Ideal Convergence in Locally Solid Riesz Spaces

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Dedicated to Professor Hari M. Srivastava

Abstract. An ideal $I$ is a family of subsets of positive integers $\mathbb{N}$ which is closed under taking finite unions and subsets of its elements. In this paper, we introduce the concepts of ideal $\tau$-convergence, ideal $\tau$-Cauchy and ideal $\tau$-bounded sequence in locally solid Riesz space endowed with the topology $\tau$. Some basic properties of these concepts has been investigated. We also examine the ideal $\tau$-continuity of a mapping defined on locally solid Riesz space.

1. Introduction

The notion of statistical convergence was introduced by Fast [10] and Steinhaus [32] independently in the same year 1951. Actually the idea of statistical convergence was formerly given under the name “almost convergence” by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [37]. The concept was formally introduced by Fast [10] and later was reintroduced by Schoenberg [31], and also independently by Buck [4]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory by (Boos [3], Fridy [11], Salát [27], topological groups (Çakalli [5, 6]), topological spaces (Di Maio and Kočinac [22]), function spaces (Caserta and Kočinac [7], Caserta, Di Maio and Kočinac [8]), locally convex spaces (Maddox[21]), measure theory (Cheng et al., [9], Millar[23]), fuzzy mathematics (Nuray and Savaş [24], Savaş [30]).

The notion of $l$-convergence($I$ denotes the ideal of subsets of $\mathbb{N}$) was initially introduced by Kostyrko et al. [18] as a generalization of statistical convergence. More applications of ideals can be seen in ([12–14], [15, 16, 28, 29, 33, 35]).

A family of sets $I \subset P(\mathbb{N})$ (power sets of $\mathbb{N}$) is called an ideal if and only if for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $\mathcal{F} \subset P(\mathbb{N})$ is a filter on $\mathbb{N}$ if and only if $\phi \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$ and each $A \in \mathcal{F}$ and each $B \supset A$, we have $B \in \mathcal{F}$. An ideal $I$ is called non-trivial ideal if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a non-trivial ideal.

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if and only if \( F = F(I) = \{ \mathbb{N} - A : A \in I \} \) is a filter on \( \mathbb{N} \). A non-trivial ideal \( I \subseteq P(\mathbb{N}) \) is called \textit{admissible} if and only if \( \{ \{ x \} : x \in \mathbb{N} \} \subseteq I \). A non-trivial ideal \( I \) is \textit{maximal} if there cannot exist any non-trivial ideal \( J \neq I \) containing \( I \) as a subset. Further details on ideals of \( P(\mathbb{N}) \) can be found in Kostyrko, et.al [18].

If we take \( I = I_f = \{ A \subseteq \mathbb{N} : A \in I \} \) then \( I_f \) is a non-trivial admissible ideal of \( \mathbb{N} \) and the corresponding convergence coincide with the statistical convergence. If we take \( I = I_0 = \{ A \subseteq \mathbb{N} : \delta(A) = 0 \} \) where \( \delta(A) \) denote the asymptotic density of the set \( A \). Then \( I_0 \) is a non-trivial admissible ideal of \( \mathbb{N} \) and the corresponding convergence coincide with the statistical convergence.

A Riesz space is an ordered vector space which is a lattice at the same time. It was first introduced by F. Riesz [25] in 1928. Riesz spaces have many applications in measure theory, operator theory and optimization. They have also some applications in economics (see [2]), and we refer to ([1, 17, 20, 36]) for more details.

Now we recall some of the basic concepts related to statistical convergence and ideal convergence.

Let \( E \subseteq \mathbb{N} \). Then the natural density of \( E \) is denoted by \( \delta(E) \) and is defined by

\[
\delta(E) = \lim_{n \to \infty} \frac{|\{ k \in E : k \leq n \}|}{n},
\]

where the vertical bar denotes the cardinality of the respective set.

**Definition 1.1.** ([22]) A sequence \( x = (x_k) \) in a topological space \( X \) is said to be \textit{statistically convergent} to \( x_0 \) if for every neighbourhood \( V \) of \( x_0 \)

\[
\delta(\{ k \in \mathbb{N} : x_k \not\in V \}) = 0.
\]

In this case, we write \( S\lim x = x_0 \) or \( (x_k) \rightarrow^S x_0 \) and \( S \) denotes the set of all statistically convergent sequences.

**Definition 1.2.** ([19]) A sequence \( x = (x_k) \) in a topological space \( X \) is said to be \textit{I-convergent} to \( x_0 \) if for every neighbourhood \( V \) of \( x_0 \)

\[
\{ k \in \mathbb{N} : x_k \not\in V \} \subseteq I.
\]

In this case, we write \( I\lim x = x_0 \) or \( (x_k) \rightarrow^I x_0 \) and \( I \) denotes the set of all ideally convergent sequences.

2. Preliminaries

Let \( X \) be a real vector space and \( \leq \) be a partial order on this space. Then \( X \) is said to be an \textit{ordered vector space} if it satisfies the following properties:

(i) if \( x, y \in X \) and \( y \leq x \), then \( y + z \leq x + z \) for each \( z \in X \).

(ii) if \( x, y \in X \) and \( y \leq x \), then \( ay \leq ax \) for each \( a \geq 0 \).

If, in addition, \( X \) is a lattice with respect to the partially ordered, then \( X \) is said to be a \textit{Riesz space} (or a \textit{vector lattice})(see[36]), if for each pair of elements \( x, y \in X \) the supremum and infimum of the set \( \{x, y\} \) both exist in \( X \).

