On the Existence of Global Solutions for a Nonlinear Klein-Gordon Equation

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Abstract. The aim of this work is to study the global existence of solutions for the Cauchy problem of a Klein-Gordon equation with high energy initial data. The proof relies on constructing a new functional, which includes both the initial displacement and the initial velocity: with sign preserving property of the new functional we show the existence of global weak solutions.

1. Introduction

The nonlinear Klein-Gordon equation with quadratic nonlinearity is
\[ u_{tt} - u_{xx} + \alpha u - \beta u^2 = 0, \] (1)
where \( \alpha \) and \( \beta \neq 0 \). Eq. (1) arises in many scientific applications such as solid state physics, nonlinear optics and quantum field theory. The Klein-Gordon equation is the first relativistic equation in quantum mechanics for the wave function of a particle with zero spin. It was proposed as a relativistic generalization of the Schrödinger equation and was investigated in many papers [1, 2, 4–6, 9, 12, 15, 23, 26].

The goal of the present paper is to investigate the existence of global solutions for the Cauchy problem of the Klein-Gordon equation with dissipation
\[ u_{tt} - \Delta u + u + u_t = |u|^{p-1} u, \quad x \in \mathbb{R}^n, \quad t > 0, \] (2)
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^n, \] (3)
where \( u_0 \) and \( u_1 \) are the initial value functions, \( n \geq 2 \) and \( 1 < p < \frac{n+2}{n-2} \) if \( n \geq 3 \), \( 1 < p < \infty \) if \( n = 2 \). Evolution equations with dissipation are studied from various aspects in many papers [3, 13, 16, 17, 19].

In the present paper, we investigate the existence of global solutions by using the potential well method [18]. Sattinger [18] investigated global existence of the initial-boundary value problem of the following nonlinear hyperbolic equation
\[ u_{tt} - \nabla^2 u + f(x, u) = 0 \]
in the case of initial energy less than the potential well depth \( d \). Then this result extended to the total energy of the initial data is less than or equal to \( d \) [24]. Very recently in a paper of Kutev et al. [8] it was proved that there exist global solutions when the total energy of initial data is greater than \( d \) and they established the existence of global weak solutions by constructing a functional which include both the initial displacement and the initial velocity. Because they showed numerically that the initial velocity plays a crucial role in the behaviour of the problem. Problem \((2), (3)\) was already treated in the \( E(0) \leq d \) case by Runzhang [25], but the functional \( f(u) \) used in their paper fails to prove the \( E(0) > d \) case. Although a strongly damped nonlinear Klein-Gordon equation is studied in [26] and a blow up result was given for the high energy initial data, i.e. \( E(0) > d \), the global existence was studied for \( E(0) \leq d \). In the present paper, we reinvestigate the problem for the case \( E(0) \leq d \), where we use a standard functional that include only the initial displacement \( u_0 \). Then, we prove that the existence of global solutions for \( E(0) > d \) can not be proved via sign invariance of this functional. A new functional which includes both the initial displacement \( u_0 \) and initial velocity \( v_1 \) will be constructed for the case of high energy initial data. Functionals depending on \( u_0 \) and \( v_1 \) are introduced for the first time in [8] and then they were successfully applied for proving the global existence to some Boussinesq-type equations in [20–22].

Throughout this paper \( H^p = H^p(\mathbb{R}^n) \) will denote the \( L^2 \) Sobolev space on \( \mathbb{R}^n \) with norm \( \| f \|_{H^p} = \|(I - \Delta)^{\frac{s}{2}} f\| = \|(1 + k^2)^{\frac{s}{2}} f\| \), where \( s \) is a real number, \( I \) is unitary operator. The notation \( \| f \|_{H^p}, \| f \|_p \) and \( \| f \|_\infty \) will be used instead of norms of \( L^p(\mathbb{R}^n), L^2(\mathbb{R}^n) \) and \( L^\infty(\mathbb{R}^n) \), respectively.

2. Global Existence for \( E(0) \leq d \)

The present section refers to two points. Firstly, we define a functional which includes only the initial displacement, and prove the existence of global solutions for \( E(0) \leq d \) by aid of the sign invariance of this functional. We then show that this functional fails to prove the global existence in the case of \( E(0) > d \).

