On Generalized Quasi Einstein Manifolds

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Abstract. Quasi Einstein manifold is a simple and natural generalization of an Einstein manifold. The object of the present paper is to study some geometric properties of generalized quasi Einstein manifolds. Two non-trivial examples have been constructed to prove the existence of a generalized quasi Einstein manifold.

1. Introduction

A Riemannian or a semi-Riemannian manifold $(M^n, g)$, $n = \text{dim} M \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n} g,$$

holds on $M$, where $S$ and $r$ denote the Ricci tensor and the scalar curvature of $(M^n, g)$ respectively. According to ([1], p. 432), (1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([1], p. 432-433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds $(M^n, g)$ realizing the following relation:

$$S(X, Y) = a g(X, Y) + b A(X) A(Y),$$

where $a, b$ are smooth functions and $A$ is a non-zero 1-form such that

$$g(X, U) = A(X),$$

for all vector fields $X$.

A non-flat Riemannian manifold $(M^n, g)$ $(n > 2)$ is defined to be a quasi Einstein manifold [3] if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and satisfies the condition (2). We shall call $A$ the associated 1-form and the unit vector field $U$ is called the generator of the manifold. Such a manifold is denoted by $(QE)_n$.

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Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetime are quasi Einstein manifolds. Also quasi Einstein manifolds can be taken as a model of the perfect fluid spacetime in general relativity[7]. So quasi Einstein manifolds have some importance in the general theory of relativity.

The study of quasi Einstein manifolds was continued by M.C.Chaki [3], S.Guha [11], U.C.De and G.C.Ghosh ([5], [6]), P.Debnath and A.Konar [9], Özgür and Sular [21], Özgür [18] and many others. In a recent paper [25] Shaikh, Kim and Hui studied Lorentzian quasi Einstein manifolds.

Several authors have generalized the notion of quasi Einstein manifold such as generalized quasi Einstein manifolds ([4], [20]), nearly quasi Einstein manifolds [8], generalized Einstein manifolds([24] and N(k)-quasi Einstein manifolds ([17], [21], [18], [27], [13]).

In 2001, Chaki [4] introduced the notion of generalized quasi Einstein manifolds. A non-flat Riemannian manifold \((M^n, g)\) \((n > 2)\) is called a generalized quasi Einstein manifold if its Ricci tensor \(S\) of type \((0, 2)\) is non-zero and satisfies the condition

\[
S(X, Y) = ag(X, Y) + bA(X)A(Y) + c(A(X)B(Y) + A(Y)B(X)),
\]

where \(a, b, c\) are certain non-zero scalars and \(A, B\) are two non-zero 1-form. The unit vector fields \(U\) and \(V\) corresponding to the 1-forms \(A\) and \(B\) respectively, defined by

\[
g(U, U) = A(X), \quad g(U, V) = B(X),
\]

for every vector field \(X\) are orthogonal, that is, \(g(U, V) = 0\). Such as \(n\)-dimensional manifold is denoted by \(G(QE)_n\). The vector fields \(U\) and \(V\) are called the generators of the manifold and \(a, b, c\) are called the associated scalars. If \(c = 0\), then the manifold reduces to a quasi Einstein manifold \((QE)_n\). It may be mentioned that De and Ghosh [5] introduced the same notion in another way. In 2008, De and Gazi [8] introduced nearly quasi Einstein manifolds \((NQE)_n\) and prove the existence of such a manifold by several examples.

A non-flat Riemannian manifold \((M^n, g)\) \((n > 2)\) is called a nearly quasi Einstein manifold if the Ricci tensor \(S\) is non-zero and satisfies the condition

\[
S(X, Y) = ag(X, Y) + bE(X, Y),
\]

where \(E\) is a symmetric tensor of type \((0, 2)\).