We shall write

\[
x \vee y = \sup \{x, y\} \quad \text{and} \quad x \wedge y = \inf \{x, y\}.
\]

For an element \( x \) of a Riesz space \( X \), the \textit{positive part} of \( x \) is defined by \( x^+ = x \vee \overline{0} \), the \textit{negative part} of \( x \) by \( x^- = -x \vee \overline{0} \) and the \textit{absolute value} of \( x \) by \( |x| = x \vee (-x) \), where \( \overline{0} \) is the zero element of \( X \).
A subset $S$ of a Riesz space $X$ is said to be solid if $y \in S$ and $|y| \leq |x|$ implies $x \in S$.

A topological vector space $(X, \tau)$ is a vector space $X$ which has a topology (linear) $\tau$, such that the algebraic operations of addition and scalar multiplication in $X$ are continuous. Continuity of addition means that the function $f: X \times X \to X$ defined by $f(x, y) = x + y$ is continuous on $X \times X$, and continuity of scalar multiplication means that the function $f: \mathbb{C} \times X \to X$ defined by $f(a, x) = ax$ is continuous on $\mathbb{C} \times X$.

Every linear topology $\tau$ on a vector space $X$ has a base $N$ for the neighborhoods of $\emptyset$ satisfying the following properties:

1. Each $Y \in N$ is a balanced set, that is, $ax \in Y$ holds for all $x \in Y$ and for every $a \in \mathbb{R}$ with $|a| \leq 1$.
2. Each $Y \in N$ is an absorbing set, that is, for every $x \in X$, there exists $a > 0$ such that $ax \in Y$.
3. For each $Y \in N$ there exists some $E \in N$ with $E + E \subseteq Y$.

A linear topology $\tau$ on a Riesz space $X$ is said to be locally solid (see[26]) if $\tau$ has a base at zero consisting of solid sets. A locally solid Riesz space $(X, \tau)$ is a Riesz space equipped with a locally solid topology $\tau$.

Recall that a first countable space is a topological space satisfying the "first axiom of countability". Specifically, a space $X$ is said to be first countable if each point has a countable neighbourhood basis (local base). That is, for each point $x$ in $X$ there exists a sequence $V_1, V_2, ...$ of open neighbourhoods of $x$ such that for any open neighbourhood $V$ of $x$ there exists an integer $j$ with $V_j$ contained in $V$.

The purpose of this article is to give certain characterizations of ideal convergent sequences in locally solid Riesz spaces and investigate some basic properties of the notions ideal $\tau$-convergence, ideal $\tau$-Cauchy, ideal $\tau$-bounded sequence and ideal $\tau$-continuity of a mapping in locally solid Riesz spaces. Finally we prove a Tauberian theorem to the locally solid Riesz spaces.

Throughout the article, the symbol $N_{\text{sol}}$ we will denote any base at zero consisting of solid sets and satisfying the conditions (1), (2) and (3) in a locally solid topology. Also we assume $I$ is a non-trivial admissible ideal of $\mathbb{N}$.

3. Ideal topological convergence in locally solid Riesz spaces

Throughout the article $X$ will denote the Hausdorff locally solid Riesz space, which satisfies the first axiom of countability.

Recently, in [1], Albayrak and Pehlivan introduced the notion of statistical convergence in locally solid Riesz spaces as follows:

**Definition 3.1.** ([1]) Let $(X, \tau)$ be a locally solid Riesz space. A sequence $(x_k)$ of points in $X$ is said to be $S(\tau)$-convergent to an element $x_0$ of $X$ if for each $\tau$-neighbourhood $V$ of zero,

$$\delta([k \in \mathbb{N} : x_k - x_0 \notin V]) = 0$$

i.e.,

$$\lim_{m \to \infty} \frac{1}{m} \|\{k \leq m : x_k - x_0 \notin V\}\| = 0.$$

In this case, we write $S(\tau) - \lim_{k \to \infty} x_k = x_0$ or $(x_k) \overset{S(\tau)}{\to} x_0$.

Now we give the definitions of $I(\tau)$-convergence and $I(\tau)$-bounded in locally solid Riesz spaces.

**Definition 3.2.** Let $(X, \tau)$ be a locally solid Riesz space. A sequence $(x_k)$ of points in $X$ is said to be $I(\tau)$-convergent to an element $x_0$ of $X$ if for each $\tau$-neighbourhood $V$ of zero,

$$[k \in \mathbb{N} : x_k - x_0 \notin V] \in I.$$
i.e.,

\[ k \in \mathbb{N} : x_k - x_0 \in V \in \mathcal{F}. \]

In this case, we write \( I(\tau) - \lim_{k \to \infty} x_k = x_0 \) or \( (x_k) \xrightarrow{I(\tau)} x_0. \)

