposition 38.6]. A bounded linear operator \(T\) acting on a Banach space \(X\) is Weyl if it is Fredholm of index zero and Browder if \(T\) is Fredholm of finite ascent and descent. For \(T \in B(X)\), let, \(E^0(T)\), and \(\pi^0(T)\) denote, the eigenvalues of finite multiplicity and poles of \(T\) respectively. The Weyl spectrum \(\sigma_w(T)\) and Browder spectrum \(\sigma_B(T)\) of \(T\) are defined by

\[\sigma_w(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl} \},\]

\[\sigma_B(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Browder} \} .\]

We have \(\pi^0(T) := \sigma(T) \setminus \sigma_B(T)\). Set \(\Delta(T) = \sigma(T) \setminus \sigma_w(T)\). According to Coburn [7], Weyl’s theorem holds for \(T\) (abbreviation, \(T \in \text{Wt}\)) if \(\Delta(T) = E^0(T)\) and that Browder’s theorem holds for \(T\) (in symbol, \(T \in \text{Bt}\)) if \(\sigma(T) \setminus \sigma_w(T) = \pi^0(T)\).
An operator $T \in B(X)$ is called B-Fredholm, $T \in BF^a_\ast(X)$, if there exist a natural number $n$, for which the induced operator $T_n : T^n(X) \to T^n(X)$ is Fredholm in usual sense, and B-Weyl, $T \in BW^a_\ast(X)$, if $T \in BF^a_\ast(X)$ and $\text{ind}(T_n) = 0$. Let $E(T)$ be the set of all eigenvalues of $T$ which are isolated in $\sigma(T)$ and $\sigma_{BW}(T) = \{ \lambda \in \mathbb{C} : \lambda \neq 0 \}$ is not B-Weyl. Set $\Delta^a(T) = \sigma(T) \setminus \sigma_{BW}(T)$. According to [12], $T \in B(X)$ satisfies property ($Bw$) (in symbol $T \in (Bw)$) if $\Delta^a(T) = \emptyset$. We say that $T$ satisfies property ($Bb$) (in symbol, $T \in (Bb)$), a variant of generalized Browder’s theorem, if $\Delta^b(T) = \pi^b(T)$. Property ($Bb$) is introduced and studied in [20] by the authors. Property ($Bw$) implies property ($Bb$) but converse is not true in general, see [20]. Let $A$ be a unital algebra. We say that $x \in A$ is Drazin invertible of degree $k$ if there exist an element $a \in A$ such that $x^k ax = x^k$, $axa = a$ and $xax = ax$. The Drazin spectrum of $a \in A$ is defined as $\sigma_D(a) = \{ \lambda \in \mathbb{C} : a - \lambda \text{ is not Drazin invertible} \}$. It is well known that $T \in B(X)$ is Drazin invertible if and only if $T$ has finite ascent and descent. Let $L_\sigma(X)$ denote the set of all finite rank operators acting on an infinite dimensional Banach space $X$. The B-Browder spectrum $\sigma_{BB}(T)$ is defined in [8] as follows:

$$\sigma_{BB}(T) = \bigcap \{ \sigma_{D}(T + F) : F \in L_\sigma(X) \text{ and } TF = FT \}$$

An operator $T \in B(X)$ has the single valued extension property (SVEP) at $\lambda_0 \in \mathbb{C}$, if for every open disc $D_{\lambda_0}$ centered at $\lambda_0$ the only analytic function $f : D_{\lambda_0} \to X$ which satisfies $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$ is the function $f \equiv 0$. We say that $T$ has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$. For more information, see [1].

The tensor product of two operators $A \in B(X)$ and $B \in B(Y)$ on $X \otimes Y$ is the operator $A \otimes B$ defined by

$$(A \otimes B)(x \otimes y) = \sum_{i} A(x_i) \otimes B(y_i)$$

for every $x_i \otimes y_i \in X \otimes Y$. Extensive study of preservation of Browder’s theorem, Weyl’s theorem, Browder’s theorem, and perturbation of Browder’s theorem are found in [10, 11, 15, 16].

