On coverings with special points and monodromy group a Weyl group of type $B_d$

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Abstract. In this paper we study Hurwitz spaces parameterizing coverings with special points and with monodromy group a Weyl group of type $B_d$. We prove that such spaces are irreducible if $k > 3d − 3$. Here, $k$ denotes the number of local monodromies that are reflections relative to long roots.

1. Introduction

The study of irreducible components of Hurwitz spaces is a classic problem in algebraic geometry and it is valuable in many applications. The Lüroth - Clebsch - Hurwitz theorem states the irreducibility of the Hurwitz space of simple coverings of $P^1$ with $n$ branch points (see [11]). This result was used by Severi in order to prove the irreducibility of the moduli space of genus $g$ curves (see [18]). Today, there were many generalizations of Lüroth - Clebsch - Hurwitz result. Let $Y$ be a smooth, connected, projective complex curve of genus $g$. Specifically, the irreducibility of Hurwitz spaces of coverings of $Y$ with monodromy group $S_d$ and with an arbitrary number of special points has been studied both when $g = 0$ and when $g > 0$ (see [1, 9, 13–15, 19, 24, 27, 28]). We point out that, for example, Harris, Graber and Starr used the result of [9] in order to prove the existence of sections of one-parameter family of complex rationally connected varieties (see [10]). Hurwitz spaces of coverings whose monodromy group is a Weyl group different from $S_d$ and their irreducible components were studied, for example, in [2, 20–23, 25, 26]. We note that coverings with monodromy group a Weyl group appear in the study of spectral curves, integrable systems and Prym - Tyurin varieties (see [6, 15, 16]). In fact, the Prym maps yield morphism from the Hurwitz spaces of coverings with monodromy group contained in a Weyl group to Siegel modular varieties which parameterize Abelian varieties. Thus, some property of these varieties can be studied by using these Hurwitz spaces.

In this paper we continue the investigation of the irreducibility of Hurwitz spaces that parameterize coverings with special fibers and with monodromy group a Weyl group of type $B_d$. In particular, we work with coverings that decompose into a sequence of type $X \twoheadrightarrow X' \twoheadrightarrow Y$ where $\pi$ is a degree two covering with $n_1$ branch points and $f$ is a degree $d$ coverings with monodromy group $S_d$. Moreover, $f$ has $n_2$ branch points, $k$ of which are simple points and $n_2 − k$ of which are special points. Furthermore, $f(D_\pi) \cap D_f = \emptyset$ where $D_\pi$ and $D_f$ denote, respectively, the branch locus of $\pi$ and $f$. 

2010 Mathematics Subject Classification. Primary 14H30; Secondary 14H10

Keywords. Hurwitz spaces; special fibers; branched coverings; monodromy; braid moves.

Received: 10 August 2013; Accepted: 14 September 2013

Communicated by Vladimir Rakocevic

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We prove that, under the hypothesis \( k > 3d - 3 \), the corresponding Hurwitz spaces are irreducible both when \( g = 0 \) and when \( g > 0 \) (see Theorems 5.5 and 5.6). In this way, we generalize the results obtained for coverings as above but with one or two special fibers by the author in \([20, 25]\). Moreover, we extend the results obtained in the case in which the monodromy group is all coverings as above but with one or two special fibers by the author in \([27]\) to coverings with monodromy group \( W \) a Weyl group of type \( B \).

**Conventions and Notations** Here, two sequences of coverings, \( X_1 \rightarrow Y \) and \( X_2 \rightarrow Y \), are equivalent if there exist two biholomorphic maps \( p : X_1 \rightarrow X_2 \) and \( p' : X_1' \rightarrow X_2' \) such that \( p' \circ \pi_1 = \pi_2 \circ p \) and \( f_2 \circ p' = f_1 \). We denote by \( [f \circ \pi] \) the equivalence class containing \( f \circ \pi \). Moreover, we denote by \( t^h \) the permutation \( h^{-1} \cdot h \) and we denote by \( \langle t_1, \ldots, t_i \rangle \) the subgroup of \( S_d \) generated by the permutations \( t_1, \ldots, t_i \).

### 2. Weyl groups of type \( B_d \)

Let \( \{\varepsilon_1, \ldots, \varepsilon_d\} \) be the standard base of \( R^d \) and let \( R \) be the root system \( \{\pm \varepsilon_i, \pm \varepsilon_i, \pm \varepsilon_i : 1 \leq i, j \leq d\} \). The Weyl group of type \( B_d \) is generated by the reflections \( s_{\varepsilon_i} \) with \( 1 \leq i \leq d \), and by the reflections \( s_{\varepsilon_i, \varepsilon_j} \) with \( 1 \leq i < j \leq d \) (see \([4]\)). We denote this group by \( W(B_d) \). We recall that the reflection \( s_{\varepsilon_i, \varepsilon_j} \) exchanges \( \varepsilon_i \) with \( \varepsilon_j \) and \( -\varepsilon_i \) with \( -\varepsilon_j \), leaving fixed each \( \varepsilon_h \) with \( h \neq i, j \). The reflection \( s_{\varepsilon_i, \varepsilon_j} \) exchanges \( \varepsilon_i \) with \( -\varepsilon_i \) and fixes all the \( \varepsilon_h \) with \( h \neq i \). Hence, identifying \( \{\pm \varepsilon_i : 1 \leq i \leq d\} \) with \( \{\pm 1, \ldots, \pm d\} \) by using the map \( \pm \varepsilon_i \rightarrow \pm i \), we can define an injective homomorphism from \( W(B_d) \) into \( S_d \) such that

\[
s_{\varepsilon_{i-j}} \rightarrow (i)(-i-j), \quad s_{\varepsilon_i} \rightarrow (i-\bar{i}), \quad s_{\varepsilon_i+\varepsilon_j} = s_{\varepsilon_i} s_{\varepsilon_j} s_{\varepsilon_{i-j}} \rightarrow (i-j)(-i-j).
\]