**Example 3.1.** Let us consider the locally solid Riesz space \( (\mathbb{R}^2, ||.||) \) with the Euclidean norm \( ||.|| \) and coordinatewise ordering. In this space, let us define a sequence \((x_k)\) by

\[
x_k = \begin{cases} 
(1 + \frac{1}{k^2}, 2 + \frac{3}{k^2}); & \text{if } k \neq n^2; \\
(4, 4); & \text{if } k = n^2.
\end{cases}
\]

for each \( n \in \mathbb{N} \). The family \( N_{sol} \) of all \( U_\varepsilon \) defined by

\[
U_\varepsilon = \{ x \in \mathbb{R}^2 : ||x|| < \varepsilon \},
\]

where \( 0 < \varepsilon \in \mathbb{R} \) constitutes a base at zero \((\bar{0} = (0,0)) \). For \( x_0 = (1,2) \), we have

\[
x_k - x_0 = \begin{cases} 
(\frac{1}{k^2}, \frac{3}{k^2}); & \text{if } k \neq n^2; \\
(3,2); & \text{if } k = n^2.
\end{cases}
\]

For each \( \tau \)-neighbourhood \( V \) of zero, there exists some \( U_\varepsilon \in N_{sol}, \varepsilon > 0 \) such that \( U_\varepsilon \subseteq V \) and

\[
\{ k \in \mathbb{N} : x_k - x_0 \notin U_\varepsilon \} \subseteq K \cup \{1, 4, 9, 16, \ldots, n^2, \ldots \},
\]

where \( K \) is a finite set. Then, we have

\[
\{ k \in \mathbb{N} : x_k - x_0 \notin V \} \subseteq \{ k \in \mathbb{N} : x_k - x_0 \notin U_\varepsilon \}
\]

i.e., \( \{ k \in \mathbb{N} : x_k - x_0 \notin V \} \subseteq K \cup \{1, 4, 9, 16, \ldots, n^2, \ldots \}. \)

Since \( I \) is admissible, so we have

\[
\{ k \in \mathbb{N} : x_k - x_0 \notin V \} \in I.
\]

Hence \( I(\tau) - \lim_{k \to \infty} x_k = (1,2) \).

**Definition 3.3.** Let \((X, \tau)\) be a locally solid Riesz space. A sequence \((x_k)\) of points in \(X\) is said to be \( I(\tau) \)-bounded in \(X\) if for each \( \tau \)-neighbourhood \( V \) of zero, there is some \( a > 0 \),

\[
\{ k \in \mathbb{N} : ax_k \notin V \} \in I.
\]

**Theorem 3.1.** Let \((X, \tau)\) be a locally solid Riesz space. Every \( I(\tau) \)-convergent sequences in \(X\) has only one limit.

**Proof.** Suppose that \( x = (x_k) \) is a sequence in \(X\) such that \( I(\tau) \)-\( \lim_k x_k = x_0 \) and \( I(\tau) \)-\( \lim_k x_k = y_0 \).

Let \( V \) be any \( \tau \)-neighbourhood of zero. Also for each \( \tau \)-neighbourhood \( V \) of zero there exists \( Y \in N_{sol} \) such that \( Y \subseteq V \). Choose any \( W \in N_{sol} \) such that \( W + W \subseteq Y \). We define the following sets:

\[
A_1 = \{ k \in \mathbb{N} : x_k - x_0 \in W \} \cap \mathcal{F}
\]

\[
A_2 = \{ k \in \mathbb{N} : x_k - y_0 \in W \} \cap \mathcal{F}.
\]

Since \( I(\tau) - \lim x_k = x_0 \) and \( I(\tau) - \lim x_k = y_0 \), we get \( A_1, A_2 \in \mathcal{F} \).

Now, let \( A = A_1 \cap A_2 \). Then we have

\[
x_0 - y_0 = x_0 - x_k + x_k - y_0 \in W + W \subseteq Y \subseteq V.
\]
Hence for each $\tau$-neighbourhood $V$ of zero we have $x_0 - y_0 \in V$. Since $(X, \tau)$ is Hausdorff, the intersection of all $\tau$-neighbourhoods $V$ of zero is the singleton set $\{0\}$. Thus we get $x_0 - y_0 = 0$ i.e., $x_0 = y_0$.

**Theorem 3.2.** Let $(X, \tau)$ be a locally solid Riesz space and let $(x_k)$ and $(y_k)$ be two sequences of points in $X$. Then the following hold:

(i) If $I(\tau)\text{-}lim_k x_k = x_0$ and $I(\tau)\text{-}lim_k y_k = y_0$ then $I(\tau)\text{-}lim_k (x_k + y_k) = x_0 + y_0$.

(ii) If $I(\tau)\text{-}lim_k x_k = x_0$ then $I(\tau)\text{-}lim_k ax_k = ax_0$ for $a \in \mathbb{R}$.

**Proof.** Let $V$ be an arbitrary $\tau$-neighbourhood of zero. Then there exists $Y \in N_{sol}$ such that $Y \subseteq V$. Choose $W \in N_{sol}$ such that $W + W \subseteq Y$. Since $I(\tau)\text{-}lim_k x_k = x_0$ and $I(\tau)\text{-}lim_k y_k = y_0$. We write

\[ B_1 = \{k \in \mathbb{N} : x_k - x_0 \in W\} \]
\[ B_2 = \{k \in \mathbb{N} : y_k - y_0 \in W\}. \]

Then we have $B_1, B_2 \in \mathcal{F}$.

Let $B = B_1 \cap B_2$. Hence we have $B \in \mathcal{F}$ and

\[(x_k + y_k) - (x_0 + y_0) = (x_k - x_0) + (y_k - y_0) \in W + W \subseteq Y \subseteq V.\]

Therefore

\[ \{k \in \mathbb{N} : (x_k + y_k) - (x_0 + y_0) \in V\} \in \mathcal{F}. \]

Since $V$ is arbitrary, we have $I(\tau)\text{-}lim(x_k + y_k) = x_0 + y_0$.