In a Riemannian manifold \((M^n, g)\) \((n > 3)\) the Weyl conformal curvature tensor \(C\) of type \((1, 3)\) is defined by

\[
C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[g(Y, Z)QX - g(X, Z)QY + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y],
\]

where \(R, S, r\) denotes the Riemann curvature tensor, the Ricci tensor of type \((0, 2)\) and the scalar curvature of the manifold respectively and \(Q\) is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor \(S\), that is, \(g(QX, Y) = S(X, Y)\). If the dimension \(n = 3\), then the conformal curvature tensor vanishes identically. The conformal curvature tensor have been studied by several authors in several ways such as ([12], [14], [15], [16], [26]) and many others.

The importance of a \(G(QE)_n\) lies in the fact that a four-dimensional semi-Riemannian manifold is relevant to study of a general relativistic fluid spacetime admitting heat flux [23], where \(U\) is taken as the velocity vector of the fluid and \(V\) is taken as the heat flux vector field.

In the present paper we have studied \(G(QE)_n\). The paper is organized as follows:

After introduction in Section 2, we study some basic results of \(G(QE)_n\). We prove that if the generator \(U\) or \(V\) is a parallel vector field, then \(G(QE)_n\) reduces to a \((QE)_n\). A necessary condition is obtained for a \(G(QE)_n\) to be conformally conservative. Section 3 is devoted to study Ricci-semisymmetric \(G(QE)_n\). In the next section we consider Ricci-recurrent \(G(QE)_n\). Finally, we construct two non-trivial examples of a \(G(QE)_n\).
2. Basic results

Suppose the generator $U$ is a parallel vector field, then $R(X,Y)U = 0$. Hence

$$S(X,U) = 0.$$  \hfill (5)

Putting $Y = U$ in (4) gives

$$S(X,U) = aA(X) + bA(U) + cB(X) = (a + b)g(X,U) + cg(X,V).$$  \hfill (6)

Using (5) in (6) we get

$$(a + b)g(X,U) + cg(X,V) = 0.$$  \hfill (7)

Putting $X = V$ in (7) yields $c = 0$. That is, $G(QE)_n$ reduces to a $(QE)_n$. Again if $V$ is a parallel vector field, then $S(X,V) = 0$. Setting $Y = V$ in (4), we obtain

$$S(X,V) = ag(X,V) + bA(X)A(V) + c(A(X)B(V) + A(V)B(X)) = ab(X) + cA(X),$$

since $A(V) = g(U,V) = 0$. \hfill (8)

Putting $X = U$ in (8) gives

$$ab(U) + cA(V) = 0$$

which implies $c = 0$, since $B(U) = g(U,V) = 0$. In this case also $G(QE)_n$ reduces to a $(QE)_n$.

This leads to the following:

**Theorem 2.1.** In a $G(QE)_n$ if either of the generators $U, V$ is parallel, then the manifold reduces to a quasi Einstein manifold.

**Corollary 2.1.** If the generator $U$ of a $G(QE)_n$ is a parallel vector field, then $a + b = 0$.

**Theorem 2.2.** In a $G(QE)_n$, $QU$ is orthogonal to $U$ iff $a + b = 0$.

**Proof.** In the equation (5) let us set $Y = U$. Then we get

$$S(X,U) = ag(X,U) + bA(X)A(U) + c(A(X)B(U) + A(U)B(X)).$$

Again putting $X = U$, we obtain $S(U,U) = a + b$ and hence $g(U,U) = a + b$, which implies that $QU$ is orthogonal to $U$ if and only if $a + b = 0$. \hfill $\square$

**Theorem 2.3.** A necessary condition for a $G(QE)_n$ to be conformally conservative is

$$2(n - 1)dc(U) = (n - 2)da(U) + (2n + 1)db(U).$$

**Proof.** A Riemannian manifold of dimension $> 3$ is said to be of conservative conformal curvature tensor if $div C = 0$ where ‘div’ denotes divergence. It is known[10] that $div C = 0$ implies

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = \frac{1}{2(n - 1)}[d\tau(X)g(Y,Z) - d\tau(Z)g(X,Y)].$$  \hfill (9)