Let \( (Z_2)^d \) be the set of the functions from \( \{1, \ldots, d\} \) into \( Z_2 \) equipped with the sum operation. Let us denote by \( \Psi \) the homomorphism from \( S_d \) in \( \text{Aut}((Z_2)^d) \) which assigns to \( t \in S_d \) the element \( \Psi(t) \in \text{Aut}((Z_2)^d) \) where

\[
[\Psi(t) \cdot a] (j) := a(j') \quad \text{for each} \quad a \in (Z_2)^d.
\]

Let \( (Z_2)^d \times S_d \) be the semidirect product of \( (Z_2)^d \) and \( S_d \) through the homomorphism \( \Psi \). Given \( (a'; t_1), (a''; t_2) \in (Z_2)^d \times S_d \), we put

\[
(a'; t_1) \cdot (a''; t_2) := (a' + \Psi(t_1) a''; t_1 t_2).
\]

Moreover, we use \( \bar{1}_j \) by denote the function of \( (Z_2)^d \) defined as

\[
I_j = 1 \quad \text{and} \quad \bar{1}_j = 0 \quad \text{for each} \quad h \neq j
\]

and we use \( z_{ij} \) to denote the function of \( (Z_2)^d \) defined as

\[
z_{ij}(t) = z_{ij}(j) = z \quad \text{and} \quad z_{ij}(h) = 0 \quad \text{for each} \quad h \neq i, j \quad \text{and} \quad z \in Z_2.
\]

We notice that the homomorphism from \( W(B_d) \) into \( (Z_2)^d \times S_d \) defined by

\[
s_{\varepsilon_{i-j}} \rightarrow (0; (i j)), \quad s_{\varepsilon_i} \rightarrow (\bar{1}_j; \bar{1}_i), \quad s_{\varepsilon_i+\varepsilon_j} \rightarrow (\bar{1}_{ij}; (i j))
\]

is an isomorphism. In what follows, we will use this isomorphism in order to identify \( W(B_d) \) by \( (Z_2)^d \times S_d \).

**Definition 2.1.** Let \( h \) be a positive integer. Let \( (c; \xi) \) be an element of \( W(B_d) \) satisfying the following: \( \xi \) is a \( h \)-cycle of \( S_d \) and \( c \) is a function that sends to \( \bar{0} \) all the indexes fixed by \( \xi \). We call an such element positive \( h \)-cycle if \( c \) is either zero or a function which sends to \( \bar{1} \) an even number of indexes. We call it negative \( h \)-cycle if it is not positive.

We recall that two cycles \( (c; \xi) \) and \( (c'; \xi') \) in \( W(B_d) \) are disjoint if \( \xi \) and \( \xi' \) are disjoint. Furthermore, all the elements in \( W(B_d) \) can be expressed as a product of disjoint positive and negative cycles. The lengths of such disjoint cycles together with their signs determine the signed cycle type of the elements of \( W(B_d) \). Two elements of \( W(B_d) \) are conjugate if and only if they have the same signed cycle type (see \([5]\)).
3. Hurwitz spaces of type $H^{W(B_d)}_{m,n,k,q_1,\ldots,q_d}(\mathcal{Y})$

Let $X$, $X'$ and $Y$ be smooth, connected, projective complex curves and let $g$ be the genus of $Y$. Let $d, n_1, n_2, k$ be integers such that $d \geq 3$, $n_1 > 0$ and $n_2 > k > 0$. In this paper we are interested in degree $2d$ coverings that decompose in a sequence of coverings, $X \rightarrow X' \rightarrow Y$, satisfying the followings:

- $\pi$ is a degree $2$ covering with $n_1$ branch points and $f$ is a degree $d$ coverings, with monodromy group $S_d$ and with $n_2$ branch points, $k$ of which are simple points and $n_2 - k$ of which are special points. Moreover, $f(D_\gamma) \cap D_f = \emptyset$ where $D_\gamma$ and $D_f$ denote, respectively, the branch locus of $\pi$ and $f$.

Let $b_0$ be a point of $Y$ and let $D$ be a finite subset of $Y$ such that $b_0 \in Y - D$. By Riemann’s existence theorem (see [8], Proposition 1.2) there is a natural one-to-one correspondence between:

- the set of equivalence classes of degree $2d$ branched coverings of $Y$ with branch locus $D$ and

- the set of equivalence classes of homomorphisms $m : \pi_1(Y - D, b_0) \rightarrow S_d$ whose images are transitive subgroups of $S_d$, where two homomorphisms $m$ and $m'$ are equivalent if there exists $h \in S_d$ such that $m'(\{y\}) = h^{-1}m(\{y\})h$ for each $\{y\} \in \pi_1(Y - D, b_0)$.

From now on, we will denote by $D$ and by $m$, respectively, the branch locus and the monodromy homomorphism of $f \circ \pi$.

Let $e_1', \ldots, e_r'$ be partitions of $d$ such that $e_i' = (e_{i1}', \ldots, e_{i\lambda_i}')$ and $e_1' \geq \cdots \geq e_r'$. Let $q_1, \ldots, q_r$ be positive integers such that $q_1 + \cdots + q_r = n_2 - k$. Let us denote by $H^{W(B_d)}_{m,n,k,q_1,\ldots,q_d}(\mathcal{Y})$ the Hurwitz space of equivalence classes of sequences of coverings, $f \circ \pi$, defined as above such that $q_i$, among the special points of $f$ have local monodromy whose cycle type is given by the partition $e_i'$, for $i = 1, \ldots, r$.