(ii) Let $V$ be an arbitrary $\tau$-neighbourhood of zero and $I(\tau)\text{-}lim_k x_k = x_0$. Then there exists $Y \in N_{sol}$ such that $Y \subseteq V$ and we have

\[ \{k \in \mathbb{N} : x_k - x_0 \in Y\} \in \mathcal{F}. \]

Since $Y$ is balanced, $x_k - x_0 \in Y$ implies that $a(x_k - x_0) \in Y$ for every $a \in \mathbb{R}$ with $|a| \leq 1$. Hence

\[ \{k \in \mathbb{N} : x_k - x_0 \in Y\} \subseteq \{k \in \mathbb{N} : ax_k - ax_0 \in Y\} \subseteq \{k \in \mathbb{N} : x_k - x_0 \in V\}. \]

Thus, we have

\[ \{k \in \mathbb{N} : x_k - x_0 \in V\} \in \mathcal{F} \]

for each $\tau$-neighbourhood $V$ of zero. Now let $|a| > 1$ and $|a|||$ be the smallest integer greater than or equal to $|a|$. Then there exists $W \in N_{sol}$ such that $|a|||W \subseteq Y$. Since $I(\tau)\text{-}lim_k x_k = x_0$ we have the set

\[ K = \{k \in \mathbb{N} : x_k - x_0 \in W\} \in \mathcal{F}. \]

Therefore

\[ |ax_k - ax_0| = |a||x_k - x_0| \leq |a|||x_k - x_0| \in |a|||W \subseteq Y \subseteq V. \]

Since $Y$ is solid, we have $ax_k - ax_0 \in Y$. This implies that $ax_k - ax_0 \in V$. Thus,

\[ \{k \in \mathbb{N} : ax_k - ax_0 \in V\} \in \mathcal{F}, \]

for each $\tau$-neighbourhood $V$ of zero. Hence $I(\tau)\text{-}lim_k ax_k = ax_0$. This completes the proof of the theorem.
Theorem 3.3. Let \((X, \tau)\) be a locally solid Riesz space. If a sequence \((x_k)\) in \(X\) is \(I(\tau)\)-convergent, then it is \(I(\tau)\)-bounded.

Proof. Suppose that \((x_k)\) is \(I(\tau)\)-convergent to a point \(x_0\) in \(X\). Let \(V\) be an arbitrary \(\tau\)-neighbourhood of zero, there exists \(Y \in N_{sol}\) such that \(Y \subseteq V\). We choose \(W \in N_{sol}\) such that \(W + W \subseteq Y\). Since \(I(\tau)\)-lim\(_k\) \(x_k = x_0\), the set
\[
A = \{k \in \mathbb{N} : x_k - x_0 \notin W\} \in I.
\]
Since \(W\) is absorbing, there exists \(a > 0\) such that \(a x_0 \in W\). Let \(b\) be such that \(|b| \leq 1\) and \(b \leq a\). Since \(W\) is solid and \(|b x_0| \leq |a x_0|\), we have \(b x_0 \in W\). Also, since \(W\) is balanced, \(x_k - x_0 \in W\) implies \(b(x_k - x_0) \in W\). Then we have
\[
bx_k = b(x_k - x_0) + b x_0 \in W + W \subseteq V,
\]
for each \(k \in \mathbb{N} - A\).

Thus
\[
k \in \mathbb{N} : bx_k \notin W \in I.
\]
Hence \((x_k)\) is \(I(\tau)\)-bounded.

Theorem 3.4. Let \((X, \tau)\) be a locally solid Riesz space and let \((x_k)\), \((y_k)\) and \((z_k)\) be three sequences of points in \(X\) such that

(i) \(x_k \leq y_k \leq z_k\), for all \(k \in \mathbb{N}\),

(ii) \(I(\tau)\)-lim\(_k\) \(x_k = x_0 = I(\tau)\)-lim\(_k\) \(z_k\),
then \(I(\tau)\)-lim\(_k\) \(y_k = x_0\).

Proof. Let \(V\) be an arbitrary \(\tau\)-neighbourhood of zero, there exists \(Y \in N_{sol}\) such that \(Y \subseteq V\). We choose \(W \in N_{sol}\) such that \(W + W \subseteq Y\). From given condition (ii), we have \(P, Q \in F\), where
\[
P = \{k \in \mathbb{N} : x_k - x_0 \in W\}
\]
and
\[
Q = \{k \in \mathbb{N} : z_k - x_0 \in W\}.
\]
Also from the given condition (i), we have
\[
x_k - x_0 \leq y_k - x_0 \leq z_k - x_0
\]
\[
\Rightarrow |y_k - x_0| \leq |x_k - x_0| + |z_k - x_0| \in W + W \subseteq Y.
\]
Since \(Y\) is solid, we have \(y_k - x_0 \in Y \subseteq V\). Thus,
\[
k \in \mathbb{N} : y_k - x_0 \in V \in F,
\]
for each \(\tau\)-neighbourhood \(V\) of zero. Hence \(I(\tau)\)-lim\(_k\) \(y_k = x_0\). This completes the proof of the theorem.