We studied necessary and sufficient conditions for Banach space operator to satisfy the property ($Bb$) in first section of this paper. Then we obtain, if Banach space operators $A \in B(X)$ and $B \in B(Y)$ satisfy property ($Bb$) implies $A \otimes B$ satisfies property ($Bb$) if and only if the B-Weyl spectrum identity $\sigma_{BW}(A \otimes B) = \sigma_{BW}(A) \cup \sigma_{BW}(B) \sigma(A)$ holds.

2. property ($Bb$)

Theorem 2.1. If $T$ satisfies property ($Bb$), then $T$ satisfies Browder’s theorem.

Proof. Suppose that $T$ satisfies property ($Bb$) ie, $\Delta^b(T) = \pi^b(T)$. Let $\lambda \in \Delta(T)$. Then $T - \lambda$ is Fredholm of index zero and hence $T - \lambda$ is B-Fredholm of index zero. Thus $\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = \Delta^b(T)$. Hence $\lambda \in \pi^b(T)$

Conversely let $\lambda \in \pi^b(T)$. Since $T$ satisfies property ($Bb$), $T - \lambda$ is B-Fredholm of index zero. Since $\sigma(T - \lambda) < \infty$, we conclude that $T - \lambda$ is Weyl. Thus $\lambda \in \Delta(T)$. This completes the proof. □

The following example shows that the converse of above theorem does not hold in general.

Example 2.2. Let $T : \tilde{P}(N) \to \tilde{P}(N)$ be an injective quasinilpotent operator which is not nilpotent. We define $S$ on Banach Space $X = \tilde{P}(N) \otimes \tilde{P}(N)$ by $S = I \otimes T$, where $I$ is the identity operator on $\tilde{P}(N)$. Then $\sigma(S) = \sigma_w(S) = \{ 0, 1 \}$ and $\sigma_{BW}(S) = \{ 0 \}$. Also $E^0(S) = \pi^0(S) = \phi$. Clearly, $S$ satisfies Browder’s theorem but not ($Bb$).

Theorem 2.3. Let $T \in B(X)$. Then the following statements are equivalent.

(i) $T \in (Bb)$;
(ii) $\sigma_{BW}(T) = \sigma_b(T)$;
(iii) $\sigma_{BW}(T) \cup E^0(T) = \sigma(T)$.

Proof. (i)$\implies$ (ii). Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Since $T$ satisfies ($Bb$), $\lambda \in \pi^0(T)$. Thus $\lambda \in \sigma(T) \setminus \sigma_b(T)$ and hence $\sigma_b(T) \subseteq \sigma_{BW}(T)$. Since the reverse inclusion is always true, we have $\sigma_b(T) = \sigma_{BW}(T)$.

(ii)$\implies$ (i). Assume that $\sigma_b(T) = \sigma_{BW}(T)$ and we will establish that $\Delta^b(T) = \pi^b(T)$. Suppose $\lambda \in \Delta^b(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_b(T)$. Hence $\lambda \in \pi^0(T)$. Conversely suppose $\lambda \in \pi^0(T)$. Since $\sigma_{BW}(T) = \sigma_b(T)$, $\lambda \in \Delta^b(T)$.


(ii) $\implies$ (iii). Let $\lambda \in \Delta^p(T)$. Since $\sigma_{BW}(T) = \sigma_b(T)$, $\lambda \in \sigma(T) \setminus \sigma_b(T)$, i.e., $\lambda \in \pi^0(T)$ which implies that $\lambda \in E^0(T)$. Thus $\sigma_{BW}(T) \cap E^0(T) \supseteq \sigma(T)$. Since $\sigma_{BW}(T) \cup E^0(T) \subseteq \sigma(T)$, always we must have $\sigma_{BW}(T) \cup E^0(T) = \sigma(T)$.