**Definition 3.1.** Let $G$ be an arbitrary group. An ordered sequence

$$(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_\gamma, \mu_\gamma) := (t_1, \ldots, t_n)$$

of elements in $G$ is a Hurwitz system if $t_i \neq \text{id}$ for each $i \in \{1, \ldots, n\}$ and $t_1 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_\gamma, \mu_\gamma]$. The subgroup of $G$ generated by $t_1, \lambda_i, \mu_i$ with $i = 1, \ldots, n$ and $s = 1, \ldots, g$ is called the monodromy group of the Hurwitz system. Two Hurwitz systems $(t_1, \lambda_1, \mu_1)$ and $(t'_1, \lambda'_1, \mu'_1)$ with elements in $G$ are equivalent if there exists $h \in G$ such that $t'_i = h^{-1}t_i h$, $\lambda'_i = h^{-1}\lambda_i h$ and $\mu'_i = h^{-1}\mu_i h$ for each $i = 1, \ldots, n$ and $s = 1, \ldots, g$.

**Remark 3.2.** We notice that an order sequence $(t_1, \ldots, t_n)$ of elements in $G$, with $t_i \neq \text{id}$ for each $i$, is a Hurwitz system if $t_1 \cdots t_n = \text{id}$.

Let $(\gamma_1, \ldots, \gamma_n, m, n, \alpha_1, \beta_1, \ldots, \alpha_\gamma, \beta_\gamma)$ be a standard generating system for $\pi_1(Y - D, b_0)$. The images via $m$ of $\gamma_1, \ldots, \gamma_n, m, n, \alpha_1, \beta_1, \ldots, \alpha_\gamma, \beta_\gamma$ determine an equivalence class of Hurwitz systems

$$(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_\gamma, \mu_\gamma)$$

with monodromy group $W(B_d)$ satisfying the following conditions: $k$ among the $t_i$ are elements of type $(z_{ij}; (i, j))$, $n_1$ are elements of type $(1; \text{id})$ and $q_i$, with $i = 1, \ldots, r$, are product of $s_i$ positive disjoint cycles whose lengths are given by the elements of the partition $e_i'$. Let us denote by $A^{e_i'}_{k,n_1,q_1,\ldots,q_d}(\mathcal{Y})$ the set of all equivalence classes of Hurwitz systems as above.

We notice that by Riemann’s existence theorem, we can identify the set of equivalence classes $[f \circ \pi] \in H^{W(B_d)}_{m,n,k,q_1,\ldots,q_d}(\mathcal{Y})$ such that $f \circ \pi$ has branch locus $D$ with the set $A^{e_i'}_{k,n_1,q_1,\ldots,q_d}(\mathcal{Y})$.

4. Braid moves

Let $n$ be a positive integer. Let $Y^{(n)}$ be the $n$-fold symmetric product of $Y$ and $A$ be the codimension 1 locus of $Y^{(n)}$ consisting of non simple divisors. In this paper we are interested in the way in which the
generators of the braid group \( \pi_1(\mathcal{Y}^{(0)} - \Delta, D) \) act on Hurwitz systems. So, we recall that such group is generated by the elementary braids \( \sigma_i \) with \( i = 1, \ldots, n - 1 \) and by the braids \( \rho_{j, s} = \tau_{j, s} \) with \( 1 \leq j \leq n \) and \( 1 \leq s \leq g \) (see [3, 7, 17]). Here, we denote by \( \sigma_i', \sigma_i^{(s)}=(\sigma_i')^{-1} \) the pair of moves associated to \( \sigma_i \). We call \( \sigma_i', \sigma_i'' \) elementary moves. The moves \( \sigma_i', \sigma_i'' \) fix all the \( \lambda_s \) all the \( \mu_s \) and all the \( t_i \) with \( h \neq i, i + 1 \). They transform \((t_i, t_{i+1})\) into
\[
(t_{i+1}, t_i^{(s)}) \quad \text{and} \quad (t_{i+1}, t_i^{(s)}),
\]
respectively (see [11]). We denote by \( \rho_{j, s}', \rho_{j, s}'' = (\rho_{j, s}')^{-1} \) and by \( \tau_{j, s}', \tau_{j, s}'' = (\tau_{j, s}')^{-1} \), respectively, the pair of moves associated to \( \rho_{j, s} \) and \( \tau_{j, s} \). We use the following result.

**Proposition 4.1 ([12], Theorem 1.8).** Let \((t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)\) be a Hurwitz system. Let \( u_0 = 1 \) and let \( u_s = [\lambda_1, \mu_1] \cdots [\lambda_s, \mu_s] \) for \( s = 1, \ldots, g \). The following formulae hold:
\[
(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g) \to (t'_1, \ldots, t'_n; \lambda'_1, \mu'_1, \ldots, \lambda'_g, \mu'_g)
\]

- For \( \rho_{i, s}' \) where \( 1 \leq i \leq n, 1 \leq s \leq g \)
  \[
  t'_j = t_j \quad \text{for each} \quad j \neq i, \quad \lambda'_j = \lambda_j \quad \text{for each} \quad l \neq s \quad \text{and}
  \]
  \[
  (t_i, \mu_s) \to (t'_i, \mu'_s) = (a_1^{-1}t_1d_1, b_1^{-1}t_i^{-1}b_1 \mu_s)
  \]
  where \( a_1 = (t_1 \cdots t_{i-1})^{-1}u_{s-1}\lambda_i(u_i^{-1}u_g)(t_{i+1} \cdots t_n)^{-1} \) and \( b_1 = (t_1 \cdots t_{i-1})^{-1}u_{s-1} \lambda_s \).

- For \( \tau_{i, s}' \) where \( 1 \leq i \leq n, 1 \leq s \leq g \)
  \[
  t'_j = t_j \quad \text{for each} \quad j \neq i, \quad \lambda'_j = \lambda_j \quad \text{for each} \quad l \neq s, \quad \mu'_j = \mu_j \quad \text{for each} \quad l \quad \text{and}
  \]
  \[
  (t_i, \lambda_s) \to (t'_i, \lambda'_s) = (c_1^{-1}t_1c_1, d_1^{-1}t_id_1 \lambda_s)
  \]
  where \( c_1 = t_{i+1} \cdots t_n(u_i^{-1}u_g)\mu_s(u_{s-1})^{-1}t_1 \cdots t_{i-1} \) and \( d_1 = t_{i+1} \cdots t_n(u_i^{-1}u_g)^{-1}\mu_s \).