4. \(I(\tau)\)-Cauchy and \(I'(\tau)\)-convergence in locally solid Riesz spaces

Definition 4.1. Let \((X, \tau)\) be a locally solid Riesz space. A sequence \((x_k)\) of points in \(X\) is said to be \(I(\tau)\)-Cauchy in \(X\) if for each \(\tau\)-neighbourhood \(V\) of zero, there is an integer \(n \in \mathbb{N}\),
\[
k \in \mathbb{N} : x_k - x_n \notin V \in I.
\]
Theorem 4.1. Let \((X, \tau)\) be a locally solid Riesz space. A sequence \((x_k)\) is \(I(\tau)\)-convergent to \(x_0\) in \(X\) if and only if for each \(\tau\)-neighbourhood \(V\) of zero there exists a subsequence \((x_{k(r)})\) of \((x_k)\) such that \(\lim_{r \to \infty} x_{k(r)} = x_0\) and

\[\{k \in \mathbb{N} : x_k - x_{k(r)} \not\in V\} \subseteq I.\]

Proof. Let \(x = (x_k)\) be a sequence in \(X\) such that \(I(\tau) - \lim_{k \to \infty} x_k = x_0\). Let \(V\) be an arbitrary \(\tau\)-neighbourhood of zero. Let \([V_n]\) be a sequence of nested base of \(\tau\)-neighbourhoods of zero. We write

\[E(i) = \{k \in \mathbb{N} : x_k - x_0 \not\in V_i\},\]

for any positive integer \(i\). Then for each \(i\), we have \(E(i+1) \subseteq E(i)\) and \(E(i) \subseteq F\). Choose \(n(1)\) such that \(r > n(1)\), then \(E(1) \neq \emptyset\). Then for each positive integer \(r\) such that \(n(1) \leq r < n(2)\), choose \(k'(r) \in E(i)\) such that \(x_{k'(r)} - x_0 \in V_1\). In general, choose \(n(p+1) > n(p)\) such that \(r > n(p+1)\), then \(E(p+1) \neq \emptyset\). Then for all \(r\) satisfying \(n(p) \leq r < n(p+1)\), choose \(k'(r) \in E(p)\) such that \(x_{k'(r)} - x_0 \in V_p\). Hence it follows that \(\lim x_{k'(r)} = x_0\).

Since \(V\) is an arbitrary \(\tau\)-neighbourhood of zero, there exists \(Y \in \mathcal{N}_{sol}\) such that \(Y \subseteq V\). Choose \(W \in \mathcal{N}_{sol}\) such that \(W + W \subseteq Y\) Now we have

\[x_k - x_{k'(r)} = x_k - x_0 + x_{k'(r)} - x_0 \in W + W \subseteq Y \subseteq V.\]

Since \(I(\tau) - \lim_{k \to \infty} x_k = x_0\) and \(I(\tau) - \lim_{r \to \infty} x_{k(r)} = x_0\) implies that

\[\{k \in \mathbb{N} : x_k - x_{k(r)} \not\in V\} \subseteq I.\]

Next suppose for an arbitrary \(\tau\)-neighbourhood \(V\) of zero there exists a subsequence \((x_{k(r)})\) of \((x_k)\) such that \(\lim_{r \to \infty} x_{k(r)} = x_0\) and

\[\{k \in \mathbb{N} : x_k - x_{k(r)} \not\in V\} \subseteq I.\]

Since \(V\) is any \(\tau\)-neighbourhood of zero, we choose \(W \in \mathcal{N}_{sol}\) such that \(W + W \subseteq V\). Then we have

\[x_k - x_0 = x_k - x_{k'(r)} + x_{k'(r)} - x_0 \in W + W \subseteq V.\]

i.e.

\[\{k \in \mathbb{N} : x_k - x_0 \not\in V\} \subseteq \{k \in \mathbb{N} : x_k - x_{k'(r)} \not\in W\} \cup \{r \in \mathbb{N} : x_{k'(r)} - x_0 \not\in W\}.\]

Therefore

\[\{k \in \mathbb{N} : x_k - x_0 \not\in V\} \subseteq I.\]

This completes the proof of the theorem.

Theorem 4.2. If \(\lim_{k \to \infty} x_k = x_0\) and \(I(\tau) - \lim_{k \to \infty} y_k = 0\), then \(I(\tau) - \lim_{k \to \infty} (x_k + y_k) = \lim_{k \to \infty} x_k\).

Proof. Let \(V\) be any \(\tau\)-neighbourhood of zero. Then there exists \(Y \in \mathcal{N}_{sol}\) such that \(Y \subseteq V\). Choose \(W \in \mathcal{N}_{sol}\) such that \(W + W \subseteq Y\). Since \(\lim_{k \to \infty} x_k = x_0\), then there exists an integer \(n_0\) such that \(k \geq n_0\) implies that \(x_k - x_0 \in W\). Hence

\[\{k \in \mathbb{N} : x_k - x_0 \not\in W\} \subseteq \mathbb{N} - \{n_0\}.\]

By assumption \(I(\tau) - \lim_{k \to \infty} y_k = 0\), then we have \(\{k \in \mathbb{N} : y_k \not\in W\} \subseteq I\). Thus

\[\{k \in \mathbb{N} : (x_k - x_0) + y_k \not\in V\} \subseteq \{k \in \mathbb{N} : x_k - x_0 \not\in W\} \cup \{k \in \mathbb{N} : y_k \not\in W\}.\]

i.e.

\[\{k \in \mathbb{N} : (x_k - x_0) + y_k \not\in V\} \subseteq I.\]
This implies that $I(\tau) - \lim_{k \to \infty} (x_k + y_k) = \lim_{k \to \infty} x_k$.

