Almost Increasing Sequences and Their New Applications II

Hüseyin Bor

P. O. Box 121, TR-06502 Bahaçelievler, Ankara, Turkey

Abstract. In this paper, we generalize a known theorem dealing with \(|C,\alpha|_k\) summability factors to the \(|C,\alpha,\beta;\delta|_k\) summability factors of infinite series. This theorem also includes some known and new results.

1. Introduction

A positive sequence \((b_n)\) is said to be an almost increasing sequence sequence if there exists a positive increasing sequence \((c_n)\) and two positive constants \(A\) and \(B\) such that \(Ac_n \leq b_n \leq Bc_n\) (see [1]). Let \(\sum a_n\) be a given infinite series with the sequence of partial sums \((s_n)\). We denote by \(t_{n}^{\alpha,\beta}\) the \(n\)th Cesàro mean of order \((\alpha,\beta)\), with \(\alpha + \beta > -1\), of the sequence \((na_n)\), that is (see [5])

\[
t_{n}^{\alpha,\beta} = \frac{1}{A_{n+1}^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_{v}^{\beta} a_{v},
\]

where

\[
A_{n}^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_{0}^{\alpha+\beta} = 1 \quad \text{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad \text{for} \quad n > 0.
\]

The series \(\sum a_n\) is said to be summable \(|C,\alpha,\beta;\delta|_k\), \(k \geq 1\) and \(\delta \geq 0\), if (see [3])

\[
\sum_{n=1}^{\infty} n^{\delta k-1} |t_{n}^{\alpha,\beta}|^k < \infty.
\]

If we take \(\delta = 0\), then \(|C,\alpha,\beta;\delta|_k\) summability reduces to \(|C,\alpha,\beta|_k\) summability (see [6]). If we set \(\beta = 0\) and \(\delta = 0\), then \(|C,\alpha,\beta,\delta|_k\) summability reduces to \(|C,\alpha|_k\) summability (see [7]). Also, if we take \(\beta = 0\), then we get \(|C,\alpha;\delta|_k\) summability (see [8]).

2. Known result

The following theorems are known dealing with an application of almost increasing sequences.

**Theorem 2.1**[11] Let \((\varphi_n)\) be a positive sequence and \((X_n)\) be an almost increasing
sequence. If the conditions

\[ \sum_{n=1}^{\infty} n \left| \Delta^2 \lambda_n \right| X_n < \infty, \]  

\[ |\lambda_n| X_n = O(1) \text{ as } n \to \infty, \]  

\[ \varphi_n = O(1) \text{ as } n \to \infty, \]  

\[ n \Delta \varphi_n = O(1) \text{ as } n \to \infty, \]  

are satisfied, then the series \( \sum a_n \lambda_n \varphi_n \) is summable \( |C, 1|_k, k \geq 1 \).

**Theorem 2.2** [4] Let \((\varphi_n)\) be a positive sequence and let \((X_n)\) be an almost increasing sequence. If the conditions (4), (5), (6) and (7) are satisfied and the sequence \((w^\alpha_n)\) defined by (see [10])

\[ w^\alpha_n = \begin{cases} \left| t^\alpha_n \right|, & \alpha = 1, \\ \max_{1 \leq v \leq n} \left| t^\alpha_v \right|, & 0 < \alpha < 1 \end{cases} \]  

satisfies the condition

\[ \sum_{n=1}^{\infty} \frac{|w^\alpha_n|^k}{vX_v^k} = O(X_n) \text{ as } n \to \infty, \]  

then the series \( \sum a_n \lambda_n \varphi_n \) is summable \( |C, \alpha, \beta|_k, 0 < \alpha \leq 1, (\alpha - 1)k > -1 \) and \( k \geq 1 \).

**Remark 2.3** It should be noted that if we take \( \alpha = 1 \), then we get Theorem 2.1. In this case, condition (10) reduces to condition (8) and the condition \('(\alpha - 1)k > -1'\) is trivial.

