Regularly Ideal Convergence and Regularly Ideal Cauchy Double Sequences in 2-Normed Spaces

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Abstract. In this paper, we introduce the notions of \((I_2, I), (I_2^*, I^*)\)-convergence and \((I_2, I), (I_2^*, I^*)\)-Cauchy double sequence in regular sense in 2-normed spaces. Also, we study some properties of these concepts.

1. Introduction, Notations and Definitions

Throughout the paper \(\mathbb{N}\) and \(\mathbb{I}\) denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [6] and Schoenberg [26]. This concept was extended to the double sequences by Mursaleen and Edely [17]. The idea of \(I\)-convergence was introduced by Kostyrko et al. [15] as a generalization of statistical convergence which is based on the structure of the ideal \(I\) of subset of the set of natural numbers [6, 7]. Nuray and Ruckle [21] independently introduced the same with another name generalized statistical convergence. Das et al. [2] introduced the concept of \(I_2\)-convergence of double sequences in a metric space and studied some properties. Dündar and Altay [4] studied the concepts of \(I_2\)-Cauchy and \(I_2^*\)-Cauchy for double sequences and they gave the relation between \(I_2\) and \(I_2^*\)-convergence of double sequences of functions defined between linear metric spaces. A lot of development have been made in this area after the works of [3, 16, 18–20, 25, 27–29].

The concept of 2-normed spaces was initially introduced by Gähler [8, 9] in the 1960’s. Since then, this concept has been studied by many authors, see for instance [10–12, 14]. Şahiner et al. [27] and Gürdal [14] studied \(I\)-convergence in 2-normed spaces. Gürdal and Açıkl [13] investigated \(I\)-Cauchy and \(I^*\)-Cauchy sequences in 2-normed spaces. Sarabadan et al. [23, 24] investigated \(I_2\) and \(I_2^*\)-convergence of double sequences in 2-normed spaces. They also examined the concepts \(I_2\)-limit points and \(I_2\)-cluster points in 2-normed spaces. Dündar and Sever [5] introduced the notions of \(I_2\) and \(I_2^*\)-Cauchy double sequences, and studied their some properties with \((AP2)\) in 2-normed spaces.

In this paper, we introduce the notions of \((I_2, I), (I_2^*, I^*)\)-convergence and \((I_2, I), (I_2^*, I^*)\)-Cauchy double sequence in regular sense in 2-normed spaces. Also, we study some properties of these concepts.

Now, we recall the concept of ideal, ideal convergence of sequences, double sequences, 2-normed space and some fundamental definitions and notations (See [1, 2, 8, 11, 13, 15, 22–24]).
A double sequence \( x = (x_{mn})_{m,n\in\mathbb{N}} \) of real numbers is said to be convergent to \( L \in \mathbb{R} \) in Pringsheim’s sense, if for any \( \varepsilon > 0 \) there exists \( N_\varepsilon \in \mathbb{N} \) such that \( |x_{mn} - L| < \varepsilon \) whenever \( m,n > N_\varepsilon \). In this case we write \( P - \lim_{m,n\to\infty} x_{mn} = L \) or \( \lim_{m,n\to\infty} x_{mn} = L \).

Let \( X \neq \emptyset \). A class \( I \) of subsets of \( X \) is said to be an ideal in \( X \) provided:

(i) \( \emptyset \in I \),
(ii) \( A, B \in I \) implies \( A \cup B \in I \),
(iii) \( A \in I \), \( B \subseteq A \) implies \( B \in I \).

\( I \) is called a nontrivial ideal if \( X \notin I \).

Let \( X \neq \emptyset \). A non empty class \( \mathcal{F} \) of subsets of \( X \) is said to be a filter in \( X \) provided:

(i) \( \emptyset \notin \mathcal{F} \),
(ii) \( A, B \in \mathcal{F} \) implies \( A \cap B \in \mathcal{F} \),
(iii) \( A \in \mathcal{F} \), \( A \subseteq B \) implies \( B \in \mathcal{F} \).

If \( I \) is a nontrivial ideal in \( X \), \( X \neq \emptyset \), then the class

\[ \mathcal{F}(I) = \{ M \subseteq X : (\exists A \in I)(M = X \setminus A) \} \]

is a filter on \( X \), called the filter associated with \( I \).

A nontrivial ideal \( I \) in \( X \) is called admissible if \( \{x\} \in I \) for each \( x \in X \).

Throughout the paper we take \( I \) as a nontrivial admissible ideal in \( \mathbb{N} \).

Let \( I \subset 2^N \) be a non trivial ideal and \( (X, \rho) \) be a metric space. A sequence \( (x_n) \) of elements of \( X \) is said to be \( I \)-convergent to \( L \in X \), if for each \( \varepsilon > 0 \) we have \( A(\varepsilon) = \{ n \in \mathbb{N} : \rho(x_n, L) \geq \varepsilon \} \in I \).

