Some Families of Differential Equations Associated with the Hermite-Based Appell Polynomials and Other Classes of Hermite-Based Polynomials

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\[ G(x, y, z; t) := A(t) \cdot \exp(xt + yt^2 + zt^3) = \sum_{n=0}^{\infty} \frac{H_{A_n}(x, y, z) t^n}{n!} \]
and investigated their many interesting properties and characteristics by using operational techniques combined with the principle of monomiality. Here, in this paper, we find the differential, integro-differential and partial differential equations for the Hermite-based Appell polynomials via the factorization method. Furthermore, we derive the corresponding equations for the Hermite-based Bernoulli polynomials and the Hermite-based Euler polynomials. We also indicate how to deduce the corresponding results for the Hermite-based Genocchi polynomials from those involving the Hermite-based Euler polynomials.

1. Introduction, Definitions and Preliminaries

A polynomial set \( \{p_n(x)\}_{n=0}^{\infty} \) is called quasi-monomial if there exist two operators \( \hat{P} \) and \( \hat{M} \), independent of \( n \), such that
\[ \hat{M} [p_n(x)] = p_{n+1}(x) \]
and
\[ \hat{P} [p_n(x)] = np_{n-1}(x) \quad (p_0(x) := 1; \ p_{-1}(x) := 0), \]

\textsuperscript{2010} Mathematics Subject Classification. Primary 33C45, 33C55, 34A35; Secondary 11B68, 33C99

Keywords. Appell and Hermite polynomials, Bernoulli and Euler polynomials, Hermite-based Appell polynomials, Hermite-based Bernoulli polynomials, Hermite-based Euler polynomials, Hermite-based Genocchi polynomials, Partial differential equations, Monomiality principle, Quasi-monomial polynomials

Received: 26 June 2013; Accepted: 18 July 2013
Communicated by Dragan S. Djordjević

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where it is assumed (as usual) that
\[ p_0(x) := 1 \quad \text{and} \quad p_{-1}(x) := 0. \]

The operators \( \hat{M} \) and \( \hat{P} \) are, respectively, called the raising and the lowering operators acting on the polynomials \( p_n(x) \). These operators satisfy the following commutation relation:
\[ [\hat{P}, \hat{M}] = I, \]
where \( I \) denotes the identity operator. Thus, clearly, the operators \( \hat{M} \) and \( \hat{P} \) display a Weyl group structure.

Many of the properties of the polynomials \( p_n(x) \) can be obtained by using the operators \( \hat{M} \) and \( \hat{P} \).

If the operators \( \hat{M} \) and \( \hat{P} \) possess a differential character, then the polynomials \( p_n(x) \) satisfy the following differential equation:
\[ \hat{M} \hat{P} \{ p_n(x) \} = np_n(x). \]

Moreover, since \( p_0(x) := 1 \), the polynomial set \( \{ p_n(x) \}_{n=0}^{\infty} \) can be constructed explicitly through the action of the operator \( \hat{M} \) on \( p_0(x) \) as follows:
\[ p_n(x) = \hat{M}^n[1] \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}), \]
where \( \mathbb{N} \) denotes (as usual) the set of positive integers. Several recent works dealing extensively with the quasi-monomiality principle include (for example) \([3],[4],[5],[6],[7],[8]\) and \([16]\).

A polynomial set \( \{ A_n(x) \}_{n=0}^{\infty} \) is called an Appell set of polynomials (see, for details, \([19, p. 398, Problem 28]\); see also \([1]\) and the recent work \([14]\) and the references cited therein) if
\[ \frac{d}{dx} [A_n(x)] = n A_{n-1}(x) \quad (n \in \mathbb{N}_0; \ A_{-1}(x) := 0) \]
or, equivalently, if
\[ A(t) \cdot e^x = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}, \quad (1) \]
where
\[ A(t) = \sum_{n=0}^{\infty} a_n t^n \quad (a_0 \neq 0). \quad (2) \]

The familiar three-variable Hermite polynomials \( \{ H_n(x, y, z) \}_{n=0}^{\infty} \) generated by
\[ \exp(x t + y t^2 + z t^3) = \sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!}, \quad (3) \]
are quasi-monomials under the action of the operators \( \hat{M} \) and \( \hat{P} \) given by
\[ \hat{M} := x + 2y \frac{\partial}{\partial x} + 3z \frac{\partial^2}{\partial x^2} \quad \text{and} \quad \hat{P} := \frac{\partial}{\partial x}, \quad (4) \]
satisfy the following differential equation:
\[ \left( 3z \frac{\partial^3}{\partial x^3} + 2y \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} - n \right) [H_n(x, y, z)] = 0 \quad (5) \]
and possess the following operational representation:

\[ H_n(x, y, z) = \exp \left( y \frac{\partial^2}{\partial x^2} + z \frac{\partial^3}{\partial x^3} \right) \{x^n\}. \]  

(6)