Now, let us define

\[
E(t) = E(u(t), u_1(t)) = \frac{1}{2} \left( \|u_1\|^2 + \|\nabla u\|^2 + \|u\|^2 \right) - \frac{1}{p+1} \|u\|_{p+1}^{p+1},
\]

(4)

\[
E(t) + \int_0^t \|u_t\|^2 \, dt = E(0)
\]

\[
f(u) = \frac{1}{2} \left( \|\nabla u\|^2 + \|u\|^2 \right) - \frac{1}{p+1} \|u\|_{p+1}^{p+1},
\]

(5)

\[
l(u) = \left( \|\nabla u\|^2 + \|u\|^2 \right) - \|u\|_{p+1}^{p+1},
\]

(6)

\[
d = \inf_{u \in N} f(u),
\]

(7)

where \( N = \{ u \in H^1 \mid I(u) = 0, \|u\|_H \neq 0 \} \), \( E(u(t), u_1(t)) \) is the total energy, \( f(u) \) is the potential energy and \( d \) is the depth of potential well which can exactly be written in terms of the Sobolev constant as

\[
d = \frac{p-1}{2(p+1)} \left( S_p^{p+1} \right)^{2/(p-1)}.
\]

(8)

Here \( S_p \) is the embedding constant from \( H^1(\mathbb{R}^n) \) into \( L^{p+1}(\mathbb{R}^n) \) given by

\[
S_p = \sup_{u \in H^1} \frac{\|u\|_{p+1}^{p+1}}{\|u\|_H}.
\]

When \( 0 < E(0) < d \), by the sign invariance of (6) one can prove the existence of global solutions of (2), (3). Existence of global solutions was proved by such functionals for problem (2), (3) in [25]. It was proved in
[25] that if \( I(u) > 0 \), then every weak solutions of the problem exist globally, and if \( I(u) < 0 \), then every weak solutions of the problem blow up in finite time.

For \( \sigma > -\frac{p-1}{2} \), define
\[
I_{\sigma}(u) = (1 - \sigma) \left( \| \nabla u \|^2 + \| u \|^2 \right) - \| u \|^{p+1}_{p+1} = I(u) - \sigma \left( \| \nabla u \|^2 + \| u \|^2 \right).
\]

Then \( D_{\sigma} \) and \( N_{\sigma} \) are defined by
\[
D_{\sigma} = \inf_{u \in N_{\sigma}} I_{\sigma}(u), \quad N_{\sigma} = \{ u \in H^1 : I_{\sigma}(u) = 0, \| u \|_{H^1} \neq 0 \}.
\]

Obviously, taking \( \sigma = 0 \), \( I_{\sigma} \) corresponds to the functional \( I(u) \). Moreover, if \( \sigma < -\frac{p-1}{2} \) then \( D_{\sigma} < 0 \). In this case for \( E(0) = D_{\sigma} < 0 \), all weak solutions of (2), (3) blow-up in a finite time.

For \( \sigma \in \left( -\frac{p-1}{2}, 1 \right) \), we have the following lemmas.

**Lemma 2.1.** Assume that \( u \in H^1(R^n) \). If \( I_{\sigma}(u) < 0 \), then \( \| u \|_{H^1} > \left( \frac{1 - \sigma}{\| \nabla u \|} \right)^{1/(p-1)} \). If \( I_{\sigma}(u) = 0 \), then \( \| u \|_{H^1} \geq \left( \frac{1 - \sigma}{\| \nabla u \|} \right)^{1/(p-1)} \) or \( \| u \|_{H^1} = 0 \).

**Proof.** First, since \( I_{\sigma}(u) < 0 \), we have \( \| u \|_{H^1} \neq 0 \). Hence, from
\[
(1 - \sigma) \| u \|_{H^1}^2 < \| u \|_{p+1}^{p+1} \leq S_p^{p+1} \| u \|_{H^1}^{p+1},
\]
we have \( \| u \|_{H^1} > \left( \frac{1 - \sigma}{\| \nabla u \|} \right)^{1/(p-1)} \).

If \( \| u \|_{H^1} = 0 \), then \( I_{\sigma}(u) = 0 \). If \( I_{\sigma}(u) = 0 \) and \( \| u \|_{H^1} \neq 0 \), then from
\[
(1 - \sigma) \| u \|_{H^1}^2 = \| u \|_{p+1}^{p+1} \leq S_p^{p+1} \| u \|_{H^1}^{p+1}
\]
it follows that \( \| u \|_{H^1} \geq \left( \frac{1 - \sigma}{\| \nabla u \|} \right)^{1/(p-1)} \). \( \square \)

**Lemma 2.2.** If \( \| u \|_{H^1} < \left( \frac{1 - \sigma}{\| \nabla u \|} \right)^{1/(p-1)} \), then \( I_{\sigma}(u) > 0 \).