- For \( \rho_{i, s}'' \) where \( 1 \leq i \leq n, 1 \leq s \leq g \)
  \[
  t'_j = t_j \quad \text{for each} \quad j \neq i, \quad \lambda'_j = \lambda_j \quad \text{for each} \quad l, \quad \mu'_j = \mu_j \quad \text{for each} \quad l \quad \text{and}
  \]
  \[
  (t_i, \mu_s) \to (t'_i, \mu'_s) = (a_2^{-1}t_1d_2, b_2^{-1}t_2b_2 \mu_s)
  \]
  where \( a_2 = t_{i+1} \cdots t_n(u_i^{-1}u_g)^{-1}\lambda_i^{-1}(u_{s-1})^{-1}t_1 \cdots t_{i-1} \) and \( b_2 = t_{i+1} \cdots t_n(u_i^{-1}u_g)^{-1} \).

- For \( \tau_{i, s}'' \) where \( 1 \leq i \leq n, 1 \leq s \leq g \)
  \[
  t'_j = t_j \quad \text{for each} \quad j \neq i, \quad \lambda'_j = \lambda_j \quad \text{for each} \quad l \neq s, \quad \mu'_j = \mu_j \quad \text{for each} \quad l \quad \text{and}
  \]
  \[
  (t_i, \lambda_s) \to (t'_i, \lambda'_s) = (c_2^{-1}t_1c_2, d_2^{-1}t_id_2 \lambda_s)
  \]
  where \( c_2 = (t_1 \cdots t_{i-1})^{-1}u_{s-1}\mu_s^{-1}(u_i^{-1}u_g)(t_{i+1} \cdots t_n)^{-1} \) and \( d_2 = (t_1 \cdots t_{i-1})^{-1}u_{s-1} \).

**Remark 4.2.** The moves \( \rho_{i, s}', \rho_{i, s}'', \tau_{i, s}' \) and \( \tau_{i, s}'' \) transform \( t_i \) into an element belonging to the same conjugacy class. Furthermore, we notice that when \( \lambda_1 = \cdots = \lambda_s = \mu_1 = \cdots = \mu_{s-1} = id \), the braid move \( \rho_{i, s}' \) transforms
\[
\mu_s \quad \text{into} \quad t_i^{-1} \mu_s,
\]
Analogously when \( \lambda_1 = \cdots = \lambda_{s-1} = \mu_1 = \cdots = \mu_{s-1} = id \), the braid move \( \tau_{i, s}'' \) transforms
\[
\lambda_s \quad \text{into} \quad t_i^{-1} \lambda_s.
\]

**Definition 4.3.** Two Hurwitz systems are said braid equivalent if one is obtained from the other by using a finite sequence of braid moves \( \sigma_i', \rho_{i, s}', \tau_{i, s}', \sigma_i' \), \( \rho_{i, s}' \), \( \tau_{i, s}' \) where \( 1 \leq i \leq n - 1, 1 \leq j \leq n \) and \( 1 \leq s \leq g \). Two ordered sequences of permutations \((t_1, \ldots, t_l)\) and \((t'_1, \ldots, t'_l)\) are said braid equivalent if \((t'_1, \ldots, t'_l)\) is obtained from \((t_1, \ldots, t_l)\) by using a finite sequence of braid moves of type \( \sigma_i', \sigma_i'' \). We denote the braid equivalence by \( \simeq \).
5. Irreducibility of $H^{W(R)}_{d,n,k,q}c_{\ldots,q_{n}}c_{\ldots,q_{r}}(Y)$

In what follows, we write $|\xi^i|$ to denote $\sum_{j=1}^{n}(e^i_j - 1)$. Moreover, we associate to the partition $\xi'$ the following element in $S_d$ having cycle type given by $\xi'$

$$e_i := (1, 2, \ldots, e^i_1)(e^i_1 + 1, \ldots, e^i_1 + e^i_2) \cdots \left(\sum_{j=1}^{n-1} e^i_{j+1} \cdots d\right).$$

Let $e$ be the following permutation of $S_d$

$$(e_1 \cdots e_in \cdots e_{r+1} \cdots e_r \cdots e_1)^{-1}$$

where $e_i$, with $i = 1, \ldots, r$, appears $q_i$ times. Let $\xi_1, \ldots, \xi_{q_r}$ be disjoint cycles of lengths $h_1, \ldots, h_{q_r}$, with $h_1 \geq h_2 \geq \cdots \geq h_{q_r}$ such that $e = \xi_1 \cdots \xi_{q_r}$. Let $\xi_j = (l^j_1 \ldots l^j_{h_j})$ where $l^j_1 < l^j_2$ for each $b = 2, \ldots, h_j$. In the sequel, we denote by $Z_j$ the sequence of transpositions $[(l^j_1, l^j_2), (l^j_1, l^j_3), \ldots, (l^j_1, l^j_{h_j})]$ and by $Z$ the concatenation $Z_1, Z_2, \ldots, Z_{q_r}$.

For a convenience of the reader we recall the following results.

**Lemma 5.1 ([19], Proposition 3).** Let $(t_1, t_2, \ldots, t_l)$ be a sequence of permutations in $S_d$ such that $t_1$ has cycle type $\xi^1$ and $t_2, \ldots, t_l$ are transpositions.