Recently, by suitably combining the generating functions (1) and (3), Khan et al. [11] defined the Hermite-based Appell polynomials \( h_A_n(x, y, z) \) by means of the following generating function:

\[ G(x, y, z; t) := A(t) \cdot \exp \left( t \hat{M} \right) [1] = \sum_{n=0}^{\infty} h_A_n(x, y, z) \frac{t^n}{n!}, \]

(7)

where the operator \( \hat{M} \) is defined in (4) and the power series \( A(t) \) is given by (2). In fact, with the aid of the Berry decoupling identity:

\[ e^{\hat{A}+\hat{B}} = \exp \left( \frac{m^2}{12} \right) \cdot \exp \left( -\frac{m}{2} \hat{A}^2 + \hat{A} \right) e^\hat{B} \quad (\{\hat{A}, \hat{B}\} = m\hat{A}^2), \]

(8)

the generating function (7) of the Hermite-based Appell polynomials \( h_A_n(x, y, z) \) can be rewritten in the following form [11, p. 759, Eq. (2.3)]:

\[ G(x, y, z; t) := A(t) \cdot \exp \left( xt + yt^2 + zt^3 \right) = \sum_{n=0}^{\infty} h_A_n(x, y, z) \frac{t^n}{n!}. \]

(9)

Some important examples of the Appell polynomials \( A_n(x) \) defined by the generating function (1) include the classical Bernoulli polynomials \( B_n(x) \), the classical Euler polynomials \( E_n(x) \) and the classical Genocchi polynomials \( G_n(x) \), together with their familiar generalizations \( \hat{B}_n^{(\alpha)}(x) \), \( E_n^{(\alpha)}(x) \) and \( G_n^{(\alpha)}(x) \) of (real or complex) order \( \alpha \) which are usually defined by means of the following generating functions (see, for details, [9, Vol. III, p. 253 et seq.,] [12, Section 2.8] and [17, p. 61 et seq.]; see also [15], [18, p. 81 et seq.] and [20] and the references to several related earlier works cited therein):

\[ \left( \frac{t}{e^t - 1} \right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!} \quad (|t| < 2\pi; \; 1^\alpha := 1), \]

(10)

\[ \left( \frac{2}{e^t + 1} \right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!} \quad (|t| < \pi; \; 1^\alpha := 1), \]

(11)

and

\[ \left( \frac{2t}{e^t + 1} \right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{t^n}{n!} \quad (|t| < \pi; \; 1^\alpha := 1), \]

(12)

so that, obviously, the classical Bernoulli polynomials \( B_n(x) \), the classical Euler polynomials \( E_n(x) \) and the classical Genocchi polynomials \( G_n(x) \) are given, respectively, by

\[ B_n(x) := B_n^{(1)}(x), \quad E_n(x) := E_n^{(1)}(x) \quad \text{and} \quad G_n(x) := G_n^{(1)}(x) \quad (n \in \mathbb{N}_0). \]

(13)

For the classical Bernoulli numbers \( B_n \), the classical Euler numbers \( E_n \) and the classical Genocchi numbers \( G_n \) of order \( n \), we have
\[ B_n := B_n(0) = B_n^{(1)}(0), \quad E_n := 2^n E_n \left( \frac{1}{2} \right) = 2^n E_n^{(1)} \left( \frac{1}{2} \right) \]

and \[ G_n := G_n(0) = G_n^{(1)}(0) \quad (n \in \mathbb{N}_0), \]

respectively.

The specific choice of the power series \( A(t) \) in each of these examples becomes obvious when we compare the generating function (1) with the generating functions (10), (11) and (12) and their special cases when \( \alpha = 1 \). We choose here to introduce the Hermite-based Bernoulli polynomials \( \mathcal{H}B_n(x, y, z) \), the Hermite-based Euler polynomials \( \mathcal{H}E_n(x, y, z) \) and the Hermite-based Genocchi polynomials \( \mathcal{H}G_n(x, y, z) \) by means of the following generating functions:

\[ \left( \frac{t}{e^{t} - 1} \right) \exp \left( xt + yt^2 + zt^3 \right) = \sum_{n=0}^{\infty} \mathcal{H}B_n(x, y, z) \frac{t^n}{n!} \quad (|t| < 2\pi), \]  

\[ \left( \frac{2}{e^{t} + 1} \right) \exp \left( xt + yt^2 + zt^3 \right) = \sum_{n=0}^{\infty} \mathcal{H}E_n(x, y, z) \frac{t^n}{n!} \quad (|t| < \pi), \]

and

\[ \left( \frac{2t}{e^{t} + 1} \right) \exp \left( xt + yt^2 + zt^3 \right) = \sum_{n=0}^{\infty} \mathcal{H}G_n(x, y, z) \frac{t^n}{n!} \quad (|t| < \pi), \]

respectively.