### 3. The main result

The aim of this paper is to generalize Theorem 2.2 in the following form:

**Theorem 3.1** Let \((\varphi_n)\) be a positive sequence and let \((X_n)\) be an almost increasing sequence. If the conditions (4), (5), (6) and (7) are satisfied and the sequence \((w^\alpha_\beta_n)\) defined by

\[ (w^\alpha_\beta_n) = \begin{cases} \left| t^\alpha_\beta_n \right|, & \alpha = 1, \beta > -1 \\ \max_{1 \leq v \leq n} \left| t^\alpha_\beta_v \right|, & 0 < \alpha < 1, \beta > -1 \end{cases} \]  

satisfies the condition

\[ \sum_{n=1}^{\infty} \frac{|w^\alpha_\beta_n|^k}{vX_v^k} = O(X_n), \text{ as } n \to \infty, \]  

then the series \( \sum a_n \lambda_n \varphi_n \) is summable \( |C, \alpha, \beta, \delta|_k, 0 < \alpha \leq 1, \delta \geq 0, (\alpha + \beta - \delta - 1)k > -1 \), and \( k \geq 1 \).
We need the following lemmas for the proof of our theorem.

**Lemma 3.2** [2]) If $0 < \alpha \leq 1$, $\beta > -1$ and $1 \leq v \leq n$, then

$$\left| \sum_{p=0}^{n} A_{n-p}^{\alpha-1} a_{p} \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_{p} \right|. \quad (13)$$

**Lemma 3.3** [9]) Under the conditions (4) and (5), we have that

$$nX_n |\Delta \lambda_n| = O(1) \text{ as } n \to \infty, \quad (14)$$

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty. \quad (15)$$

**4. Proof of Theorem 3.1** Let $(T_{n}^{\alpha, \beta})$ be the $n$th $(C, \alpha, \beta)$ mean, with $0 < \alpha \leq 1$ and $\beta > -1$, of the sequence $(n a_n \lambda_n q_n)$. Then, by (1), we have that

$$T_{n}^{\alpha, \beta} = \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_{v}^{\beta} a_{v} \lambda v q_{v}. \quad (16)$$

Thus, applying Abel's transformation first and then using Lemma 3.2, we have that

$$T_{n}^{\alpha, \beta} = \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta (\lambda v q_{v}) \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_{p}^{\beta} a_{p} + \frac{\lambda v q_{v}}{A_{n}^{\alpha+\beta}} \sum_{p=1}^{n} A_{n-p}^{\alpha-1} A_{p}^{\beta} a_{v}. \quad \text{ (17)}$$

$$|T_{n}^{\alpha, \beta}| \leq \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n-1} \left| \Delta (\lambda v q_{v}) \right| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_{p}^{\beta} \left| a_{p} \right| + \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n-1} \left| \Delta \lambda v \right| \sum_{p=1}^{n} A_{n-p}^{\alpha-1} A_{p}^{\beta} \left| a_{v} \right| \quad \text{ (18)}$$

$$+ \frac{\left| \lambda v q_{v} \right|}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_{v}^{\beta} \left| a_{v} \right| \quad \text{ (19)}$$

$$\leq \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n-1} A_{v}^{\beta} \omega v^{\beta} \left| \lambda v \right| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_{p}^{\beta} \left| a_{p} \right| + \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n-1} A_{v}^{\alpha+\beta} \omega v^{\alpha+\beta} \left| \lambda v \right| \sum_{p=1}^{n} A_{n-p}^{\alpha-1} A_{p}^{\beta} \left| a_{v} \right| \quad \text{ (20)}$$

$$= T_{n1}^{\alpha, \beta} + T_{n2}^{\alpha, \beta} + T_{n3}^{\alpha, \beta}. \quad \text{ (21)}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-1} |T_{n,r}^{\alpha, \beta}| < \infty, \text{ for } r = 1, 2, 3. \quad \text{ (22)}$$
When $k > 1$, we can apply Hölder’s inequality with indices $k$ and $k'$, where $\frac{1}{k} + \frac{1}{k'} = 1$, we get