(iii) $\implies$ (ii). Suppose $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Since $\sigma_{BW}(T) \cup E^0(T) = \sigma(T)$, $\lambda \in E^0(T)$. In particular $\lambda$ is an isolated point of $\sigma(T)$. Then by [4, Theorem 4.2] that $\lambda \notin \sigma_b(T)$ and this implies that $\lambda \in \pi(T)$ and so $a(T - \lambda) = d(T - \lambda) < \infty$. So, it follows from [1, Theorem 3.4] that $\beta(T - \lambda) = a(T - \lambda) < \infty$. Hence $\lambda \in \pi^0(T)$. Therefore, $\lambda \notin \sigma_b(T)$. Since the other inclusion is always verified, we have $\sigma_{BW}(T) = \sigma_b(T)$. This completes the proof. □

Theorem 2.4. Let $T \in B(X)$. If $T$ satisfies property (Bb). Then the following statements are equivalent.

(i) $T \in (Bw)$;

(ii) $\sigma_{BW}(T) \cap \pi^0(T) = \emptyset$;

(iii) $E^0(T) = \pi^0(T)$.

Proof. (i) $\implies$ (ii). Suppose (i) holds, that is, $\Delta^p(T) = E^0(T)$; then it follows that $\sigma_{BW}(T) \cap \pi^0(T) = \emptyset$.

(ii) $\implies$ (iii). Suppose $\sigma_{BW}(T) \cap \pi^0(T) = \emptyset$ and let $\lambda \in \pi^0(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Since $T \in (Bb)$, we must have $\lambda \in \pi^0(T)$ and hence $E^0(T) \subseteq \pi^0(T)$. Since the reverse inclusion is trivial, we have $E^0(T) = \pi^0(T)$.

(iii) $\implies$ (i). Since $T$ satisfies property (Bb) and $E^0(T) = \pi^0(T)$, we conclude that $T \in (Bw)$. □

3. property(Bb) and Tensor product

Let $SF_+(X)$ denote the set of upper semi B-Fredholm operators and let $\sigma_{SBF}(T) = \{\lambda \in \mathbb{C} : \lambda \notin SF_+(X)\}$. We write $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : \lambda \notin \sigma_{SBF}(T) \text{ or } \text{ind}(T - \lambda) > 0\}$.

The quasinilpotent part $H_0(T - \lambda I)$ and the analytic core $K(T - \lambda I)$ of $T - \lambda I$ are defined by

$$
H_0(T - \lambda I) := \{x \in X : \lim_{n \to \infty} \|\lambda^nx\|^\frac{1}{\delta} = 0\},
$$

and

$$
K(T - \lambda I) := \{x \in X : \text{there exists a sequence } \{x_n\} \subseteq X \text{ and } \delta > 0 \text{ for which } x = x_0(T - \lambda I)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta\|x\| \text{ for all } n = 1, 2, \cdots\}.
$$

We note that $H_0(T - \lambda I)$ and $K(T - \lambda I)$ are generally non-closed hyper-invariant subspaces of $T - \lambda I$ such that $(T - \lambda I)^n(0) \subseteq H_0(T - \lambda I)$ for all $n = 0, 1, \cdots$ and $(T - \lambda I)K(T - \lambda I) = K(T - \lambda I)$. Recall that if $\lambda \in \text{iso}(\sigma(T))$, then $H_0(T - \lambda I) = \chi_T(\text{closure}(\lambda I))$, where $\chi_T$ is the glocal spectral subspace consisting of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus \lambda \to X$ that satisfies $(T - \mu)f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus \lambda$, see, Duggal [9].

Lemma 3.1. Let $A \in B(X)$ and $B \in B(Y)$. Then

$$
\sigma_{BW}(A \otimes B) \subseteq \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A) \subseteq \sigma_w(A)\sigma(B) \cup \sigma_w(B)\sigma(A) \subseteq \sigma_b(\sigma(A)\sigma(B) + \sigma_w(B)\sigma(A)) = \sigma_b(A \otimes B).
$$