Putting $X = Y = U$ and $Z = V$ in (9) we get

$$(\nabla_U S)(U,V) - (\nabla_V S)(U,U) = \frac{1}{2(n - 1)}[d\tau(U)g(U,V) - d\tau(V)g(U,U)].$$  \hfill (10)
From (4) we obtain
\[ r = an + b \] (11)
and
\[ S(U, V) = c. \] (12)
Using (11) and (12) in (10), we get
\[ \nabla_U c - \nabla_V (a + b) = \frac{1}{2(n - 1)} [ -nda(U) - db(U) ]. \]
That is,
\[ 2(n - 1)dc(U) - (n - 2)da(U) - (2n + 1)db(U) = 0. \]
This completes the proof. \( \square \)

3. Ricci-semisymmetric \( G(QE)_n \)

A Riemannian manifold is said to be Ricci-semisymmetric if \( R \cdot S = 0 \) holds. In this section we study
Ricci-semisymmetric \( G(QE)_n \), and prove the following theorem:

**Theorem 3.1.** A Ricci-semisymmetric \( G(QE)_n \) is either nearly quasi Einstein manifold \( N(QE)_n \) or, \( A(R(X, Y)V) = 0. \)

**Proof.** Suppose that \( R \cdot S = 0 \). Then we get
\[ S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0. \]
Now using (4) we get
\[ \begin{align*}
ag(R(X, Y)Z, W) + ba(R(X, Y)Z)A(W) + c[A(R(X, Y)Z)B(W) \\
+ A(W)B(R(X, Y)Z)] + ag(Z, R(X, Y)W) + ba(Z)A(R(X, Y)W) \\
+ c[A(Z)B(R(X, Y)W) + A(R(X, Y)W)B(Z)] &= 0.
\end{align*} \] (13)
Taking \( W = U \) and \( Z = V \) in (13), we obtain
\[ ba(R(X, Y)V) = 0, \text{since } B(R(X, Y)V) = g(R(X, Y)V, V) = 0. \]
Then either \( b = 0 \) or, \( A(R(X, Y)V) = 0. \)
If \( b = 0 \), from (4) we get
\[ S(X, Y) = ag(X, Y) + c[A(X)B(Y) + A(Y)B(X)] = ag(X, Y) + cE(X, Y), \]
where \( E(X, Y) = A(X)B(Y) + A(Y)B(X) \) is a symmetric tensor. Hence either the manifold is a nearly quasi Einstein manifold \( N(QE)_n \) or, \( A(R(X, Y)V) = 0. \) \( \square \)
4. Nature of the associated 1-forms of a $G(QE)_n$

In this section, we assume that the associated scalars $a, b, c$ are constants and we enquire under what conditions the associated 1-forms $A, B$ to be closed. Let us suppose that the manifold $G(QE)_n$ satisfies Codazzi type of Ricci tensor, that is, the Ricci tensor satisfies


(14)

Using (4) in (14) we get

$$b[(\nabla_X A)YA(Z) + A(Y)(\nabla_X A)Z] + c[(\nabla_X A)YB(Z) + A(Y)(\nabla_X B)Y]$$

$$= b[(\nabla_Y A)XA(Z) + A(X)(\nabla_Y A)Z] + c[(\nabla_Y A)XB(Z) + A(X)(\nabla_Y B)X].$$

(15)

Putting $Z = U$ in (15) and using $(\nabla_X A)U = 0$, since $U$ is a unit vector, we obtain

$$b[(\nabla_X A)Y - (\nabla_Y A)X] = c[A(X)(\nabla_Y B)U + (\nabla_Y B)X - A(Y)(\nabla_X B)U - (\nabla_X B)Y].$$

(16)

Now suppose $\nabla_Y U \perp V$, then

$$(\nabla_X B)U = 0.$$  

(17)

Using (17) in (16), we get

$$b(dA)(X, Y) = -c(dB)(X, Y).$$

Hence we can state the following:

**Theorem 4.1.** If a $G(QE)_n$ with associated scalars as constants satisfies Codazzi type of Ricci tensor, then the associated 1-form $A$ is closed if and only if $B$ is closed, provided $\nabla_Y U \perp V$.