**Proof.** By \( \| u \|_{H^1} < \left( \frac{1 - \sigma}{\| \nabla u \|} \right)^{1/(p-1)} \), we obtain
\[
\| u \|_{H^1}^{p+1} \leq S_p^{p+1} \| u \|_{H^1}^{p+1} < (1 - \sigma) \| u \|_{H^1}^2
\]
from which follows \( I_{\sigma}(u) > 0 \). \( \square \)

**Theorem 2.3.** Let \( D_{\sigma} \) be defined as above. Then for \( \sigma > -\frac{p-1}{2} \), we have
\[
D_{\sigma} = \frac{p-1 + 2\sigma}{2(p+1)} \left( \frac{1 - \sigma}{S_p^{p+1}} \right)^{2/(p-1)}.
\]

If we write \( D_{\sigma} \) in terms of \( d \), we obtain
\[
D_{\sigma} = \frac{p-1 + 2\sigma}{2(p+1)} \left[ 1 - \sigma \right]^{2/(p-1)} \frac{2 (p+1)}{p-1} d.
\]
Proof. If \( u \in N_{\sigma} \), we have by Lemma 2.1 that \( \| u \|_{H^1} \geq \left( \frac{1 - \sigma}{S_p} \right)^{(p-1)/2} \). In the proof of Lemma 2.1 the inequality (9) is an equality if \( u \) is a minimizer of the embedding \( H^1 \) into \( L^{p+1} \). Since \( \| u \|_{p+1} = S_p \| u \|_{H^1} \) is attained only for \( \bar{u} = (\cosh (\frac{p-1}{2}) x)^{-\frac{1}{p+1}} \) for \( n = 1 \) [11], and for the ground state solution of (2), (3) for \( n > 1 \) [14] and it has constant sign, we have

\[
\inf_{u \in N_{\sigma}} \| u \|_{H^1} = \left( \frac{1 - \sigma}{S_p} \right)^{(p-1)/2}.
\]

Hence from

\[
\inf_{u \in N_{\sigma}} \| u \|_{H^1} = \inf_{u \in N_{\sigma}} \left( \frac{1}{2} \| u \|_{H^1}^2 - \frac{1}{p+1} \| u \|_{p+1} \right)
\]

\[
= \inf_{u \in N_{\sigma}} \left( \frac{1}{2} - \frac{(1 - \sigma)}{p+1} \right) \| u \|_{H^1}^2 + \frac{1}{p+1} I_\sigma (u)
\]

\[
= \left( \frac{1}{2} - \frac{(1 - \sigma)}{p+1} \right) \inf_{u \in N_{\sigma}} \| u \|_{H^1}^2,
\]

and by definition of \( D_\sigma \) we obtain \( D_\sigma = \frac{p-1+2\sigma}{2(p+1)} \left( \frac{1 - \sigma}{S_p} \right)^{2(p-1)/2} \). \( \square \)

We can also state the following properties of \( D_\sigma \), which can be proved easily.

i) \( D_\sigma \) is strictly increasing on \( \sigma \in \left( -\frac{p-1}{2}, 0 \right) \cup (1, \infty) \) and strictly decreasing on \( (0, 1) \).

ii) \( \lim_{\sigma \to 1} D_\sigma = 0 \), and \( D_{\sigma_0} = 0 \), where \( \sigma_0 = -\frac{p-1}{2} \).

The following theorems show the invariance of \( I_\sigma \) under the flow of (2), (3) for \( 0 < E(0) < d \) and \( E(0) = d \), respectively, and can be proved by contradiction as in [20].

**Theorem 2.4.** Assume that \( u_0 \in H^1 (R^n), u_1 \in L^2 (R^n) \). Let \( 0 < E(0) < d \). Then the sign of \( I_\sigma \) is invariant under the flow of (2), (3) for \( \sigma \in (\sigma_1, \sigma_2) \), where \( \sigma_1 \) and \( \sigma_2 \) are the corresponding minimal negative and minimal positive roots of equation \( D_\sigma = E(0) \).