Proof. Since $\sigma_{BW}(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T)$, the inclusion

$$
\sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A) \subseteq \sigma_w(A)\sigma(B) \cup \sigma_w(B)\sigma(A) \subseteq \sigma_b(A)\sigma(B) \cup \sigma_w(B)\sigma(A) = \sigma_b(\sigma(A)\sigma(B) + \sigma_w(B)\sigma(A)) = \sigma_b(A \otimes B),
$$

is evident. Also we have $\sigma_b(A)\sigma(B) \cup \sigma_b(B)\sigma(A) = \sigma_b(A \otimes B)$ is true so it is enough to prove the inclusion $\sigma_{BW}(A \otimes B) \subseteq \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A)$. Let $\lambda \notin \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A)$. Since $\sigma_{SBF}(A \otimes B) \subseteq \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A)$, we have $\lambda \notin \sigma_{SBF}(A \otimes B)$. Now we have to prove that $\text{ind}(A \otimes B - \lambda) \leq 0$. If $\text{ind}(A \otimes B - \lambda) > 0$, 

Then \(\alpha(A \otimes B - \lambda) \leq \infty\) and so \(\beta(A \otimes B - \lambda) \leq \infty\). Let \(E = \{(\mu_i, \nu_i) \in \sigma(A)\sigma(B) : 1 \leq i \leq p, \mu_i\nu_i = \lambda\}\). Then we have by [14, Theorem 3.5] that

\[
\text{ind} (A \otimes B - \lambda) = \sum_{j=n+1}^{p} \text{ind}(A - \mu_j) \dim H_0(B - \nu_j) + \sum_{j=1}^{n} \text{ind}(B - \nu_j) \dim H_0(A - \mu_j).
\]

Since \(\text{ind} (A - \mu_i) < 0\) and \(\text{ind}(B - \nu_i) < 0\), we have a contradiction. Hence we have \(\lambda \notin \sigma_{BW}(A \otimes B)\). This completes the proof. \(\square\)

**Lemma 3.2.** Let \(A \in \mathcal{B}(X)\) and \(B \in \mathcal{B}(Y)\). If \(A \otimes B\) satisfies property (Bb), then \(\sigma_{BW}(A \otimes B) = \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A)\).

**Proof.** It follows from Theorem 2.3 that \(A \otimes B\) satisfies property (Bb) if and only if \(\sigma_{BW}(A \otimes B) = \sigma_B(A \otimes B)\). Thus the required result is an immediate consequence of Lemma 3.1. \(\square\)

The following theorem gives a sufficient condition for the equality \(\sigma_{BW}(A \otimes B) = \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A)\) to hold. The equality \(\sigma_{SBF}(A \otimes B) = \sigma_{SBF}(A)\sigma(B) \cup \sigma_{SBF}(B)\sigma(A)\) follows as in lemma 2 of [11] is useful for our proof of Theorem 3.3

**Theorem 3.3.** If \(A\) and \(B\) satisfy property (Bb), then the following conditions are equivalent:

1. \(A \otimes B\) satisfies property (Bb);
2. \(\sigma_{BW}(A \otimes B) = \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A)\);
3. \(A\) has SVEP at points \(\lambda \in BF_+(A)\) and \(\nu \in BF_+(B)\) such that \(\lambda = \mu \nu \notin \sigma_{BW}(A \otimes B)\).

**Proof.** (i) \(\implies\) (ii), is clear from Lemma 3.2.

(ii) \(\implies\) (i). Let (ii) satisfied. since \(A\) and \(B\) satisfy Bb, it follows that

\[
\sigma_{BW}(A \otimes B) = \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A).
\]

(ii) \(\implies\) (iii). Let \(\lambda \in \sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B).\) Since \(A\) and \(B\) satisfy Bb, we have \(\lambda \in \sigma(A \otimes B) \setminus \sigma_B(A \otimes B).\) Then for every factorization \(\lambda = \nu \mu\) of \(\lambda\), we have \(\mu \in SBF_+(A)\) and \(\nu \in SBF_+(B)\) we have that \(\rho(A - \mu)\) and \(\eta(B - \nu)\) are finite. Hence, \(A\) and \(B\) have SVEP at \(\mu\) and \(\nu\), respectively.