$$\sum_{n=2}^{m+1} n^{\delta k - 1} T_{m,n}^{\alpha,\beta,k} \leq \sum_{n=2}^{m+1} n^{\delta k - 1} (A_n^{\alpha,\beta})^{-k} \left\{ \sum_{v=1}^{n-1} A_v^{\alpha,\beta} w_v^{\alpha,\beta} |\Delta \varphi_v| |\lambda_v| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+(\alpha+\beta-\delta)k}} \sum_{v=1}^{n-1} (w_v^{\alpha,\beta} k |\Delta \varphi_v| |\lambda_v|) \left\{ \sum_{v=1}^{n-1} 1 \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{2+(\alpha+\beta-\delta)k}} \sum_{v=1}^{n-1} (w_v^{\alpha,\beta} k |\Delta \varphi_v| |\lambda_v|) \int_{c}^{\infty} \frac{dx}{x^{2+(\alpha+\beta-\delta)k}}$$

$$= O(1) \sum_{v=1}^{m} v^{\alpha,\beta} k |\lambda_v| |\Delta \lambda_v| \left\{ \sum_{v=1}^{n-1} \frac{w_v^{\alpha,\beta} k |\Delta \varphi_v| |\lambda_v|}{v X_v^{k-1}} \right\}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3. Again, we get that

$$\sum_{n=2}^{m+1} n^{\delta k - 1} T_{m,n}^{\alpha,\beta,k} \leq \sum_{n=2}^{m+1} n^{\delta k - 1} (A_n^{\alpha,\beta})^{-k} \left\{ \sum_{v=1}^{n-1} A_v^{\alpha,\beta} w_v^{\alpha,\beta} |\varphi_{v+1}| |\Delta \lambda_v| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+(\alpha+\beta-\delta)k}} \sum_{v=1}^{n-1} (w_v^{\alpha,\beta} k |\varphi_{v+1}| |\Delta \lambda_v|) \left\{ \sum_{v=1}^{n-1} 1 \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{2+(\alpha+\beta-\delta)k}} \sum_{v=1}^{n-1} (w_v^{\alpha,\beta} k |\varphi_{v+1}| |\Delta \lambda_v|) \int_{c}^{\infty} \frac{dx}{x^{2+(\alpha+\beta-\delta)k}}$$

$$= O(1) \sum_{v=1}^{m} v^{\alpha,\beta} k |\lambda_v| |\Delta \lambda_v| \left\{ \sum_{v=1}^{n-1} \frac{w_v^{\alpha,\beta} k |\varphi_{v+1}| |\Delta \lambda_v|}{v X_v^{k-1}} \right\}$$
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\[ = O(1) \sum_{v=1}^{m} v|\Delta \lambda_v|^{(1)} + O(1) m|\Delta \lambda_m| \sum_{v=1}^{m} v|\Delta \lambda_v|^{(1)} \]

\[ = O(1) \sum_{v=1}^{m} v|\Delta \lambda_v|^{(1)} + O(1) m|\Delta \lambda_m| \sum_{v=1}^{m} v|\Delta \lambda_v|^{(1)} \]

by hypotheses of Theorem 3.1 and Lemma 3.3. Finally, as in $\tau_{n,1}^{a,b}$, we have that

\[ \sum_{n=1}^{m} n^{|\lambda_v|^k} = \sum_{n=1}^{m} n^{|\lambda_v|^k} = O(1) \sum_{n=1}^{m} n^{|\lambda_v|^k} \]

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3. This completes the proof of Theorem 3.1. It should be noted that, if we take $\beta=0$, $\delta=0$ and $\alpha=1$, then we get Theorem 2. 1. If we set $\delta=0$, then we get a result concerning the $|C,\alpha,\beta|$ summability factors of infinite series. Also, if we take $\beta=0$ and $\delta=0$, then we obtain Theorem 2. 2. Finally, if we take $k=1$, $\delta=0$ and $\beta=0$, then we get a new result dealing with the $|C,\alpha|$ summability factors of infinite series.

References