In their special cases when \( z = 0 \), the generating functions (15), (16) and (17) would reduce immediately to the generating functions of the two-dimensional Bernoulli, Euler and Genocchi polynomials. In particular, the special case of the generating function (15) when \( z = 0 \) was investigated by Bretti and Ricci [2] who also derived the differential, integro-differential and partial differential equations of the two-dimensional Appell polynomials (see, for details, [2]). The above-mentioned investigation for the extended two-dimensional Appell polynomials was presented by Yılmaz and Özarslan [21]. Earlier, in the year 2002, He and Ricci [10] made use of the factorization method in order to derive the differential equations for the one-variable Appell polynomials. Moreover, the Hermite-based Apostol-Bernoulli polynomials, the Hermite-based Apostol-Euler polynomials and the Hermite-based Apostol-Genocchi polynomials, as well as their unification, were introduced and studied recently by Özarslan [13].

The main idea of the so-called factorization method is to find the lowering operator \( L^- \) and the raising operators \( L^+ \) and then use such relationships as follows:

\[ L^-_{n+1} L^+_{n} \{ \mathcal{H}A_n(x, y, z) \} = \mathcal{H}A_n(x, y, z). \]  

The object of this paper is to find the differential, integro-differential and partial differential equations for the Hermite-based Appell polynomials \( \mathcal{H}A_n(x, y, z) \) defined by (9) via the factorization method. We also derive the corresponding equations for the Hermite-based Bernoulli polynomials \( \mathcal{H}B_n(x, y, z) \) defined by (15) and the Hermite-based Euler polynomials \( \mathcal{H}E_n(x, y, z) \) defined by (16). It is fairly straightforward to obtain, in a similar manner, the analogous results for the Hermite-based Genocchi polynomials \( \mathcal{H}G_n(x, y, z) \) defined by (17). Alternatively, of course, one can freely use the following relationships:

\[ \mathcal{H}G_n(x, y, z) = n \mathcal{H}G_{n-1}(x, y, z) \]

and

\[ \mathcal{H}E_n(x, y, z) = \left( \frac{1}{n + 1} \right) \mathcal{H}G_{n+1}(x, y, z), \]
In Section 3, we find the differential Appell polynomials $H_n(x, y, z)$ defined by (15) and the three-dimensional Hermite-based Euler polynomials $E_n(x, y, z)$ defined by (16) and the raising operators for the three-dimensional Hermite-based Bernoulli polynomials $B_n(x, y, z)$ defined by (15) and the three-dimensional Hermite-based Euler polynomials $E_n(x, y, z)$ defined by (16).

In this section, we begin by deriving the recurrence relations and the shift operators for the Hermite-based Appell polynomials $H_n(x, y, z)$ and also list the corresponding results for the Hermite-Bernoulli polynomials $B_n(x, y, z)$ and the Hermite-Euler polynomials $E_n(x, y, z)$ as their special cases.

2. Recurrence Relations and Shift Operators

In this section, we begin by deriving the recurrence relations and the shift operators for the Hermite-based Appell polynomials $H_n(x, y, z)$ defined by the generating function (9).

**Theorem 1.** The Hermite-based Appell polynomials $H_n(x, y, z)$ defined by the generating function (9) satisfy the following recurrence relations:

$$H_{n+1}(x, y, z) = (x + \alpha_0) H_n(x, y, z) + \sum_{k=1}^{n} \binom{n}{k} \alpha_k H_{n-k}(x, y, z) + 2ny H_{n-1}(x, y, z) + 3n(n - 1)z H_{n-2}(x, y, z),$$

(21)

where

$$H_{-1}(x, y, z) := 0 \quad \text{and} \quad H_{-2}(x, y, z) := 0$$

(22)

and the coefficients $\{\alpha_k\}_{k \in \mathbb{N}_0}$ are given by the following expansion:

$$A'(t) = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}$$

(23)

The shift operators are given by

$$z L_n^- := \frac{1}{n} D_x,$$

(24)

$$y L_n^- := \frac{1}{n} D_y^{-1} D_y,$$

(25)

$$z L_n^- := \frac{1}{n} D_z^{-2} D_z,$$

(26)

$$z L_n^+ := x + \alpha_0 + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^k + 2y D_x + 3z D_x^2,$$

(27)

$$y L_n^+ := x + \alpha_0 + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_y D_x^k + 2y D_y^{-1} D_y + 3z D_y^{-2} D_y^2 + 3z D_x^{-2} D_x^2,$$

(28)

and

$$z L_n^+ := x + \alpha_0 + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_z^{-2} D_x^k + 2y D_x^{-2} D_z + 3z D_x^{-4} D_z^2,$$

(29)

where

$$D_x := \frac{\partial}{\partial x}, \quad D_y := \frac{\partial}{\partial y}, \quad D_z := \frac{\partial}{\partial z} \quad \text{and} \quad D_x^{-1} := \int_{0}^{x} f(\xi) d\xi.$$
Proof. Upon differentiating both sides of the generating relation (9) with respect to \( t \), we have

\[
\frac{\partial}{\partial t} [\mathcal{G}(x, y, z; t)] = \mathcal{G}(x, y, z; t) \left( \frac{A'(t)}{A(t)} + x + 2yt + 3zt^2 \right). \tag{31}
\]

We now substitute the corresponding series forms for \( \mathcal{G}(x, y, z; t) \) from (9) and for the quotient:

\[
\frac{A'(t)}{A(t)}
\]

from (23). By equating the coefficients of \( t^n \) in the equation resulting from (31), we obtain the recurrence relation (21) asserted by Theorem 1.