**Theorem 2.5.** Let all the assumptions of Theorem 2.4 hold and that \( E(0) = d \). Then the sign of \( I_\sigma \) (recall that when \( E(0) = d \), we have \( \sigma_1 = \sigma_2 = 0 \)) is invariant with respect to (2), (3) for every \( t \in [0, \infty) \).

Now, we give a lemma for \( \sigma > 1 \), which states similar results to Lemmas 2.1, 2.2, and can be proved similarly.

**Lemma 2.6.** Assume that \( u \in H^1 (R^n) \). Let \( \sigma > 1 \). If \( I_\sigma (u) > 0 \), then \( \| u \|_{H^1} \geq s (\sigma) \). If \( I_\sigma (u) = 0 \), then \( \| u \|_{H^1} \geq s (\sigma) \) or \( \| u \|_{H^1} = 0 \), where \( s (\sigma) = \left( \frac{\sigma}{\sigma_m} \right)^{(p-1)/2} \). Moreover, if \( \| u \|_{H^1} < s (\sigma) \), then \( I_\sigma (u) \leq 0 \) and \( I_\sigma (u) = 0 \) if and only if \( \| u \|_{H^1} = 0 \).

**Theorem 2.7.** Assume that \( u_0 \in H^1 (R^n), u_1 \in L^2 (R^n) \). If \( E(0) > 0 \), then \( I_\sigma (u(t)) \leq 0 \) for every \( t > 0 \) and \( \sigma \geq \sigma_m \), where \( \sigma_m \) is the maximal positive root of \( D_\sigma = E(0) \).

**Proof.** We give the proof of the theorem for \( \sigma = \sigma_m \) and \( \sigma > \sigma_m \) separately. First, we prove the theorem for \( \sigma = \sigma_m \). By contradiction, assume that there exists some \( t' > 0 \) such that \( I_{\sigma_m} (u(t')) > 0 \). By Lemma 2.1, we have \( \| u \|_{H^1} > 0 \) and there exists a value \( \sigma > \sigma_m \) such that \( I_{\sigma} (u(t')) = 0 \). Then, by (4), \( D_{\sigma_m} = E(0) \geq \inf I (u(t')) \geq \inf I (u) = D_\sigma \). A contradiction occurs, which proves the theorem for \( \sigma = \sigma_m \). For \( \sigma \geq \sigma_m, I_{\sigma_m} (u(t)) \geq I_\sigma (u(t)) \) implies that the theorem is true for every \( \sigma \geq \sigma_m \). \( \square \)
The following Corollary gives a more precise result for subcritical initial energy.

**Corollary 2.8.** Suppose \( u_0 \in H^1 (\mathbb{R}^n), u_1 \in L^2 (\mathbb{R}^n) \). Let \( 0 < E(0) < d \) and \( I_0 (u_0) > 0 \). Then,

\[
0 < I_0 (u(t)) \leq \sigma_m \| u \|_{H^1}^p
\]

for every \( t > 0 \).

**Proof.** We know that for \( I_0 (u(t)) > 0 \), the solution \( u(x,t) \) of problem (2), (3) is globally defined. Since \( E(0) = D_{\sigma_m} \) for some \( \sigma_m > 1 \) then by Theorem 2.7 we have \( I_{\sigma_m} (u(t)) \leq 0 \) for every \( t \in [0, \infty) \). Thus we get the inequality (11) from below and from above. \( \square \)

**Remark 2.9.** We tried to characterize the behavior of solutions for \( E(0) > d \) in terms of initial displacement. We constituted the new functional \( I_\sigma (u) \) and proved the sign invariance of \( I_\sigma (u) \) for \( 0 < E(0) < d \) and \( E(0) = d \). But the case \( E(0) > d \) is still an open question, because from Theorem 2.7, we concluded that in this case \( I_\sigma (u) \) is always non-positive.

### 3. Main Results

We will introduce our new functional which will be used for global existence of solutions with high energy initial data.

\[
\tilde{M}(v, \omega) = \left( \| \nabla v \|^2 + \| v \|^2 \right) - \| v \|_{p+1}^{q+1} - (\omega, \omega)
\]

for every \( v \in H^1 \) and \( \omega \in L^2 \). For simplicity we denote

\[
M(u, t) = \tilde{M}(u(, t), u_t (, t)).
\]

The sign invariance of this new functional can be stated as follows.