Next suppose the 1-form $A$ is closed. Then

$$(\nabla_X A)Y - (\nabla_Y A)X = 0.$$  

(18)

which implies

$$g(\nabla_X U, Y) + g(\nabla_Y U, X) = 0,$$

Hence the vector field $U$ is irrotational. Putting $X = U$ in (18), we get

$$g(\nabla_U U, Y) + g(\nabla_Y U, U) = 0.$$  

Since $U$ is a unit vector, $g(\nabla_Y U, U) = 0$. Hence

$$g(\nabla_U U, Y) = 0$$

which implies $\nabla_U U = 0$, that is, the integral curves of the vector field $U$ are geodesic.

Thus we can state the following:

**Corollary 4.1.** If a $G(QE)_n$ with associated scalars as constants satisfies Codazzi type of Ricci tensor, then the vector field $U$ is irrotational and the integral curves of the vector field $U$ are geodesic provided 1-form $B$ is closed and $\nabla_Y U \perp V$. 

5. Ricci-recurrent $G(QE)_n$

A Riemannian manifold is said to be Ricci-recurrent [22] if the Ricci tensor is non-zero and satisfies the condition

$$ (\nabla_X S)(Y, Z) = D(X)S(Y, Z), $$

where $D$ is a non-zero 1-form.

Let $(M^n, g)$ be a $G(QE)_n$ manifold. If $U$ is a parallel vector field, then $\nabla_X U = 0$, from which it follows that $R(X, Y)U = 0$. Therefore $S(Y, U) = 0$. Then from Theorem 1 and Corollary 1, we get $c = 0$ and $a + b = 0$. Therefore we can rewrite the equation (4) in the following form:

$$ S(X, Y) = a[g(X, Y) - A(X)A(Y)]. $$

Taking the covariant derivative of the above equation with respect to $Z$, we obtain

$$ (\nabla_Z S)(X, Y) = da(Z)[g(X, Y) - A(X)A(Y)], $$

since $\nabla_X U = 0$ implies that $(\nabla_Z A)(X) = 0$. Therefore $(\nabla_Z S)(X, Y) = \frac{da(Z)}{a} S(X, Y)$, i.e., the manifold $(M^n, g)$ is Ricci-recurrent.

Conversely, suppose that $G(QE)_n$ is Ricci-recurrent. Then

$$ (\nabla_X S)(Y, Z) = D(X)S(Y, Z), \quad D(X) \neq 0. $$

But

$$ (\nabla_X S)(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). $$

Therefore

$$ D(X)S(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \quad (19) $$

Putting $Y = Z = U$ in (19), we obtain

$$ D(X)(a + b) = X(a + b) - S(\nabla_X U, U) - S(U, \nabla_X U). \quad (20) $$

From the equation (4), we obtain

$$ S(\nabla_X U, U) = ag(\nabla_X U, U) + bA(\nabla_X U) + cB(\nabla_X U) = (a + b)A(\nabla_X U) + cB(\nabla_X U). $$

Hence from (20), we get

$$ X(a + b) - D(X)(a + b) = 2(a + b)A(\nabla_X U) + 2cB(\nabla_X U). \quad (21) $$

Since $A(U) = 1$ implies $g(\nabla_X U, U) = 0$, i.e., $A(\nabla_X U) = 0$, therefore from (21) $B(\nabla_X U) = 0$ if and only if $d(a + b)(X) = (a + b)D(X)$. But $B(\nabla_X U) = 0$ implies that either $U$ is a parallel vector field or $\nabla_X U \perp V$.