We now use this recurrence relation (21) to find the shift operators \( _x \mathcal{L}_n^+ \), \( _y \mathcal{L}_n^+ \) and \( _z \mathcal{L}_n^+ \) with respect to \( x \), \( y \) and \( z \). First of all, in order to find the shift operator \( _x \mathcal{L}_n^+ \), we differentiate both sides of the generating relation (9) with respect to \( x \) and equate the coefficients of \( t^n \), so that we have

\[
\frac{\partial}{\partial x} \left[ _h A_n(x, y, z) \right] = n \ _h A_{n-1}(x, y, z).
\]

Thus, clearly, the operator given by (24) satisfies the following relation:

\[
_x \mathcal{L}_n^+ \left[ _h A_n(x, y, z) \right] = _h A_{n-1}(x, y, z).
\]

By taking the derivative with respect to \( y \) in the generating relation (9), we have

\[
\frac{\partial}{\partial y} \left[ _h A_n(x, y, z) \right] = n(n - 1) \ _h A_{n-2}(x, y, z) = n \frac{\partial}{\partial x} \left[ _h A_{n-1}(x, y, z) \right],
\]

so that

\[
D_x^{n-1} D_y \left[ _h A_n(x, y, z) \right] = n \ _h A_{n-1}(x, y, z),
\]

and, therefore, we get (25).

Upon differentiating both sides of the generating relation (9) with respect to \( z \), we have

\[
\frac{\partial}{\partial z} \left[ _h A_n(x, y, z) \right] = n(n - 1)(n - 2) \ _h A_{n-3}(x, y, z) = n \frac{\partial^2}{\partial x^2} \left[ _h A_{n-1}(x, y, z) \right],
\]

so that

\[
D_x^{n-2} D_z \left[ _h A_n(x, y, z) \right] = n \ _h A_{n-1}(x, y, z),
\]

which yields (26).

Next, in order to obtain the raising operator \( _x \mathcal{L}_n^+ \), we use the following relations:

\[
_h A_{n-k}(x, y, z) = \left( _x \mathcal{L}_{n-k+1}^+ _x \mathcal{L}_{n-k+2}^+ \cdots _x \mathcal{L}_{n-k-1}^+ _x \mathcal{L}_n^+ \right) \left[ _h A_n(x, y, z) \right]
\]

\[
= \frac{(n - k)!}{n!} D_x^n \left[ _h A_n(x, y, z) \right],
\]

\[
_h A_{n-1}(x, y, z) = \ _x \mathcal{L}_n^+ \left[ _h A_n(x, y, z) \right]
\]

\[
= \frac{1}{n} D_x \left[ _h A_n(x, y, z) \right]
\]
and

\[ H^{n-2}(x, y, z) = \left( z^n \mathcal{L}_n \right) [H^{n}(x, y, z)] \]

\[ = \frac{1}{n(n-1)} D_x^2 [H^{n}(x, y, z)]. \]

By substituting from these relations into the recurrence relation (21), we have

\[ H^{n+1}(x, y, z) = \left( x + a_0 + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^k + 2yD_x + 3zD_x^2 \right) [H^{n}(x, y, z)]. \]

which yields the raising operator (27).

To obtain the raising operator (28), we use the following relations:

\[ H^{n+1}(x, y, z) = \left( x^n \mathcal{L}_n \right) [H^{n}(x, y, z)] \]

\[ = \frac{(n-1)!}{n!} D_x^{n+1} [H^{n}(x, y, z)]. \]

\[ H^{n-1}(x, y, z) = (x^{n-1}) [H^{n}(x, y, z)] \]

\[ = \frac{1}{n} D_x^{n-1} [H^{n}(x, y, z)] \]

and

\[ H^{n-2}(x, y, z) = (x^{n-2}) [H^{n}(x, y, z)] \]

\[ = \frac{1}{n(n-1)} D_x^{n-2} [H^{n}(x, y, z)]. \]

Upon substituting from these relations into the recurrence relation (21), we get

\[ H^{n+1}(x, y, z) = \left( x + a_0 + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^k + 2yD_x + 3zD_x^2 \right) [H^{n}(x, y, z)], \]

which leads us to the raising operator (28).

The derivation of the raising operator (29) would similarly make use of the following relations:

\[ H^{n+1}(x, y, z) = \left( x^n \mathcal{L}_n \right) [H^{n}(x, y, z)] \]

\[ = \frac{(n-1)!}{n!} D_x^{n-2} [H^{n}(x, y, z)]. \]

\[ H^{n-1}(x, y, z) = (x^{n-1}) [H^{n}(x, y, z)] \]

\[ = \frac{1}{n} D_x^{n-1} [H^{n}(x, y, z)] \]

and

\[ H^{n-2}(x, y, z) = (x^{n-2}) [H^{n}(x, y, z)] \]

\[ = \frac{1}{n(n-1)} D_x^{n-2} [H^{n}(x, y, z)]. \]
Corollary 1. The recurrence relations of the Hermite-based Bernoulli polynomials $H B_n(x, y, z)$ defined by the generating functions (15) and (16), respectively. The shift operators are given by

$$
H A_{n+1}(x, y, z) = \left( x + a_0 + \sum_{k=1}^{n} \frac{D^k}{k!} \right) H A_n(x, y, z) + 2y D_x^2 D_z + 3z D_x^2 D_z^2
$$

and, consequently, we have the raising operator (29).