Theorem 4.3. Let $(X, \tau)$ be a locally solid Riesz space and let $x = (x_k)$ be a sequence in $X$. If there is a $I(\tau)$-convergent sequence $y = (y_k)$ in $X$ such that $\{k \in \mathbb{N} : y_k \neq x_k \notin V\} \in I$ then $x$ is also $I(\tau)$-convergent.

Proof. Suppose that $\{k \in \mathbb{N} : y_k \neq x_k \notin V\} \in I$ and $I(\tau)-\lim_k y_k = x_0$. Then for an arbitrary $\tau$-neighborhood $V$ of zero, we have

$$\{k \in \mathbb{N} : y_k - x_0 \notin V\} \in I.$$  

Now,

$$\{k \in \mathbb{N} : x_k - x_0 \notin V\} \subseteq \{k \in \mathbb{N} : y_k \neq x_k \notin V\} \cup \{k \in \mathbb{N} : y_k - x_0 \notin V\}.$$  

Therefore we have

$$\{k \in \mathbb{N} : x_k - x_0 \notin V\} \in I.$$  

This completes the proof of the theorem.

Definition 4.2. Let $(X, \tau)$ be a locally solid Riesz space. A sequence $x = (x_k)$ in $X$ is said to be $I'(\tau)$-convergent to $x_0$ if there exists a set $K = \{k_1 < k_2 < \cdots < k_r < \cdots\} \subseteq \mathbb{N}$ with $K \in F$ such that $\lim_k x_k = x_0$. In this case we write $I'(\tau)$-lim$_k x_k = x_0$.

Theorem 4.4. Let $(X, \tau)$ be a locally solid Riesz space. A sequence $x = (x_k)$ in $X$ is $I(\tau)$-convergent to $x_0$ if and only if it is $I'(\tau)$-convergent to $x_0$.

Proof. Suppose that $I'(\tau)$-lim$_k x_k = x_0$. Let $V$ be an arbitrary $\tau$-neighborhood $V$ of zero. Since $I'(\tau)$-lim$_k x_k = x_0$, there is a set $K = \{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ with $K \in F$ and $n \in \mathbb{N}$ such that $k \geq n$ and $k \in K$ imply $x_k - x_0 \in V$. Then

$$K_1 = \{k \in \mathbb{N} : x_k - x_0 \notin V\} \subseteq \mathbb{N} - \{k_{n+1}, k_{n+2}, \ldots\}.$$  

Therefore

$$K_1 \in I.$$  

Hence $x$ is $I(\tau)$-convergent to $x_0$.

Next suppose that $x$ is $I(\tau)$-convergent to $x_0$. For a fix countable local base $V_1 \supset V_2 \supset \ldots$ at $x_0$. For each $j \in \mathbb{N}$, we put

$$K_j = \{k \in \mathbb{N} : x_k - x_0 \notin V_j\}$$

and

$$M_j = \{k \in \mathbb{N} : x_k - x_0 \in V_j\}.$$  

Then $K_j \in I$ and

$$M_1 \supset M_2 \supset \ldots \supset M_j \supset M_{j+1} \supset \ldots$$ \hspace{1cm} (1)  

and

$$M_j \in F, \ j = 1, 2, 3, \ldots$$ \hspace{1cm} (2)
Now we show that for $k \in M_j$, $(x_k)$ is convergent to $x_0$. Suppose that $(x_k)$ is not convergent to $x_0$. Therefore $x_k - x_0 \notin V_j$ for infinitely many terms. Let

$$M_j = \{k \in \mathbb{N} : x_k - x_0 \notin V_j, j > f\}.$$ 

Then $M_j \in I$, by using (1) we get $M_j \subset M_j$. Hence $M_j \in I$ which contradicts (2). Therefore $(x_k)$ is convergent to $x_0$. Hence $x$ is $I^J(\tau)$-convergent to $x_0$. This completes the proof of the theorem.

**Theorem 4.5.** The sequential method $I(\tau)$ is regular, i.e. if $\tau - \lim x_k = x_0$ then $I(\tau) - \lim x_k = x_0$.

**Proof.** Proof of the theorem is straightforward.

**Theorem 4.6.** The sequential method $I(\tau)$ is subsequential.

**Proof.** Proof of the theorem follows from the Theorem 4.4.

5. *$I$*-sequentially continuous in locally solid Riesz spaces

**Definition 5.1.** Let $(X_1, \tau_1)$ and $(X_2, \tau_2)$ be locally solid Riesz spaces and $S \subset X_1$. A function $f : S \rightarrow X_2$ is said to be $I$-sequentially continuous at a point $x_0 \in S$ if $x_k \overset{I(\tau_1)}{\rightarrow} x_0$ in $S$ implies that $f(x_k) \overset{I(\tau_2)}{\rightarrow} f(x_0)$ in $X_2$.

**Theorem 5.1.** Let $(X, \tau)$ be a locally solid Riesz space. Any $I$-sequentially continuous function at a point $x_0$ is $\tau$-continuous at $x_0$.

**Proof.** Let $f$ be any $I$-sequentially continuous function at a point $x_0$. Since any proper admissible ideal is a regular subsequential method by Theorem 4.5 and 4.6, it follows that $f$ is $\tau$-continuous.

**Corollary 5.2.** Let $(X, \tau)$ be a locally solid Riesz space. Any $I$-sequentially continuous function at a point $x_0$ is $I$-continuous at $x_0$.