Throughout the paper we take \( I_2 \) as a non trivial admissible ideal in \( \mathbb{N} \times \mathbb{N} \).

A non trivial ideal \( I_2 \subset 2^N \times \mathbb{N} \) is called strongly admissible if \( \{ i \} \times \mathbb{N} \) and \( \mathbb{N} \times \{ i \} \) belong to \( I_2 \) for each \( i \in \mathbb{N} \).

It is evident that a strongly admissible ideal is also admissible.

Let \( I_1^0 = \{ A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i,j \geq m(A) \Rightarrow (i,j) \notin A) \} \). Then \( I_1^0 \) is a nontrivial strongly admissible ideal and clearly an ideal \( I_2 \) is strongly admissible if and only if \( I_1^0 \subset I_2 \).

Let \( (X, \rho) \) be a linear metric space and \( I_2 \subset 2^N \times \mathbb{N} \) be a strongly admissible ideal. A double sequence \( x = (x_{mn}) \) in \( X \) is said to be \( I_2 \)-convergent to \( L \in X \), if for any \( \varepsilon > 0 \) we have \( A(\varepsilon) = \{ (m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon \} \in I_2 \) and is written \( I_2 \)-convergent.

If \( I_2 \subset 2^N \times \mathbb{N} \) is a strongly admissible ideal, then usual convergence implies \( I_2 \)-convergence.

Let \( I_2 \) be an ideal of \( \mathbb{N} \times \mathbb{N} \) and \( I \) be an ideal of \( \mathbb{N} \), then a double sequence \( x = (x_{mn}) \in C \), which is the set of complex numbers, is said to be regularly \( (I_2, I) \)-convergent \((r(I_2, I) \)-convergent\), if it is \( I_2 \)-convergent in Pringsheim’s sense and for every \( \varepsilon > 0 \), the following statements hold: \( \{ m \in \mathbb{N} : |x_{mn} - L_m| \geq \varepsilon \} \in I \) for some \( L_m \in C \), for each \( n \in \mathbb{N} \) and \( \{ n \in \mathbb{N} : |x_{mn} - K_m| \geq \varepsilon \} \in I \) for some \( K_m \in C \), for each \( m \in \mathbb{N} \).

We say that an admissible ideal \( I \subset 2^N \) satisfies the property \((AP)\), if for every countable family of mutually disjoint sets \( \{ A_1, A_2, \ldots \} \) belonging to \( I \), there exists a countable family of sets \( \{ B_1, B_2, \ldots \} \) such that \( A_j \Delta B_j \) is a finite set for \( j \in \mathbb{N} \) and \( B = \bigcup_{j=1}^\infty B_j \in I \). (hence \( B_j \in I \) for each \( j \in \mathbb{N} \)).

We say that an admissible ideal \( I_2 \subset 2^N \times \mathbb{N} \) satisfies the property \((AP_2)\), if for every countable family of mutually disjoint sets \( \{ A_1, A_2, \ldots \} \) belonging to \( I_2 \), there exists a countable family of sets \( \{ B_1, B_2, \ldots \} \) such that \( A_j \Delta B_j \in I_1^0 \), i.e., \( A_j \Delta B_j \) is included in the finite union of rows and columns in \( \mathbb{N} \times \mathbb{N} \) for each \( j \in \mathbb{N} \) and \( B = \bigcup_{j=1}^\infty B_j \in I_2 \) (hence \( B_j \in I_2 \) for each \( j \in \mathbb{N} \)).

Let \( X \) be a real vector space of dimension \( d \), where \( 2 \leq d < \infty \). A 2-norm on \( X \) is a function \( \|\cdot,\cdot\| : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) which satisfies (i) \( \|x, y\| = 0 \) if and only if \( x \) and \( y \) are linearly dependent; (ii) \( \|x, y\| = \|y, x\| \); (iii) \( \|ax, y\| = |a| \|x, y\|, a \in \mathbb{R} \); (iv) \( \|x, y + z\| \leq \|x, y\| + \|x, z\| \). The pair \( (X, \|\cdot,\cdot\|) \) is then called a 2-normed space.

As an example of a 2-normed space we may take \( X = \mathbb{R}^2 \) being equipped with the 2-norm \( \|x, y\| := \text{the area of the parallelogram spanned by the vectors} \ x \ y, \text{ which may be given explicitly by the formula} \)

\[ \|x, y\| = |x_1 y_2 - x_2 y_1|, \ x = (x_1, x_2), \ y = (y_1, y_2). \]
The sequence \((x_n)_{n \in \mathbb{N}}\) in a 2-normed space \((X, \| \cdot, \|)\) is said to be convergent to \(L \in X\), if for each \(\varepsilon > 0\) and nonzero \(z \in X\), \(\|x_n - L, z\| < \varepsilon\). In this case we write \(\lim_{n \to \infty} (x_n) = L\).