**Remark 1.** By appropriately choosing $A(t)$ in Theorem 1, we can deduce the following corollaries for the Hermite-based Bernoulli polynomials $H B_n(x, y, z)$ and the Hermite-based Euler polynomials $H E_n(x, y, z)$ defined by the generating functions (15) and (16), respectively:

**Corollary 1.** The recurrence relations of the Hermite-based Bernoulli polynomials $H B_n(x, y, z)$ are given by

$$
H B_{n+1}(x, y, z) = \left( x + \frac{1}{2} \right) H B_n(x, y, z) + 2ny H B_{n-1}(x, y, z)
$$

$$
+ 3(n-1)z H B_{n-2}(x, y, z) - \frac{1}{n+1} \sum_{k=2}^{n+1} \binom{n+1}{k} H B_{n-k+1}(x, y, z)B_k,
$$

where $B_k$ denotes the Bernoulli number of order $k$ and

$$
H B_{-n}(x, y, z) := 0 \quad (n \in \mathbb{N}).
$$

The shift operators are given by

$$
x L^- = \frac{1}{n} D_x,
$$

$$
y L^- = \frac{1}{n} D_x^{-1} D_y,
$$

$$
z L^- = \frac{1}{n} D_x^2 D_z,
$$

$$
x L^+ = x - \frac{1}{2} + 2y D_x + 3z D_x^2 - \sum_{k=2}^{n+1} B_k \frac{D_x^{k-1}}{k!},
$$

$$
y L^+ = x - \frac{1}{2} + 2y D_x^{-1} D_y + 3z D_x^{2} D_y^2 - \sum_{k=2}^{n+1} B_k \frac{D_x^{1-k} D_y^k}{k!}
$$

and

$$
z L^+ = x - \frac{1}{2} + 2y D_x^{-2} D_z + 3z D_x^4 D_z^2 - \sum_{k=2}^{n+1} B_k \frac{D_x^{2-k} D_z^k}{k!}.
$$

**Corollary 2.** The recurrence relations of the Hermite-based Euler polynomials $H E_n(x, y, z)$ are given by

$$
H E_{n+1}(x, y, z) = \left( x - \frac{1}{2} \right) H E_n(x, y, z) + \frac{1}{2} \sum_{k=1}^{n} \binom{n}{k} \phi_k H E_{n-k}(x, y, z)
$$

$$
+ 2ny H E_{n-1}(x, y, z) + 3zn(n-1) H E_{n-2}(x, y, z).
$$

The shift operators are given by

$$
x L^- = \frac{1}{n} D_x,
$$

$$
y L^- = \frac{1}{n} D_x^{-1} D_y,
$$

$$
z L^- = \frac{1}{n} D_x^2 D_z
$$
Theorem 3. The Hermite-based Appell polynomials satisfy the following integro-diiferential equations for the Hermite-based Appell polynomials via the factorization method. Moreover, we list the corresponding differential, integro-differential and partial differential equations for the Hermite-based Bernoulli polynomials and the Hermite-based Euler polynomials.

\[
\begin{align*}
\Delta L_n^- &= \frac{1}{n} D_x^2 D_y^2, \\
\Delta L_n^+ &= x - \frac{1}{2} + 2yD_x + 3zD_x^2 + \frac{1}{2} \sum_{k=1}^n \frac{e_k}{k!} D_x^k, \\
y L_n^+ &= x - \frac{1}{2} + 2yD_x^2 D_y + 3zD_x^2 D_y^2 + \frac{1}{2} \sum_{k=1}^n \frac{e_k}{k!} D_x^{2k} D_y^k
\end{align*}
\]

and

\[
\begin{align*}
\Delta L_n^+ &= x - \frac{1}{2} + 2yD_x^2 D_y + 3zD_x^2 D_y^2 + \frac{1}{2} \sum_{k=0}^n \frac{e_k}{k!} D_x^{2k} D_y^k,
\end{align*}
\]

where \( e_k \) are the coefficients that are linked with the Euler numbers \( E_k \) by

\[
e_k = -\frac{1}{2^k} \sum_{j=0}^k \frac{\binom{k}{j}}{j!} E_{k-j}.
\]

Remark 2. The results asserted by Corollary 2 can easily be restated in terms of the Hermite-based Genocchi polynomials \( \mathcal{G}_n(x, y, z) \) defined by the generating function (17) by simply making use of the relationships (19) and (20). The details involved are being omitted here.


In this section, we obtain differential, integro-differential and partial differential equations for the Hermite-based Appell polynomials via the factorization method. Moreover, we list the corresponding equations for the Hermite-based Bernoulli polynomials and the Hermite-based Euler polynomials.