(iii) \(\implies\) (ii). Suppose (iii) holds. We have to prove that \(\sigma_B(A \otimes B) \subseteq \sigma_{BW}(A \otimes B)\). Let \(\lambda \in \sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B)\). Then \(\lambda \in BF_+(A \otimes B)\) and \(\text{ind}(A \otimes B) \leq 0\). Then by the hypothesis and by equality \(\sigma_{SBF}(A \otimes B) = \sigma_{SBF}(A)\sigma(B) \cup \sigma_{SBF}(B)\sigma(A)\), we conclude that \(\mu \notin \sigma_B(A \otimes B)\) and \(\nu \notin \sigma_B(A \otimes B)\). Thus \(\lambda \notin \sigma_B(A \otimes B)\). \(\square\)

**Theorem 3.4.** Let \(A \in \mathcal{B}(X)\) and \(B \in \mathcal{B}(Y)\). If \(A^*\) and \(B^*\) have SVEP, then \(A \otimes B\) satisfies property (Bb).

**Proof.** The hypothesis \(A^*\) and \(B^*\) have SVEP implies

\[
\sigma_w(A) = \sigma_{BW}(A), \quad \sigma_w(B) = \sigma_{BW}(B)
\]

and

\[
A, B \quad \text{and} \quad A \otimes B \quad \text{satisfy Browder’s theorem.}
\]

Hence, Browder’s theorem transfer from \(A\) and \(B\) to \(A \otimes B\). Thus,

\[
\sigma_B(A \otimes B) = \sigma_w(A \otimes B) = \sigma(A)\sigma(B) \cup \sigma_w(A)\sigma(B)
\]

\[
= \sigma(A)\sigma_{BW}(B) \cup \sigma_{SW}(A)\sigma(B) = \sigma_{BW}(A \otimes B)
\]

Therefore,

\[
\pi^0(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_w(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B),
\]

i.e., \(A \otimes B\) satisfies property (Bb). \(\square\)
An operator $T \in B(X)$ is polaroid if every $\lambda \in \text{iso}^0(T)$ is a pole of the resolvent operator $(T - \lambda I)^{-1}$. $T \in B(X)$ polaroid implies $T^*$ polaroid. It is well known that if $T$ or $T^*$ has SVEP and $T$ is polaroid, then $T$ and $T^*$ satisfy Weyl’s theorem.

**Theorem 3.5.** Suppose that the operators $A \in B(X)$ and $B \in B(Y)$ are polaroid.

(i) If $A^*$ and $B^*$ have SVEP, then $A \otimes B$ satisfies property (Bw).

(ii) If $A$ and $B$ have SVEP, then $A^* \otimes B^*$ satisfies property (Bw).

**Proof.** (i) The hypothesis $A^*$ and $B^*$ have SVEP implies

$$\sigma_w(A) = \sigma_{BW}(A), \quad \sigma_w(B) = \sigma_{BW}(B)$$

and

$$A, B \quad \text{and} \quad A \otimes B \quad \text{satisfy Browder’s theorem.}$$

Hence, Browder’s theorem transfer from $A$ and $B$ to $A \otimes B$. Thus,

$$\sigma_{B}(A \otimes B) = \sigma_{w}(A \otimes B) = \sigma(A)\sigma_{w}(B) \cup \sigma_{w}(A)\sigma(B)$$

$$= \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) = \sigma_{BW}(A \otimes B)$$

Evidently, $A \otimes B$ is polaroid by Lemma 2 of [10]; combining this with $A \otimes B$ satisfies Browder’s theorem, it follows that $A \otimes B$ satisfies $Wt$, i.e., $\sigma(A \otimes B) \setminus \sigma_w(A \otimes B) = E^0(A \otimes B)$. But then

$$E^0(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_w(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B),$$

i.e., $A \otimes B$ satisfies property (Bw).