As statistical limit is an $I$-sequential method we get:

**Corollary 5.3.** Let $(X, \tau)$ be a locally solid Riesz space. A function is statistically continuous at a point $x_0$ if and only if it is $\tau$-continuous at $x_0$.

**Theorem 5.4.** Let $(X_1, \tau_1)$ and $(X_2, \tau_2)$ be locally solid Riesz spaces. If a function $f : X_1 \rightarrow X_2$ is uniformly continuous, then $f$ is $I$-continuous.

**Proof.** Let the function $f : X_1 \rightarrow X_2$ be uniformly continuous and $x_k \overset{I(\tau_1)}{\rightarrow} x_0$ holds in $X_1$. Let us denote the zeros of $X_1$ and $X_2$ by $\overline{O}_1$ and $\overline{O}_2$, respectively. Let $V$ be an arbitrary $\tau_2$-neighbourhood of $\overline{O}_2$. Since $f$ is uniformly continuous, there exists some $\tau_1$-neighborhood $W$ of $\overline{O}_1$ such that

$$x - y \in W \Rightarrow f(x) - f(y) \in V \text{ for all } x, y \in X_1. \quad (3)$$

Since $x_k \overset{I(\tau_1)}{\rightarrow} x_0$, we have $A \subset F$, where $A = \{k \in \mathbb{N} : x_k - x_0 \in W\}$. Using (3) we have

$$f(x_k) - f(x_0) \in V \text{ for each } k \in A.$$ 

Then we get

$$A \subset \{k \in \mathbb{N} : f(x_k) - f(x_0) \in V\},$$
and hence
\[ \{ k \in \mathbb{N} : f(x_k) - f(x_0) \in V \} \in \mathcal{F}. \]

Thus \( f(x_k) \overset{\text{i.o.}}{\to} f(x_0) \). This shows that \( f \) is \( I \)-continuous.

**Theorem 5.5.** Let \((X, \tau)\) be a locally solid Riesz space. Then the following mappings
(i) \((X, \tau) \times (X, \tau) \to (X, \tau) : (x, y) \mapsto x \lor y, \)
(ii) \((X, \tau) \times (X, \tau) \to (X, \tau) : (x, y) \mapsto x \land y, \)
(iii) \((X, \tau) \to (X, \tau) : (x, y) \mapsto |x|, \)
(iv) \((X, \tau) \to (X, \tau) : (x, y) \mapsto x^-, \)
(v) \((X, \tau) \to (X, \tau) : (x, y) \mapsto x^+ \)
are all \( I \)-continuous.

**Proof.** (i) Let \( I(\tau \times \tau) - \lim(x_k, y_k) = (x, y) \) and \( V \) be an arbitrary \( \tau \)-neighbourhood of zero in \( X \). Then there exists a \( Y \in N_{\text{sol}} \) such that \( Y \subseteq V \). Let us choose \( W \in N_{\text{sol}} \) such that \( W + W \subseteq Y \). Since \( I(\tau \times \tau) - \lim(x_k, y_k) = (x, y) \), we have
\[ A = \{ k \in \mathbb{N} : (x_k - x, y_k - y) \in W \times W \} \in \mathcal{F}. \]

Also we have
\[ |x_k \lor y_k - x \lor y| \leq |x_k - x| + |y_k - y| \in W + W \subseteq Y \] for each \( k \in A \).

Since \( Y \) is solid, we have
\[ x_k \lor y_k - x \lor y \in Y, \] for each \( k \in A \).

Then we get
\[ A \subseteq \{ k \in \mathbb{N} : x_k \lor y_k - x \lor y \in V \} \]
and hence
\[ \{ k \in \mathbb{N} : x_k \lor y_k - x \lor y \in V \} \in \mathcal{F}. \]

Therefore we have \( I(\tau) - \lim(x_k \lor y_k) = (x \lor y) \).

(ii) Let \( V \) be an arbitrary \( \tau \)-neighbourhood of zero in \( X \). Then there exists a \( Y \in N_{\text{sol}} \) such that \( Y \subseteq V \). Let us choose \( W \in N_{\text{sol}} \) such that \( W + W \subseteq Y \). Let \( I(\tau \times \tau) - \lim(x_k, y_k) = (x, y) \). Then we have
\[ A = \{ k \in \mathbb{N} : (x_k - x, y_k - y) \in W \times W \} \in \mathcal{F}. \]

Also we have
\[ |x_k \land y_k - x \land y| = |(-x_k) \lor (-y_k)| + |(-x) \lor (-y)| \]
\[ \leq |(-x) - (-x_k)| + |(-y) - (-y_k)| \]
\[ = |x_k - x| + |y_k - y| \in W + W \subseteq Y \] for each \( k \in A \).

Since \( Y \) is solid, we have
\[ x_k \land y_k - x \land y \in Y, \] for each \( k \in A \).