Theorem 2. The Hermite-based Appell polynomials satisfy the following differential equation:

\[
\left( x + a_0 \right) D_x + \sum_{k=1}^n \frac{\alpha_k}{k!} D_x^{k+1} + 2y D_x^2 + 3z D_x^3 - n \right) [\mathcal{A}_n(x, y, z)] = 0.
\]

Proof. Using the following factorization relation:

\[
\Delta L_n^+ \Delta L_n^- [\mathcal{A}_n(x, y, z)] = \mathcal{A}_n(x, y, z)
\]

and the shift operators (24) and (27), we get the desired result (32).

Theorem 3. The Hermite-based Appell polynomials satisfy the following integro-differential equations:

\[
\begin{align*}
\left( x + a_0 \right) D_y + \sum_{k=1}^n \frac{\alpha_k}{k!} D_x^{-k} D_y^{k+1} + 2D_x^{-1} D_y \\
+ 2y D_x^{-1} D_y^2 + 3z D_x^{-2} D_y^3 - (n + 1) D_x \right) [\mathcal{A}_n(x, y, z)] = 0,
\end{align*}
\]

\[
\begin{align*}
\left( x + a_0 \right) D_x + \sum_{k=1}^n \frac{\alpha_k}{k!} D_x^{-2k} D_y^{k+1} + 2y D_x^{-2} D_y^2 \\
+ 3D_x^{-4} D_y^2 + 3z D_x^{-4} D_y^3 - (n + 1) D_x \right) [\mathcal{A}_n(x, y, z)] = 0,
\end{align*}
\]
Theorem 4. The Hermite-based Appell polynomials satisfy the following partial differential equations:

\[
\left( x + \alpha_0 \right) D_y + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{2k} D_y^k + 2D_x^{-2} D_z
+ 2yD_x^{-2} D_z D_y + 3zD_x^{-4} D_y^2 - (n + 1)D_x \right) [H_{A_n}(x, y, z)] = 0 \tag{35}
\]

and

\[
\left( x + \alpha_0 \right) D_z + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x D_y^k + 2yD_x^{-1} D_y D_z
+ 3D_x^{-2} D_y^2 + 3zD_x^{-4} D_y D_z - (n + 1)D_x^2 \right) [H_{A_n}(x, y, z)] = 0. \tag{36}
\]

Proof. Using the following factorization relation:

\[
\mathcal{L}_{n+1}^+ \mathcal{L}_n^- H_{A_n}(x, y, z) = H_{A_n}(x, y, z)
\]

and the shift operators (25), (26), (28) and (29), we get the integro-differential equations (33) and (34), respectively. Again, by using the above factorization relation together with the shift operators (25) and (29), we get the integro-differential equation (35). To obtain the integro-differential equation (36), we use the shift operators (26) and (28) in the above factorization relation. \qed

Theorem 4. The Hermite-based Appell polynomials satisfy the following partial differential equations:

\[
\left( x + \alpha_0 \right) D_x^{2n} D_z + 2nD_x^{2n-1} D_z + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{2n-2k} D_y^{k+1} + 2yD_x^{2n-2} D_y
+ 3D_x^{2n-1} D_y^2 + 3zD_x^{2n-4} D_y^3 - (n + 1)D_x^{2n+2} \right) [H_{A_n}(x, y, z)] = 0, \tag{37}
\]

\[
\left( x + \alpha_0 \right) D_x^n D_y + nD_x^{n-1} D_y + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{n-k} D_y^{k+1} + 2D_x^{n} D_y
+ 2yD_x^{n-1} D_y^2 + 3zD_x^{n-2} D_y^3 - (n + 1)D_x^{n+1} \right) [H_{A_n}(x, y, z)] = 0, \tag{38}
\]

\[
\left( x + \alpha_0 \right) D_x^{2n} D_y + 2nD_x^{2n-1} D_y + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_y D_x^{2n-2k} D_y^{k+1} + 2D_x^{2n-2} D_y
+ 2yD_x^{2n-2} D_z D_y + 3zD_x^{2n-4} D_y^2 D_z - (n + 1)D_x^{2n+1} \right) [H_{A_n}(x, y, z)] = 0 \tag{39}
\]

and

\[
\left( x + \alpha_0 \right) D_x^{n} D_z + nD_x^{n-1} D_z + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x D_y^{n-k} D_y^k + 2yD_x^{n-1} D_y D_z
+ 3D_x^{n-2} D_y^2 + 3zD_x^{n-2} D_y D_z - (n + 1)D_x^{n+2} \right) [H_{A_n}(x, y, z)] = 0. \tag{40}
\]
Proof. If we differentiate the integro-differential equation (34) $2n$ times with respect to $x$, we get the partial differential equation (37). Similarly, by taking the derivatives of the integro-differential equation (33) $n$ times with respect to $x$, we get the partial differential equation (38). In order to derive the partial differential equation (39), we take the derivatives of the integro-differential equation (35) $2n$ times with respect to $x$. Similarly, in order to obtain the partial differential equation (40), we take the derivatives of the integro-differential equation (36) $n$ times with respect to $x$. 