(ii) In this case $\sigma(A) = \sigma(A^*)$, $\sigma(B) = \sigma(B^*)$, $\sigma_w(A^*) = \sigma_{BW}(A^*)$, $\sigma_w(B^*) = \sigma_{BW}(B^*)$, $\sigma(A \otimes B) = \sigma(A^* \otimes B^*)$, polaroid property transfer from $A, B$ to $A^* \otimes B^*$, and Browder’s theorem transfer from $A, B$ to $A \otimes B$. Hence

$$\sigma_{B}(A^* \otimes B^*) = \sigma_{w}(A^* \otimes B^*) = \sigma(A^*)\sigma_{w}(B^*) \cup \sigma_{w}(A^*)\sigma(B^*)$$

$$= \sigma(A^*)\sigma_{BW}(B^*) \cup \sigma_{BW}(A^*)\sigma(B^*)$$

$$= \sigma_{BW}(A^* \otimes B^*).$$

Thus, since $A^* \otimes B^*$ polaroid and $A \otimes B$ satisfies Browder’s theorem imply $A^* \otimes B^*$ satisfy $Wt$,

$$E^0(A^* \otimes B^*) = \sigma(A^* \otimes B^*) \setminus \sigma_w(A^* \otimes B^*) = \sigma(A^* \otimes B^*) \setminus \sigma_{BW}(A^* \otimes B^*),$$

i.e., $A^* \otimes B^*$ satisfies property (Bw). 

4. Perturbations

Let $[A, Q] = AQ - QA$ denote the commutator of the operators $A$ and $Q$. If $Q_1 \in B(X)$ and $Q_2 \in B(Y)$ are quasinilpotent operators such that $[Q_1, A] = [Q_2, B] = 0$ for some operators $A \in B(X)$ and $B \in B(Y)$, then

$$(A + Q_1) \otimes (B + Q_2) = (A \otimes B) + Q,$$

where $Q = Q_1 \otimes B + A \otimes Q_2 + Q_1 \otimes Q_2 \in B(X \otimes Y)$ is a quasinilpotent operator. If in the above, $Q_1$ and $Q_2$ are nilpotents then $(A + Q_1) \otimes (B + Q_2)$ is the perturbation of $A \otimes B$ by a commuting nilpotent operator.

A bounded operator $T$ on $X$ is called finite isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of $T$ of finite multiplicity, i.e $\text{iso}^0(T) \subseteq E^0(T)$. Recall that an operator $T \in B(X)$ satisfies generalized Browder’s theorem (in symbol, $T \in gBt$) if $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$. Note that from Theorem 2.1 of [3] that an operator $T$, $T \in Bt$ if and only if $T \in gBt$. The following lemma from [12] is useful in the proof of the following results.
Lemma 4.1. Let $T \in B(X)$. Then the following statements are equivalent:

(i) $T$ satisfies property $(Bw)$;

(ii) generalized Browder’s theorem holds for $T$ and $\pi(T) = E^0(T)$.

Proposition 4.2. Let $Q_1 \in B(X)$ and $Q_2 \in B(Y)$ be quasinilpotent operators such that $[Q_1, A] = [Q_2, B] = 0$ for some operators $A \in B(X)$ and $B \in B(Y)$. If $A \otimes B$ is finitely isoloid, then $A \otimes B$ satisfies property $(Bw)$ implies $(A + Q_1) \otimes (B + Q_2)$ satisfies property $(Bw)$.

Proof. Recall that $\sigma((A + Q_1) \otimes (B + Q_2)) = \sigma(A \otimes B), \sigma_w((A + Q_1) \otimes (B + Q_2)) = \sigma_{Bw}(A + Q_1) \otimes (B + Q_2)) = \sigma_w((A \otimes B)$ and that the perturbation of an operator by a commuting quasinilpotent has SVEP if and only if the operator has SVEP. If $A \otimes B$ satisfies property $(Bw)$, then

$$E^0(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_{Bw}(A \otimes B)$$

$$= \sigma((A + Q_1) \otimes (B + Q_2)) \setminus \sigma_{Bw}((A + Q_1) \otimes (B + Q_2)).$$