Then we get
\[ A \subseteq \{ k \in \mathbb{N} : x_k \land y_k - x \land y \in V \} \]
and hence
\[ \{ k \in \mathbb{N} : x_k \land y_k - x \land y \in V \} \in \mathcal{F}. \]
Therefore we have \( I(\tau) - \lim (x_k \land y_k) = (x \land y). \)

(iii) Let \( V \) be an arbitrary \( \tau \)-neighbourhood of zero in \( X \). Then there exists a \( Y \in N_{sol} \) such that \( Y \subseteq V \). Let us choose \( W \in N_{sol} \) such that \( W + W \subseteq Y \). Let \( I(\tau) - \lim x_k = x \). Then we have
\[ A = \{ k \in \mathbb{N} : x_k - x \in W \} \in \mathcal{F}. \]
Also we have
\[ |x_k| - |x| = |[x_k \lor (-x_k)] - [x \lor (-x)]| \]
\[ \leq |x_k - x| + |(-x_k) - (-x)| \in W + W \subseteq Y \text{ for each } k \in A. \]
Since \( Y \) is solid, we have
\[ |x_k| - |x| \in Y, \text{ for each } k \in A. \]
Then we get
\[ A \subseteq \{ k \in \mathbb{N} : |x_k| - |x| \in V \} \]
and hence
\[ \{ k \in \mathbb{N} : |x_k| - |x| \in V \} \in \mathcal{F}. \]
Therefore we have \( I(\tau) - \lim |x_k| = |x| \).

(iv) Let \( V \) be an arbitrary \( \tau \)-neighbourhood of zero in \( X \). Then there exists a \( Y \in N_{sol} \) such that \( Y \subseteq V \). Let us choose \( W \in N_{sol} \) such that \( W + W \subseteq Y \). Let \( I(\tau) - \lim x_k = x \). Then we have
\[ A = \{ k \in \mathbb{N} : x_k - x \in W \} \in \mathcal{F}. \]
Also we have
\[ |x_k^- - x^-| = |[(x_k) \lor 0] - [(-x) \lor 0]| \]
\[ \leq |(-x_k) - (-x)| + |0 + 0| = |x_k - x| \in W \subseteq Y \text{ for each } k \in A. \]
Since \( Y \) is solid, we have
\[ x_k^- - x^- \in Y, \text{ for each } k \in A. \]
Then we get
\[ A \subseteq \{ k \in \mathbb{N} : x_k^- - x^- \in V \} \]
and hence
\[ \{ k \in \mathbb{N} : x_k^- - x^- \in V \} \in \mathcal{F}. \]
Therefore we have \( I(\tau) - \lim x_k^- = x^- \).
Let $V$ be an arbitrary $\tau$-neighbourhood of zero in $X$. Then there exists a $Y \in N_{sol}$ such that $Y \subseteq V$. Let us choose $W \in N_{sol}$ such that $W + W \subseteq Y$. Let $l(\tau) - \lim x_k = x$. Then we have

$$ A = \{k \in \mathbb{N} : x_k - x \in W\} \in \mathcal{F}. $$

Also we have

$$ |x_k^+ - x^+| = |(x_k \vee 0) - (x \vee 0)| $$

$$ \leq |x_k - x| + |0 + 0| = |x_k - x| \in W \subseteq Y \text{ for each } k \in A. $$

Since $Y$ is solid, we have

$$ x_k^+ - x^+ \in Y, \text{ for each } k \in A. $$

Then we get

$$ A \subseteq \{k \in \mathbb{N} : x_k^+ - x^+ \in V\} $$

and hence

$$ \{k \in \mathbb{N} : x_k^+ - x^+ \in V\} \in \mathcal{F}. $$

Therefore we have $l(\tau) - \lim x_k^+ = x^+$.

**Definition 5.2.** Let $(X, \tau)$ be a locally solid Riesz space. A sequence $(x_k)$ in $X$ is called slowly oscillating if, for each $\tau$-neighbourhood $V$ of zero, there exists a positive integer $n_0$ and $\delta > 0$ such that if $n_0 \leq k \leq n \leq (1 + \delta)k$, then $x_k - x_n \in V$.

Now we give a Tauberian theorem.

**Theorem 5.6.** Let $(X, \tau)$ be a locally solid Riesz space. If $(x_k)$ is statistically convergent and slowly oscillating, then it is convergent.

**Proof.** Let $S(\tau) - \lim x_k = x_0$. Then we have a subsequence $(i_m)$ with $1 \leq i_1 \leq i_2 \leq ... \leq i_m \leq ...$ of those indices $n$ for which $y_n = x_n$. Since

$$ \lim_{k \to \infty} \frac{1}{k} ||n \leq k : x_n \neq y_n|| = 0. $$

Then we have

$$ \lim_{m \to \infty} \frac{1}{i_m} ||n \leq i_m : x_n = y_n|| = \lim_{m \to \infty} \frac{m}{i_m} = 1. $$

Consequently, it follows that

$$ \lim_{m \to \infty} \frac{i_{m+1}}{i_m} = \lim_{m \to \infty} \frac{i_{m+1}}{m+1} \cdot \frac{m+1}{m} \cdot \frac{m}{i_m} = 1. \quad (4) $$

By the definition of $(i_m)$, we get

$$ \lim_{m \to \infty} x_{i_m} = \lim_{m \to \infty} y_{i_m} = x_0. \quad (5) $$

By (4) and (5) we conclude that for each closed $\tau$-neighbourhood $V$ of zero, there exists a positive integer $n_0$ such that if $m > n_0$ then $(x_k - x_{i_m}) \in V$ whenever $i_m < k < i_{m+1}$. Since $V$ is arbitrary, it follows that

$$ \lim_{m \to \infty} (x_m - x_{i_m}) = 0. $$

By (5.3), we have $(x_m)$ is convergent to $x_0$. This completes the proof of the theorem.
References