If $1 - 1 + |\xi^1| \geq 2d$ then $(t_1, t_2, \ldots, t_l)$ is braid equivalent to

$$(t'_1, t'_2, \ldots, t'_{l-2}, t'_{l-1}, t'_l)$$

where $t'_i$ has cycle type $\xi^1$, $t'_2, \ldots, t'_l$ are transpositions, $t'_{l-1} = t'_l$ and

$$\langle t'_1, t'_2, \ldots, t'_{l-2} \rangle = \langle t'_1, \ldots, t'_{l-2}, t'_{l-1}, t'_l \rangle.$$

**Lemma 5.2 ([12], Main Lemma 2.1).** Let $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_r, \mu_r)$ be a Hurwitz system of permutations in $S_d$. Suppose that $t_1, t_{n+1} = id$. Let $H$ be the subgroup of $S_d$ generated by $(t_1, \ldots, t_{n-1}, t_{n+2}, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_r, \mu_r)$. Then for every $h \in H$ the given Hurwitz system is braid equivalent to

$$(t_1, \ldots, t_{n-1}, t^h_1, t^h_{n+1}, t_{n+2}, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_r, \mu_r).$$

**Proposition 5.3 ([14], Theorem 2.3).** Let $[t_1, \ldots, t_{n_r}]$ be an equivalence class of Hurwitz systems of permutations in $S_d$, with monodromy group $S_0$, satisfying the followings: $k$ among the $t_i$ are transpositions and $q_i$ among the $t_i$ are permutations whose cycle type is given by the partition $\xi'$ of $d$, for $i = 1, \ldots, r$. If $k > 3d - 3$, $[t_1, \ldots, t_{n_r}]$ is braid equivalent to the class $[\tilde{t}_1, \ldots, \tilde{t}_{n_r}]$ where

$$\tilde{t}_1 = \ldots = \tilde{t}_{q_1} = e_1, \ \tilde{t}_{q_1+1} = \ldots = \tilde{t}_{q_{r+1}} = e_{r+1}$$

with $j = 1, \ldots, r - 1$. Moreover the sequence $(\tilde{t}_{n_r-k+1}, \ldots, \tilde{t}_{n_r})$ is equal to

$$(Z, (1, 2), \ldots, (1, 2), (2, 3), (2, 3), \ldots, (d - 1, d), (d - 1, d))$$

where $(1, 2)$ appears an even number of times.

Now, by using Proposition 5.3, we show that any two class in $A^{e}_{k,n,q}c_{\ldots,q_{n}}c_{\ldots,q_{r}},0$ are braid equivalent.

**Proposition 5.4.** If $k > 3d - 3$, each equivalence class in $A^{e}_{k,n,q}c_{\ldots,q_{n}}c_{\ldots,q_{r}},0$ is braid equivalent to a class of the form:

$$[0; \bar{t}_1, \ldots, 0; \bar{t}_{n_r}, (\bar{t}_1; id), \ldots, (\bar{t}_1; id)]$$

where $(\bar{t}_1, \ldots, \bar{t}_{n_r})$ is the sequence in Proposition 5.3.
Proof. Let \([\ell] \in A_{k,n-d}^{\sigma} g \leq 0\). We act with elementary moves of type \(\sigma'_i\) and move on the right all the elements \((\vec{1}; id)\). In this way, we have that \([\ell]\) is braid equivalent to a class of the form
\[
[(\sigma; t'_1), \ldots, (\sigma; t'_{n_2}), (\vec{1}; id), \ldots, (\vec{1}; id)].
\]
We notice that the equivalence class \([t'_1, \ldots, t'_{n_2}]\) satisfies all the hypothesis in Proposition 5.3, so it is braid equivalent to the class \([\bar{t}_1, \ldots, \bar{t}_{n_2}]\). Then, \([\ell]\) is braid equivalent to a class of type
\[
[(\sigma; \bar{t}_1), \ldots, (\sigma; \bar{t}_{n_2}), (\vec{1}; id), \ldots, (\vec{1}; id)].
\]

**Step 1.** We show that \([\ell]\) is braid equivalent to a class of the form
\[
[(\sigma; \bar{t}_1), \ldots, (\sigma; \bar{t}_{n_2}), (\vec{1}; id), \ldots, (\vec{1}; id)].
\]
Let \(i\) and \(j\) be two arbitrary indexes in \([1, \ldots, d]\) such that \(i < j\) and \(j \neq i + 1\). We notice that the sequence
\[
((\sigma; (i, i + 1)), (\sigma; (i, i + 1)), \ldots, (\sigma; (j - 1, j)), (\sigma; (j - 1, j)))
\]
is braid equivalent to the sequence
\[
((\sigma; (i, j)), (\sigma; (i, j)), (\sigma; (i, i + 1)), (\sigma; (i, i + 1)), \ldots, (\sigma; (j - 2, j - 1)), (\sigma; (j - 2, j - 1))).
\]
In fact, if the elements of the pair \((\sigma; (j - 1, j)), (\sigma; (i, j))\) are at the places \(h, h + 1\) and the elements of the pair \((\sigma; (i, i + 1)), (\sigma; (i, i + 1))\) are at the places \(l, l + 1\), in order to obtain the claim we can act with the sequence of moves
\[
\sigma'_{h-1}, \sigma'_{h-2}, \sigma''_{h-3}, \sigma'_{h-2}, \sigma''_{h-3}, \ldots, \sigma'_{l+1}, \sigma'_{l+2}, \sigma''_{l+1}.
\]
Now, we can bring the elements of the pair \((\sigma; (i, j)), (\sigma; (i, j))\) to the places \(n_2 - 1\) and \(n_2\) by using the sequence of moves \(\sigma'_{n_2-1}, \sigma'_{n_2-2}, \sigma''_{n_2-3}, \ldots, \sigma'_{n_2-1}, \sigma''_{n_2-2}\).
This ensures that acting by suitable elementary moves on the sequence \((\sigma; (1, 2)), (\sigma; (1, 2)), \ldots, (\sigma; (d - 1, d)), (\sigma; (d - 1, d))\) we can replace it with
\[
(\sigma; (1, 2)), (\sigma; (1, 2)), \ldots, (\sigma; (u - 2, u - 1)), (\sigma; (u - 2, u - 1)), (\sigma; (u, u + 1)),
\]
\[
(\sigma; (u, u + 1)), \ldots, (\sigma; (d - 1, d)), (\sigma; (d - 1, d)), (\sigma; (1, u)), (\sigma; (1, u))
\]
where \(u\) is an arbitrary index in \([1, \ldots, d]\). Let \((\vec{1}; id)\) be the element that occupies the place \(n_2 + 1\). Then, we choose \(u = v\) and we act with \(\sigma''_{n_2}\) in order to replace \((\vec{1}; id)\) with \((\vec{1}; id)\). Now, we move this element to the last place. If the elements \((\sigma; (u, u + 1)), (\sigma; (u, u + 1))\) are in the places \(h + 2, h + 3\), then we use the moves
\[
\sigma'_{n_2-2}, \sigma''_{n_2-2}, \sigma'_h, \sigma''_h, \sigma'_h, \sigma''_h, \sigma'_h, \sigma''_h, \sigma'_h, \sigma''_h
\]
in order to obtain again a sequence of the type
\[
((\sigma; (1, 2)), (\sigma; (1, 2)), \ldots, (\sigma; (u - 1, u)), (\sigma; (u - 1, u)), \ldots, (\sigma; (d - 1, d)), (\sigma; (d - 1, d))).
\]
Now, we can proceed as above for all the elements of type \((\vec{1}; id)\). In this way, we obtain the claim.