**Theorem 3.1.** Let \( u_0 \in H^1 (\mathbb{R}^n), u_1 \in L^2 (\mathbb{R}^n) \) and \( E(0) > 0 \). For \( \sigma > \sigma_m \), assume that

\[
(u_1, u_0) + \frac{1}{2} \| u_0 \|^2 + \frac{(p + 1) \sigma}{p - 1 + (p + 3) \sigma} E(0) \leq 0.
\]

If \( M(u, 0) \) is positive, then \( M(u, t) \) is positive for every \( t \in [0, \infty) \).

**Proof.** [Proof] We prove the theorem by contradiction. Let us define

\[
\theta(t) = \| u \|^2 + \int_0^t \| u \|^2 \, dt.
\]

Then

\[
\theta'(t) = 2 (u_t, u) + \| u \|^2,
\]

\[
\theta''(t) = 2 \| u_t \|^2 + 2 (u_{tt}, u) + 2 (u_t, u)
\]

\[
= 2 \| u_t \|^2 + 2 \left[ \| u \|_{p+1}^{q+1} - \| \nabla u \|^2 - \| u \|^2 - (u_t, u) \right] + 2 (u_t, u)
\]

\[
= -2M(u, t).
\]

To get a contradiction, let us assume that there exists some \( t' > 0 \) such that \( M(u, t') = 0 \). Since \( \theta''(t) < 0 \), we conclude that \( \theta'(t) \) is strictly decreasing on \([0, t')\). Moreover, (13) implies \( \theta'(0) < 0 \) and therefore \( \theta'(t) < 0 \) in
for every $t \in M$.

The proof of this theorem follows from adding some arguments to the local existence result.

Proof. Thereby $I_{E}(0)$ completes the proof.

Therefore

\[ \left\| \nabla \theta^{0} \right\| ^{2} \geq \sigma_{m}^{-1} I_{0} (u (t')) \geq \sigma^{-1} \left\| u_{t} (t') \right\| ^{2}. \]

The use of this inequality in (14) gives

\[ E (0) \geq \left( \frac{1}{2} + \frac{1}{p + 1} + \frac{p - 1}{2 (p + 1) \sigma} \right) \left\| u_{t} (t') \right\| ^{2} \]

\[ = \frac{(p + 3) \sigma + p - 1}{2 (p + 1) \sigma} \left[ \left\| (u_{t} (t') + u (t')) \right\| ^{2} - 2 \left( u_{t} (t'), u (t') \right) - \left\| u (t') \right\| ^{2} \right]. \]

From the monotonicity of $\theta (t)$ and $\theta' (t)$, we get

\[ E (0) > \frac{(p + 3) \sigma + p - 1}{(p + 1) \sigma} \left[ - (u_{1}, u_{0}) - \frac{1}{2} \left\| u_{0} \right\| ^{2} \right] \]

which contradicts with (13). Thus the proof is completed. \( \square \)

**Theorem 3.2.** Let $1 < p < \infty$ for $n = 2$; $1 < p < \frac{n+2}{n-2}$ for $n \geq 3$ and $u_{0} \in H^{1} (\mathbb{R}^{n})$, $u_{1} \in L^{2} (\mathbb{R}^{n})$. Suppose that $E (0) > 0$, $M (u, 0) > 0$ and (13) holds for some $\sigma > \sigma_{m}$. Then, the weak solution of problem (2),(3) is globally defined for every $t \in [0, \infty)$.

Proof. [Proof] The proof of this theorem follows from adding some arguments to the local existence result of Proposition 1.1 of [25]. $M (u, 0) > 0$ implies from the sign preserving property of $M (u, t)$ that $M (u, t) > 0$, thereby $l_{0} (u) > 0$ for every $t > 0$. From energy identity, we have

\[ E (0) \geq \frac{1}{2} \left\| u_{t} \right\| ^{2} + \frac{p - 1}{2 (p + 1)} \left( \left\| \nabla u \right\| ^{2} + \left\| u \right\| ^{2} \right) + \frac{1}{p + 1} I (u) \]

\[ \geq \frac{1}{2} \left\| u_{t} \right\| ^{2} + \frac{p - 1}{2 (p + 1)} \left( \left\| \nabla u \right\| ^{2} + \left\| u \right\| ^{2} \right). \]

Therefore $\left\| u_{t} \right\| _{L^{2}}$ and $\left\| u \right\| _{L^{2}}$ are bounded for every $t > 0$. The previously mentioned local existence theory completes the proof. \( \square \)

**References**