We prove that $E^0(A \otimes B) = E^0((A + Q_1) \otimes (B + Q_2))$. Observe that if $\lambda \in isoo(A \otimes B)$, then $A^* \otimes B^*$ has SVEP at $\lambda$; equivalently, $(A' + Q_1') \otimes (B' + Q_2')$ has SVEP at $\lambda$. Let $\lambda \in E^0(A \otimes B)$; then $\lambda \in \sigma((A + Q_1) \otimes (B + Q_2)) \setminus \sigma_{Bw}((A + Q_1) \otimes (B + Q_2))$. Since $(A' + Q_1') \otimes (B' + Q_2')$ has SVEP at $\lambda$, it follows that $\lambda \notin \sigma_{Bw}((A + Q_1) \otimes (B + Q_2))$ and $\lambda \in isoo((A + Q_1) \otimes (B + Q_2))$. Thus, $\lambda \in E^0((A + Q_1) \otimes (B + Q_2))$. Hence $E^0(A \otimes B) \subseteq E^0((A + Q_1) \otimes (B + Q_2))$. Conversely, if $\lambda \in E^0((A + Q_1) \otimes (B + Q_2))$, then $\lambda \in isoo(A \otimes B)$, and this, since $A \otimes B$ is finitely isoloid implies that $\lambda \in E^0(A \otimes B)$. Hence $E^0(A \otimes B) \subseteq E^0((A + Q_1) \otimes (B + Q_2)) \subseteq E^0(A \otimes B)$.

From [6], we recall that an operator $R \in B(X)$ is said to be Riesz if $R - \lambda I$ is Fredholm for every non-zero complex number $\lambda$, that is, $\Pi(R)$ is quasi-nilpotent in $C(X)$ where $C(X) := B(X)/K(X)$ is the Calkin algebra and $\Pi$ is the canonical mapping of $B(X)$ into $C(X)$. Note that for such operator, $\pi^0(R) = \sigma(R) \setminus \{0\}$, and its restriction to one of its closed subspace is also a Riesz operator, see [6]. The situation for perturbations by commuting Riesz operators is a bit more delicate. The equality $\sigma(T) = \sigma(T + R)$ always hold for operators $T, R \in B(X)$ such that $R$ is Riesz and $[T, R] = 0$; the tensor product $T \otimes R$ is not a Riesz operator (the Fredholm spectrum $\sigma_{Bw}(T \otimes R) = \sigma(T) \sigma_{Bw}(R) = \sigma_{Bw}(T) \sigma(R) = \sigma_{Bw}(T) \sigma(R) \setminus \{0\}$ for a particular choice of $T$ only). However, $\sigma_w$ (also, $\sigma_{Bw}$) is stable under perturbation by commuting Riesz operators [17, 18], and so $T$ satisfies Browder’s theorem if and only if $T + R$ satisfies Browder’s theorem. Thus, if $T, R \in B(X)$ (such that $R$ is Riesz and $[T, R] = 0$), then $\pi^0(T) = \sigma(T) \setminus \sigma_{Bw}(T) = \sigma(T + R) \setminus \sigma_{Bw}(T + R) \subseteq \pi^0(T + R)$, which are finite rank poles of the resolvent of $T$. If we now suppose additionally that $T$ satisfies property $(Bw)$, then

$$E^0(T) = \sigma(T) \setminus \sigma_{Bw}(T) = \sigma(T + R) \setminus \sigma_{Bw}(T + R) \quad (1)$$

and a necessary and sufficient condition for $T + R$ to satisfy property $(Bw)$ is that $E^0(T) = E^0(T + R)$. One such condition, namely $T$ is finitely isoloid.

Proposition 4.3. Let $T, R \in B(X)$, where $R$ is Riesz, and $T$ is finitely isoloid. Then $T$ satisfies property $(Bw)$ implies $T + R$ satisfies property $(Bw)$.

Proof. Observe that if $T$ satisfies property $(Bw)$, then identity (1) holds. Let $\lambda \in E^0(T)$. Then, $\lambda \in E^0(T) \cap \sigma(T) = E^0(T + R - \lambda I)$ and has SVEP at $\lambda$. Since $\lambda \in \sigma(T + R) \setminus \sigma_{Bw}(T + R)$, it implies that $T + R - \lambda I$ is Fredholm of index 0 and so $\lambda \in E^0(T + R)$. Hence, $E^0(T) \subseteq E^0(T + R)$. Now let $\lambda \in E^0(T + R)$. Then $\lambda \in E^0(T + R) \cap \sigma(T + R) = E^0(T + R) \cap \sigma(T) \subseteq isoo(T)$, which by the finite isoloid property of $T$ implies $\lambda \in E^0(T)$. Thus, $E^0(T) \subseteq E^0(T + R)$.