Remark 3. Just as we indicated in Remark 1, by suitably specializing the function $A(t)$ in Theorems 2, 3 and 4, we can deduce the following corollaries which provide the differential, integro-differential and partial differential equations for the Hermite-based Bernoulli polynomials $H_{B}(x, y, z)$ and the Hermite-based Euler polynomials $H_{E}(x, y, z)$ defined by the generating functions (15) and (16), respectively.

Corollary 3. The Hermite-based Bernoulli polynomials satisfy the following differential equation:

$$\left( x - \frac{1}{2} \right) D_x + 2y D_y^2 + 3z D_z^3 - \sum_{k=2}^{n+1} \frac{B_k}{k!} D_x^{k-1} - n \right) [H_{B}(x, y, z)] = 0,$$

where $B_k$ denotes the Bernoulli number of order $k$.

Corollary 4. The Hermite-based Bernoulli polynomials satisfy the following integro-differential equations:

$$\left( x - \frac{1}{2} \right) D_y + 2y D_y^{-1} D_y + 2y D_y^{-1} D_y^2 + 3z D_z^{-2} D_z^3 - \sum_{k=2}^{n+1} \frac{B_k}{k!} D_y^{1-k} D_y - (n + 1)D_y \right) [H_{B}(x, y, z)] = 0,$$

$$\left( x - \frac{1}{2} \right) D_x + 2y D_x^{-2} D_x^2 + 3D_x^{-4} D_x^2 + 3z D_z^{-3} D_z^3 - \sum_{k=2}^{n+1} \frac{B_k}{k!} D_x^{2-k} D_x^2 - (n + 1)D_x \right) [H_{B}(x, y, z)] = 0,$$

$$\left( x - \frac{1}{2} \right) D_y + 2y D_y^{-2} D_y + 2y D_y^{-2} D_y D_y + 3D_x^{-2} D_x^2 D_y - \sum_{k=2}^{n+1} \frac{B_k}{k!} D_y^{2-k} D_y^{1-k} D_y - (n + 1)D_y \right) [H_{B}(x, y, z)] = 0,$$

and

$$\left( x - \frac{1}{2} \right) D_z + 2y D_y^{-1} D_y D_z + 3D_x^{-3} D_x^2 + 3z D_z^{-2} D_y^2 D_z - \sum_{k=2}^{n+1} \frac{B_k}{k!} D_x^{1-k} D_y^{1-k} D_z - (n + 1)D_z \right) [H_{B}(x, y, z)] = 0,$$

where $B_k$ denotes the Bernoulli number of order $k$. 
Corollary 5. The Hermite-based Bernoulli polynomials satisfy the following partial differential equations:

\[
\left( x - \frac{1}{2} \right) D_x^n D_y + n D_x^{n-1} D_y + 2 D_x^{n-1} D_y + 2 y D_x^{n-1} D_y^2 + 3 z D_x^{n-2} D_y^3
- \sum_{k=2}^{n+1} \frac{B_k}{k!} D_x^{n-k+1} D_y - (n + 1) D_x^{n+1} \right) [H_B(x, y, z)] = 0, \tag{46}
\]

\[
\left( x - \frac{1}{2} \right) D_x^n D_y + 2 n D_x^{n-1} D_y + 2 y D_x^{n-1} D_y^2 + 3 D_x^{n-1} D_y^2 + 3 z D_x^{n-1} D_y^3
- \sum_{k=2}^{n+1} \frac{B_k}{k!} D_x^{n-k+2} D_y^2 - (n + 1) D_x^{n+2} \right) [H_B(x, y, z)] = 0, \tag{47}
\]

\[
\left( x - \frac{1}{2} \right) D_x^n D_y + 2 n D_x^{n-1} D_y + 2 y D_x^{n-1} D_y^2 + 2 y D_x^{n-2} D_y + 3 z D_x^{n-3} D_y^2 D_y
- \sum_{k=2}^{n+1} \frac{B_k}{k!} D_x^{n-k+2} D_y^3 - (n + 1) D_x^{n+3} \right) [H_B(x, y, z)] = 0 \tag{48}
\]

and

\[
\left( x - \frac{1}{2} \right) D_x^n D_y + n D_x^{n-1} D_y + 2 y D_x^{n-1} D_y + 3 D_x^{n-2} D_y^2 + 3 z D_x^{n-2} D_y^2 D_y
- \sum_{k=2}^{n+1} \frac{B_k}{k!} D_x^{n-k+1} D_y^3 - (n + 1) D_x^{n+2} \right) [H_B(x, y, z)] = 0, \tag{49}
\]

where \(B_k\) denotes the Bernoulli number of order \(k\).

Corollary 6. The differential equation satisfied by the Hermite-based Euler polynomials is given by

\[
\left( x - \frac{1}{2} \right) D_x + 2 y D_x^2 + 3 z D_x^3 + \frac{1}{2} \sum_{k=1}^{n} \frac{e_k}{k!} D_x^{k+1} - n \right) [H_E(x, y, z)] = 0, \tag{50}
\]

where \(e_k\) is given, in terms of the Euler number \(E_k\), as in Corollary 2.