**Step 2.** By Step 1, \([\ell]\) is braid equivalent to a class of the form
\[
[(b_1; \bar{t}_1), \ldots, (b_{n_2}; \bar{t}_{n_2}), (\vec{1}; id), (\vec{1}; id), \ldots, (\vec{1}; id)].
\]
Now we claim that \([\ell]\) is braid equivalent to a class of type
\[
[(0; i_1), \ldots, (0; i_{n_2-d+1}), (\sigma; \bar{t}_{n_2-d+1}), \ldots, (\sigma; \bar{t}_{n_2}), (\vec{1}; id), \ldots, (\vec{1}; id)].
\]
Let \(i_1, i_2, \ldots, i_l\) be the indexes which \(b_1\) sends to \(\vec{1}\). We suppose that \(i_1 < i_2 < \cdots < i_{l-1} < i_l\). We notice that, by Step 1, we can assume that the element at the place \(n_2 + 1\) is \((\vec{1}; id)\). In fact, in order to obtain the claim it
is sufficient to choose \( u = i_l \). By using elementary moves of type \( \sigma_i'' \), we move \((\hat{1}; i_l; \text{id})\) to the place 2. We act two times with the moves \( \sigma_i'' \) and so we replace the pair \((b_1; \hat{1}_l), (\hat{1}_l; \text{id})\) with \((b_1; \hat{1}_l), (\hat{1}_l; i_l; \text{id})\) where \( b_1 \) is a function that sends to \( \hat{1} \) the indexes \( i_1, i_2, \ldots, i_{l-1}, i_l - 1 \). Here, \( i_l - 1 \) and \( i_l + 1 \) are, respectively, the index that precede and the index that follow \( i_l \) in \( \hat{1}_l \). Now, we move the element \((1_l, i_l; \text{id})\) to the place \( n_2 + 1 \). By Step 1, we can replace \((\hat{1}_n, i_l; \text{id})\) with \((1_l; i_l; \text{id})\).

Since \( b_1 \) is a function which sends to \( \hat{1} \) an even number of indexes (see Definition 2.1), acting as above, after a finite number of steps, we can replace the element \((b_1; \hat{1}_l)\) with \((0; \hat{1}_l)\).

We can proceed as done for \((b_1; \hat{1}_l)\), also for all the elements of type \((\ast; \hat{1}_j)\) with \( j = 2, \ldots, n_2 - 2(d - 1) \). In this way, we obtain the claim.

**Step 3.** By Step 2, \([\ast]\) is braid equivalent to the class

\[
[(0; \hat{1}_l), \ldots, (0; \hat{1}_{n_2 - 2(d - 1)}), (\ast; (1_2)), (\ast; (1_2)), \ldots, (\ast; (d - 1, d)), (\ast; (d - 1, d)), (\hat{1}_l; \text{id}), \ldots, (\hat{1}_l; \text{id})].
\]

Since \( n_l \) is even, one has

\[
(0; \hat{1}_l) \cdots (0; \hat{1}_{n_2 - 2(d - 1)})(\ast; (1_2)) \ast (1_2) \cdots (\ast; (d - 1, d))(\ast; (d - 1, d)) = (0; \text{id})
\]

From this it follows that the sequence \(((0; (1_2)), (\ast; (1_2)), \ldots, (\ast; (d - 1, d)), (\ast; (d - 1, d)))\) is equal to either

\[
((0; (1_2)), (0; (1_2)), \ldots, (0; (d_1 - 1, d)), (0; (d_1 - 1, d)))
\]

or

\[
((\hat{1}_l; (1_2)), (\hat{1}_l; (1_2)), \ldots, (\hat{1}_{d - 1 l}; (d_1 - 1, d)), (\hat{1}_{d - 1 l}; (d_1 - 1, d))).
\]

In the first case, we have the claim. So, we analyze the second case. We use the moves \( \sigma''_n, \sigma''_{n - 1}, \ldots, \sigma''_{n_2 - 2(d - 1) + 3} \) in order to shift one element of type \((1_l; \text{id})\) to the right of the pair \(((\hat{1}_l; (1_2)), (\hat{1}_l; (1_2)))\). We use the moves \( \sigma''_{n_2 - 2(d - 1) + 2}, \sigma''_{n_2 - 2(d - 1) + 2}, \sigma''_{n_2 - 2(d - 1) + 1} \) in order to replace the sequence \(((\hat{1}_l; (1_2)), (\hat{1}_l; (1_2)), (\hat{1}_l; \text{id}))\) with \(((0; (1_2)), (\hat{1}_l; (1_2)), (\hat{1}_l; \text{id}))\).