Thus we can state the following:

**Theorem 5.1.** A $G(QE)_n$ is a Ricci-recurrent manifold provided the generator $U$ is a parallel vector field. Conversely, if a $G(QE)_n$ is a Ricci-recurrent manifold, then either the vector field $U$ is parallel or, $\nabla_X U \perp V$. 
6. Examples of generalized quasi Einstein manifolds

Example 6.1. We consider a Riemannian manifold $(\mathbb{R}^4, g)$ endowed with the metric $g$ given by

$$ds^2 = g_{ij}dx^i dx^j = (1 + 2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

where $q = e^{x_1}$ and $k$ is a non-zero constant and $i, j = 1, 2, 3, 4$.

The only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{11} = \frac{q}{1 + 2q}, \quad \Gamma^1_{12} = -\frac{q}{1 + 2q}, \quad \Gamma^1_{33} = -\frac{q}{1 + 2q},$$

$$\Gamma^1_{44} = -\frac{q}{1 + 2q}, \quad \Gamma^2_{12} = \frac{q}{1 + 2q}, \quad \Gamma^3_{13} = \frac{q}{1 + 2q},$$

$$\Gamma^4_{14} = \frac{q}{1 + 2q},$$

$$R_{1221} = R_{1331} = R_{1441} = \frac{q}{1 + 2q},$$

$$R_{2332} = R_{2442} = R_{3443} = \frac{q^2}{1 + 2q},$$

$$R_{11} = \frac{3q}{(1 + 2q)^2},$$

$$R_{22} = R_{33} = R_{44} = \frac{q}{1 + 2q},$$

The scalar curvature is $\frac{6q(1 + 3q)}{(1 + 2q)^2}$ which is non-zero and non-constant. We take scalars $a, b$ and $c$ as follows:

$$a = \frac{q}{(1 + 2q)^2}, \quad b = \frac{3q}{(1 + 2q)^3} - \frac{q}{(1 + 2q)^2}, \quad c = \frac{q}{1 + 2q}.$$

We choose the 1-forms as follows:

$$A_i(x) = \begin{cases} \sqrt{1 + 2q}, & \text{for } i = 1 \\ 0, & \text{for } i = 2, 3, 4 \end{cases}$$

and

$$B_i(x) = \begin{cases} \sqrt{\frac{1 + 2q}{3}}, & \text{for } i = 2, 3, 4 \\ 0, & \text{for } i = 1 \end{cases}$$

We have,

$$R_{11} = a_{11} + bA_1A_1 + c(A_1B_1 + A_1B_1),$$

$$R_{22} = a_{22} + bA_2A_2 + c(A_2B_2 + A_2B_2),$$

$$R_{33} = a_{33} + bA_3A_3 + c(A_3B_3 + A_3B_3),$$

$$R_{44} = a_{44} + bA_4A_4 + c(A_4B_4 + A_4B_4).$$
R.H.S. of (22) is \( \frac{3q}{(1 + 2q)} = R_{11} = L.H.S \) of (22).
R.H.S. of (23) is \( \frac{q}{(1 + 2q)} = R_{22} = L.H.S \) of (23).
Similarly we can show that the (24) and (25) are also true. We shall now show that the 1-forms are unit and orthogonal.

\[
g^{ij}A_iA_j = g^{11}A_1A_1 + g^{22}A_2A_2 + g^{33}A_3A_3 + g^{44}A_4A_4 = 1,
\]

\[
g^{ij}B_iB_j = g^{11}B_1B_1 + g^{22}B_2B_2 + g^{33}B_3B_3 + g^{44}B_4B_4 = 1
\]

and

\[
g^{ij}A_iB_j = g^{11}A_1B_1 + g^{22}A_2B_2 + g^{33}A_3B_3 + g^{44}A_4B_4 = 0.
\]

So, the manifold under consideration is a generalized quasi Einstein manifold.

**Example 2.** We consider the 3-dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3\} \), where \((x, y, z)\) are the standard coordinates in \( \mathbb{R}^3 \). Let \([e_1, e_2, e_3]\) be linearly independent global frame on \( M \) given by

\[
e_1 = \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.
\]

Let \( g \) be the Riemannian metric defined by \( g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0 \) and \( g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1 \).