Theorem 4.4. Let $A \in B(X)$ and $B \in B(Y)$ be finitely isoloid operators which satisfy property $(Bw)$. If $R_1 \in B(X)$ and $R_2 \in B(Y)$ are Riesz operators such that $[A, R_1] = [B, R_2] = 0$, $\sigma(A + R_1) = \sigma(A)$ and $\sigma(B + R_2) = \sigma(B)$, then $A \otimes B$ satisfies property $(Bw)$ implies $(A + R_1) \otimes (B + R_2)$ satisfies property $(Bw)$ if and only if generalized Browder’s theorem transforms from $A + R_1$ and $B + R_2$ to their tensor product.
Proof. The hypotheses imply (by Proposition 4.3) that both $A + R_1$ and $B + R_2$ satisfy property (BW). Suppose that $A \otimes B$ satisfies property (BW). Then $\sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B) = E^0(A \otimes B)$. Evidently $A \otimes B$ satisfies generalized Browder’s theorem, and so the hypothesis $A$ and $B$ satisfy property (BW) implies that generalized Browder’s theorem transfers from $A$ and $B$ to $A \otimes B$. Furthermore, since $\sigma(A + R_1) = \sigma(A)$, $\sigma(B + R_2) = \sigma(B)$, and $\sigma_{BW}$ is stable under perturbations by commuting Riesz operators,

$$
\sigma_{BW}(A \otimes B) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) = \sigma(A + R_1)\sigma_{BW}(B + R_2) \cup \sigma_{BW}(A + R_1)\sigma(B + R_2).
$$

Suppose now that generalized Browder’s theorem transfers from $A + R_1$ and $B + R_2$ to $(A + R_1) \otimes (B + R_2)$. Then

$$
\sigma_{BW}(A \otimes B) = \sigma_{BW}((A + R_1) \otimes (B + R_2))
$$

and

$$
E^0(A \otimes B) = \sigma((A + R_1) \otimes (B + R_2)) \setminus \sigma_{BW}((A + R_1) \otimes (B + R_2)).
$$

Let $\lambda \in E^0(A \otimes B)$. Then $\lambda \neq 0$, and hence there exist $\mu \in \sigma(A + R_1) \setminus \sigma_{BW}(A + R_1)$ and $\nu \in \sigma(B + R_2) \setminus \sigma_{BW}(B + R_2)$ such that $\lambda = \mu \nu$. As observed above, both $A + R_1$ and $B + R_2$ satisfy property (BW); hence $\mu \in E^0(A + R_1)$ and $\nu \in E^0(B + R_2)$. This, since $\lambda \in \sigma(A \otimes B) = \sigma((A+R_1)\otimes(B+R_2))$, implies $\lambda \in E^0((A+R_1)\otimes(B+R_2))$. Conversely, if $\lambda \in E^0((A+R_1)\otimes(B+R_2))$, then $\lambda \neq 0$ and there exist $\mu \in E^0(A + R_1) \subseteq \sigma(A)$ and $\nu \in E^0(B + R_2) \subseteq \sigma(B)$ such that $\lambda = \mu \nu$. Recall that $E^0((A+R_1)\otimes(B+R_2)) \subseteq E^0(A+R_1)E^0(B+R_2)$. Since $A$ and $B$ are finite isolation, $\mu \in E^0(A)$ and $\nu \in E^0(B)$. Hence, since $\sigma((A + R_1) \otimes (B + R_2)) = \sigma(A \otimes B)$, $\lambda = \mu \nu \in E^0(A \otimes B)$. To complete the proof, we observe that if the implication of the statement of the theorem holds, then (necessarily) $(A + R_1) \otimes (B + R_2)$ satisfies generalized Browder’s theorem. This, since $A + R_1$ and $B + R_2$ satisfy generalized Browder’s theorem, implies generalized Browder’s theorem transfers from $A + R_1$ and $B + R_2$ to $(A + R_1) \otimes (B + R_2)$. 

References