By using the moves

\[
\sigma''_{n_2 - 2(d - 1) + 3}, \sigma''_{n_2 - 2(d - 1) + 4}, \sigma''_{n_2 - 2(d - 1) + 5}, \sigma''_{n_2 - 2(d - 1) + 6}, \ldots, \sigma''_{n_2}, \sigma''_{n_2 + 1}
\]

we replace

\[
((\hat{1}_l; \text{id}), (\hat{1}_l; (2, 3)), (\hat{1}_l; (2, 3)), \ldots, (\hat{1}_{d - 1 l}; (d_1 - 1, d)), (\hat{1}_{d - 1 l}; (d_1 - 1, d)))
\]

with

\[
((0; (2, 3)), (\hat{1}_l; (2, 3)), \ldots, (0; (d_1 - 1, d)), (\hat{1}_{d - 1 l}; (d_1 - 1, d)), (\hat{1}_l; \text{id}))
\]

Now, we apply the sequence of moves \( \sigma''_{n_2}, \sigma''_{n_2 - 1}, \ldots, \sigma''_{n_2 - 2(d - 1) + 4}, \sigma''_{n_2 - 2(d - 1) + 5}, \sigma''_{n_2 - 2(d - 1) + 6}, \ldots, \sigma''_{n_2}, \sigma''_{n_2 + 1} \). In this way, we have that the above sequence is braid equivalent to

\[
((0; (1_2)), (1_l; \text{id}), (0; (2, 3)), (0; (2, 3)), \ldots, (0; (d_1 - 1, d)), (0; (d_1 - 1, d))).
\]

We obtain the claim by using the moves

\[
\sigma''_{n_2 - 2(d - 1) + 3}, \sigma''_{n_2 - 2(d - 1) + 4}, \ldots, \sigma''_{n_2 - 1}, \sigma''_{n_2}, \sigma''_{n_2 + 1}.
\]

\[ \square \]

The purpose of this paper is to show that the space \( H_{d_n, k_n, d_1, \ldots, d_2}^{W(B_l)}(Y) \) is irreducible. We notice that such space is smooth. So, if we prove that it is connected then we also prove that it is irreducible. Let

\[ \delta : H_{d_n, k_n, d_1, \ldots, d_2}^{W(B_l)}(Y) \to Y^{(m + n_2)} - \Delta \]
be the map which assigns to each equivalence class \([f \circ \pi]\) the branch locus of \(f \circ \pi\). The topology defined on \(H_{d,n,k;\lambda_o}^e(\mathfrak{S}_e' \cdots \mathfrak{S}_e')\) (Y) is such that \(\bar{6}\) is a topological covering map (see [8]). Therefore the braid group \(\pi_1(Y^{(m+n)} - \Delta, D)\) acts on \(A_{k,n,k;\lambda_o}^e(\mathfrak{S}_e' \cdots \mathfrak{S}_e')\). The orbits of this action are in one-to-one correspondence with the connected components of \(H_{d,n,k;\lambda_o}^e(\mathfrak{S}_e' \cdots \mathfrak{S}_e')(Y)\). So, if we prove that \(\pi_1(Y^{(m+n)} - \Delta, D)\) acts transitively on \(A_{k,n,k;\lambda_o}^e(\mathfrak{S}_e' \cdots \mathfrak{S}_e')\), then we also prove that \(H_{d,n,k;\lambda_o}^e(\mathfrak{S}_e' \cdots \mathfrak{S}_e')(Y)\) is connected. We notice that, in order to check the transitivity of this action, it is sufficient to prove that any class in \(A_{k,n,k;\lambda_o}^e(\mathfrak{S}_e' \cdots \mathfrak{S}_e')\) is braid equivalent to a given normal form. Hence, an immediate consequence of the previous proposition is the following theorem.

**Theorem 5.5.** If \(k > 3d - 3\), then the Hurwitz space \(H_{d,n,k;\lambda_o}^e(\mathfrak{S}_e' \cdots \mathfrak{S}_e')(P^1)\) is irreducible.

From Proposition 5.4, it follows also the following result.

**Theorem 5.6.** Let \(Y\) be a smooth, connected, projective complex curve of genus \(\geq 1\). If \(k > 3d - 3\), then the Hurwitz space \(H_{d,n,k;\lambda_o}^e(\mathfrak{S}_e' \cdots \mathfrak{S}_e')(Y)\) is irreducible.

**Proof.** In order to obtain the claim it is sufficient to prove that each equivalence class in \(A_{k,n,k;\lambda_o}^e(\mathfrak{S}_e' \cdots \mathfrak{S}_e')\) is braid equivalent to a class of the form \([I'; (0; id), (0; id), \ldots, (0; id), (0; id)]\). In fact, \([I']\) belongs to \(A_{k,n,k;\lambda_o}^\circ(\mathfrak{S}_e' \cdots \mathfrak{S}_e')\) and so the theorem follows by Proposition 5.4. Let \([I; \lambda, \mu]\) be an arbitrary transposition of \([1, \ldots, d]\). Using suitable elementary moves \(\sigma_i\), we shift on the right the elements of the form \((1; id)\). We act with elementary moves \(\omega_i\) in order to bring to the first place an element of type \((\ast; \eta)\), where \(\eta\) is a permutation with cycle type given by the partition \(e_1^\prime\) of \(d\). Now, we move to the places \(2, \ldots, k + 1\) the elements of type \((z_i; (i, j))\). In this way, we have that our class is braid equivalent to

\[
[I_1, \ldots, I_{n_2}, (\hat{1}; id), \ldots, (\hat{1}; id); \lambda_1, \mu_1, \ldots, \lambda_p, \mu_p]
\]

where \(I_i = (\ast; I_i')\), \(\lambda_k = (\ast; \lambda_k')\), \(\mu_k = (\ast; \mu_k')\), \(I_i' = \eta\) and \(I_2', \ldots, I_{n_2}'\) are transpositions.