Let \( V \) be the Levi-Civita connection with respect to the Riemannian metric \( g \) and \( R \) be the curvature tensor of \( g \). Then we have

\[
[e_1, e_2] = e_3, [e_1, e_3] = 0, [e_2, e_3] = 0.
\]

The Riemannian connection \( V \) of the metric \( g \) is given by

\[
2g(V_xY, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [X, Z]) + g(Z, [X, Y]),
\]

which is known as Koszul’s formula. This formula yields

\[
V_{e_1}e_1 = 0, \quad V_{e_1}e_2 = \frac{1}{2}e_3, \quad V_{e_1}e_3 = -\frac{1}{2}e_2,
\]

\[
V_{e_2}e_1 = -\frac{1}{2}e_3, \quad V_{e_2}e_3 = 0, \quad V_{e_2}e_2 = \frac{1}{2}e_1,
\]

\[
V_{e_3}e_1 = -\frac{1}{2}e_2, \quad V_{e_3}e_2 = \frac{1}{2}e_1, \quad V_{e_3}e_3 = 0.
\]

It is known that

\[
R(X, Y)Z = V_XV_YZ - V_YV_XZ - V_{[X,Y]}Z.
\]

With the help of the above results and using (27), we can easily calculate the non-vanishing components of the curvature tensor as follows:

\[
R(e_1, e_2)e_3 = \frac{1}{4}e_2, \quad R(e_1, e_3)e_3 = \frac{1}{4}e_1, \quad R(e_1, e_2)e_2 = -\frac{3}{4}e_1,
\]

\[
R(e_1, e_3)e_2 = -\frac{1}{4}e_3, \quad R(e_1, e_2)e_1 = -\frac{1}{4}e_3, \quad R(e_1, e_2)e_1 = \frac{3}{4}e_2,
\]

\[
R(e_2, e_3)e_1 = -\frac{1}{4}e_3, \quad R(e_1, e_3)e_1 = \frac{1}{4}e_3, \quad R(e_1, e_2)e_1 = \frac{3}{4}e_2.
\]
and the components which can be obtained from these by the symmetric properties from which, we can easily calculate the non-vanishing components of the Ricci tensor $S$ as follows:

\[ S(e_1, e_1) = -\frac{1}{2}, \quad S(e_2, e_2) = -\frac{1}{2}, \quad S(e_3, e_3) = \frac{1}{2}, \]

and the scalar curvature is $-\frac{1}{2}$. Since $\{e_1, e_2, e_3\}$ is a frame field, any vector field $X, Y \in \chi(M)$ can be written as

\[ X = a'_1 e_1 + b'_1 e_2 + c'_1 e_3, \]

and

\[ Y = a'_2 e_1 + b'_2 e_2 + c'_2 e_3, \]

where $a'_i, b'_i, c'_i \in \mathbb{R}^+$ such that $a'_1 a'_2 + b'_1 b'_2 + c'_1 c'_2 \neq 0$. Hence

\[ S(X, Y) = -\frac{1}{2} (a'_1 a'_2 + b'_1 b'_2 - c'_1 c'_2) \]

\[ g(X, Y) = a'_1 a'_2 + b'_1 b'_2 + c'_1 c'_2 \]

We choose the associated scalars as follows:

\[ a = 1, \quad b = -\frac{3}{2} \quad \text{and} \quad c = -\frac{1}{2} \]

We also choose two associated 1-forms as follows:

\[ A(X) = \left( a'_1 a'_2 + b'_1 b'_2 \right)^{\frac{1}{2}}, \quad \forall X. \]

\[ B(X) = \frac{c'_1 c'_2}{2 \left( a'_1 a'_2 + b'_1 b'_2 \right)^{\frac{1}{2}}}, \quad \forall X. \]

By virtue of the definition and chosen of two scalars and 1-forms, we can say that $(M^3, g)$ is a generalized quasi Einstein manifold whose associated scalars are constants.

References