Corollary 7. The Hermite-based Euler polynomials satisfy the following integro-differential equations:

\[
\left( x - \frac{1}{2} \right) D_x + 2 y D_x^{-1} D_y + 2 y D_x^{-1} D_y^2 + 3 z D_x^{-2} D_y^3
+ \frac{1}{2} \sum_{k=1}^{n} \frac{e_k}{k!} D_x^{-k} D_y^{k+1} - (n + 1) D_x \right) [H_E(x, y, z)] = 0, \tag{51}
\]

\[
\left( x - \frac{1}{2} \right) D_x + 2 y D_x^{-2} D_y^2 + 3 D_x^{-3} D_y^2 + 3 z D_x^{-3} D_y^3
+ \frac{1}{2} \sum_{k=1}^{n} \frac{e_k}{k!} D_x^{-2k} D_y^{k+1} - (n + 1) D_x^2 \right) [H_E(x, y, z)] = 0, \tag{52}
\]
Corollary 8. The Hermite-based Euler polynomials satisfy the following partial differential equations:

\[
\left(x - \frac{1}{2}\right) D_y + 2yD_y^2D_z + 2yD_x^{-2}D_xD_y + 3zD_x^{-4}D^2_xD_y
\]
\[
+ \frac{1}{2} \sum_{k=1}^{n} \frac{\epsilon_k}{k!} D_x^{-2k}D^k_yD_y - (n + 1)D_x \right) [H_x(x, y, z)] = 0
\]

(53)

and

\[
\left(x - \frac{1}{2}\right) D_z + 2yD_x^{-1}D_yD_z + 3D_x^2D_y^2 + 3zD_x^{-2}D^2_yD_z
\]
\[
+ \frac{1}{2} \sum_{k=1}^{n} \frac{\epsilon_k}{k!} D_x^{-2k}D^k_yD_y - (n + 1)D_x \right) [H_y(x, y, z)] = 0
\]

(54)

Corollary 8. The Hermite-based Euler polynomials satisfy the following partial differential equations:

\[
\left(x - \frac{1}{2}\right) D^n_xD_y + nD_x^{n-1}D_y + 2D_x^{-1}D_y + 2yD_x^{-1}D_y^2 + 3zD_x^{-2}D_y^3
\]
\[
+ \frac{1}{2} \sum_{k=1}^{n} \frac{\epsilon_k}{k!} D_x^{-2k}D^k_yD_y - (n + 1)D_x \right) [H_x(x, y, z)] = 0,
\]

(55)

\[
\left(x - \frac{1}{2}\right) D^n_xD_y + 2nD_x^{2n-1}D_y + 3D_x^2D_y^2 + 3zD_x^{2n-2}D^2_yD_z + 3D_x^{2n-4}D^2_z
\]
\[
+ \frac{1}{2} \sum_{k=1}^{n} \frac{\epsilon_k}{k!} D_x^{2n-2k}D^k_yD_y - (n + 1)D_x \right) [H_y(x, y, z)] = 0
\]

(56)

\[
\left(x - \frac{1}{2}\right) D^n_xD_y + 2nD_x^{2n-1}D_y + 3D_x^2D_y^2 + 3zD_x^{2n-2}D^2_yD_z + 3D_x^{2n-4}D^2_zD_y
\]
\[
+ \frac{1}{2} \sum_{k=1}^{n} \frac{\epsilon_k}{k!} D_x^{2n-2k}D^k_yD_y - (n + 1)D_x \right) [H_x(x, y, z)] = 0
\]

(57)

and

\[
\left(x - \frac{1}{2}\right) D^n_yD_z + nD_y^{n-1}D_z + 2yD_y^{n-1}D_yD_z + 3D_y^{n-2}D^2_y + 3zD_y^{n-2}D^2_yD_y
\]
\[
+ \frac{1}{2} \sum_{k=1}^{n} \frac{\epsilon_k}{k!} D_y^{n-2k}D^k_yD_y - (n + 1)D_y \right) [H_y(x, y, z)] = 0,
\]

(58)

\(\epsilon_k\) being given, in terms of the Euler number \(E_n\), as in Corollary 2.

4. Further Remarks and Observations

For the Hermite-based Appell polynomials defined by means of the generating function (9), many interesting properties and characteristics were investigated earlier by using operational techniques combined with the principle of monomiality. Here, in our present investigation, we have found the differential, integro-differential and partial differential equations for the Hermite-based Appell polynomials via the factorization method. We have also derived the corresponding equations for the Hermite-based Bernoulli polynomials and the Hermite-based Euler polynomials which are defined by the generating functions (15) and (16), respectively. Moreover, just as we indicated in Remark 2, we can easily deduce the corresponding results for the Hermite-based Genocchi polynomials defined by the generating function (17) from those involving the Hermite-based Euler polynomials by means of such simple relationships as (19) and (20).
References