We observe that the condition \(k > 3d - 3\) ensures that \(k + |I'| \geq 2d\). So, by Lemma 5.1, we have that the sequence of permutations \((\eta, I_2', \ldots, I_{n_2}')\) is braid equivalent to a sequence of type \((\eta', I_2', \ldots, I_{n_2}')\) where \(\eta'\) has cycle type \(e_1', e_2', \ldots, e_{k_1}'\) are transpositions, \(I_k'' = I_{n_2}''\) and

\[
(\eta', I_2', \ldots, I_{n_2}') = (\eta', I_2', \ldots, I_{n_2}').
\]

Now, we notice that \((\eta', I_2', \ldots, I_{n_2}'; \lambda_1', \mu_1', \ldots, \lambda_p', \mu_p')\) is the Hurwitz system of a degree \(d\) branched covering of \(Y\) with monodromy group \(S_d\). So, by Lemma 5.2, it is braid equivalent to a system of type \((\ast, v; \ldots; \lambda_1', \mu_1'; \ldots, \lambda_p', \mu_p')\) where \(v\) is an arbitrary transposition of \(S_d\). From this, it follows that our class is braid equivalent to a class of type

\[
[\ldots, (\ast; v), (\ast; v), \ldots, (\hat{1}; id), \ldots; \lambda_1, \mu_1, \ldots, \lambda_p, \mu_p].
\]

Now, in order to obtain the claim it is sufficient to choose \(v = (i, h)\), to move one element of type \((\ast; v)\) to the place \(n_2\) and to act with \(\sigma_n\).

**Step 2.** Now, we claim that \([I; \lambda, \mu]\) is braid equivalent to a class of type \([I'; (0; id), (0; id), \ldots, (0; id), (0; id)]\).

Acting by suitable elementary moves \(\sigma_i\) we have that our class is braid equivalent to

\[
[I_1, \ldots, I_{n_2}, (\hat{1}; id), \ldots, (\hat{1}; id); \lambda_1, \mu_1, \ldots, \lambda_p, \mu_p]
\]

where \(I_i = (\ast; I_i)\), \(\lambda_k = (\ast; \lambda_k')\) and \(\mu_k = (\ast; \mu_k')\).
We notice that \((l_1, \ldots, l_n; \lambda_1', \mu_1', \ldots, \lambda_g', \mu_g')\) is the Hurwitz system of a degree \(d \geq 3\) covering of \(Y\), with monodromy group \(S_d\) and with \(n_2\) branch points, \(k\) of which are simple points and \(n_2 - k\) of which are special points. Moreover, \(q_i\) among of the special points have local monodromies with cycle type given by the partition \(e_i^j\) of \(d\). Since, under the condition \(k > 3d - 3\), the Hurwitz space parameterizing coverings as above is irreducible (see [27], Theorem 2), the Hurwitz system \((l_1, \ldots, l_n; \lambda_1', \mu_1', \ldots, \lambda_g', \mu_g')\) is braid equivalent to a system of type

\[(l_1, \ldots, l_n; \lambda_1; \mu_1, \ldots, \mu_g)\]

Hence, \([\lambda_1; \mu_1, \ldots, \mu_g]\) is braid equivalent to a class of the form

\[[l_1, \ldots, l_n, (\lambda_1; \mu_1), \ldots; (a_1; \mu_1), (b_1; \mu_1), \ldots, (a_g; \mu_g), (b_g; \mu_g)]\].

We notice that if \(a_i = 0\) and \(b_v = 0\) for each \(1 \leq s, v \leq g\) we have the claim. So, let \(a_1 \neq 0\) and \(i\) be one of the indexes that \(a_1\) sends to \(\bar{1}\).

By Step 1, \([l_1, \ldots, l_n, (\lambda_1; \mu_1), \ldots; (a_1; \mu_1), (b_1; \mu_1), \ldots, (a_g; \mu_g), (b_g; \mu_g)]\) is braid equivalent to the class

\[[\ldots, (\lambda_1; \mu_1), \ldots; (a_1; \mu_1), (b_1; \mu_1), \ldots, (a_g; \mu_g), (b_g; \mu_g)]\].

Acting with elementary moves \(\sigma''\) we bring to the first place the element \((\lambda_1; \mu_1)\) and then we use the move \(\tau_{11}^{-1}\) to replace \((a_1; \mu_1)\) with \((\lambda_1; \mu_1)(a_1; \mu_1)\) where \(\bar{1} + a_1\) is a function that sends \(i\) to \(0\).

So reasoning for all the indexes that \(a_1\) sends to \(\bar{1}\), after a finite number of steps, we obtain that our class is braid equivalent to

\[[\ldots; (0; \mu_1), (b_1; \mu_1), \ldots, (a_g; \mu_g), (b_g; \mu_g)]\].

If \(a_1 = 0\), \(b_1 \neq 0\) and \(b_1\) sends \(i\) to \(\bar{1}\), we again use elementary moves of type \(\sigma''\) to shift \((\lambda_1; \mu_1)\) to the first place. We act by the braid move \(\rho_{11}^{-1}\), and so we transform \((b_1; \mu_1)\) into \((\lambda_1; \mu_1)(b_1; \mu_1)\) where the function \(\bar{1} + b_1\) sends \(i\) to \(0\). Following this line for all the indexes that \(b_1\) sent to \(\bar{1}\), we can replace our class with

\[[\ldots; (0; \mu_1), (0; \mu_1), \ldots, (a_g; \mu_g), (b_g; \mu_g)]\].

We notice that if \(a_s \neq 0\) and \(a_t = b_t = 0\), for each \(l \leq s - 1\), in order to obtain the claim one can reason in the same way but this time applying the braid move \(\tau_{1n}^{-1}\). Analogously if \(b_s \neq 0\), \(a_t = b_t = 0\), for each \(l \leq s - 1\), and \(a_s = 0\) one can apply the braid move \(\rho_{1n}^{-1}\) to transform \((b_t; \mu_1)\) into \((0; \mu_1)\). \(\square\)

References