The double sequence \((x_{mn})_{m,n \in \mathbb{N}}\) in 2-normed space \((X, \| \cdot, \|)\) is said to be convergent to \(L \in X\) in Pringsheim’s sense, if for each \(\varepsilon > 0\) and nonzero \(z \in X\), \(\|x_{mn} - L, z\| < \varepsilon\). In this case we write \(P - \lim_{m,n \to \infty} (x_{mn}) = L\) or \(P - \lim_{m,n \to \infty} x_{mn} = L\).

Let \(I \subset 2^\mathbb{N}\) be a nontrivial ideal. The sequence \((x_n)\) in 2-normed space \((X, \| \cdot, \|)\) is said to be \(I\)-convergent to \(L \in X\), if for each \(\varepsilon > 0\) and nonzero \(z \in X\), \(A(\varepsilon) = \{ n \in \mathbb{N} : \|x_n - L, z\| \geq \varepsilon \} \in I\). In this case we write \(I - \lim_{n \to \infty} (x_n) = L\) or \(I - \lim_{n \to \infty} x_n = L\).

Let \(I \subset 2^\mathbb{N}\) be a nontrivial ideal. The sequence \((x_n)\) in 2-normed space \((X, \| \cdot, \|)\) is said to be \(I^*\)-convergent to \(L \in X\), if there exists a set \(M = \{ m_1, m_2, \ldots, m_k, \ldots \} \subset \mathbb{N}\), \(M \in F(I)\) such that \(\lim_{k \to \infty} 2^\mathbb{N} \|x_{m_k} - L, z\| = 0\), for each nonzero \(z \in X\).

Let \((X, \| \cdot, \|)\) be a linear 2-normed space and \(I \subset 2^\mathbb{N}\) be an admissible ideal. The sequence \((x_n)\) is said to be \(I\)-Cauchy sequence in \(X\), if for each \(\varepsilon > 0\) and nonzero \(z \in X\) there exists a number \(N = N(\varepsilon, z)\) such that \(\|x_n - x_N, z\| \geq \varepsilon\) for each \(n \in \mathbb{N}\).

Let \((X, \| \cdot, \|)\) be a linear 2-normed space and \(I \subset 2^\mathbb{N}\) be an admissible ideal. The sequence \((x_n)\) is said to be \(I^*\)-Cauchy sequence in \(X\), if there exists a set \(M = \{ m_1, m_2, \ldots, m_k, \ldots \} \subset \mathbb{N}\), \(M \in F(I)\) such that \(\lim_{k \to \infty} 2^\mathbb{N} \|x_{m_k} - x_N, z\| = 0\), for each nonzero \(z \in X\).

Let \(I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}\) be a strongly admissible ideal. A double sequence \(x = (x_{mn})_{m,n \in \mathbb{N}}\) in 2-normed space \((X, \| \cdot, \|)\) is said to be \(I_2\)-convergent to \(L \in X\), if for each \(\varepsilon > 0\) and nonzero \(z \in X\), \(A(\varepsilon) = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L, z\| \geq \varepsilon \} \in I_2\). In this case we write \(I_2 - \lim_{m,n \to \infty} (x_{mn}) = L\).

Let \(I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}\) be a strongly admissible ideal. A double sequence \(x = (x_{mn})_{m,n \in \mathbb{N}}\) in 2-normed space \((X, \| \cdot, \|)\) is said to be \(I_2^*\)-convergent to \(L \in X\), if there exists a set \(M \in F(I_2)\) such that \(\lim_{m,n \to \infty} 2^{\mathbb{N}\times\mathbb{N}} \|x_{mn} - L, z\| = 0\), for each nonzero \(z \in X\). In this case we write \(I_2^* - \lim_{m,n \to \infty} x_{mn} = L\).

Let \((X, \| \cdot, \|)\) be a linear 2-normed space and \(I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}\) be a strongly admissible ideal. A double sequence \(x = (x_{mn})\) in \(X\) is said to be \(I_2^*\)-Cauchy if for each \(\varepsilon > 0\) and nonzero \(z \in X\) there exist \(s = s(\varepsilon, z)\), \(t = t(\varepsilon, z) \in \mathbb{N}\) such that

\[ A(\varepsilon) := \{ (m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{st}, z\| \geq \varepsilon \} \in I_2. \]

Let \((X, \| \cdot, \|)\) be a linear 2-normed space and \(I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}\) be a strongly admissible ideal. A double sequence \(x = (x_{mn})\) in \(X\) is said to be \(I_2^*\)-Cauchy if there exists a set \(M \in F(I_2)\) such that \(\varepsilon > 0\) and for all \((m, n), (s, t) \in M\),

\[ \|x_{mn} - x_{st}, z\| < \varepsilon, \text{ for each nonzero } z \in X, \]

where \(m, n, s, t > k_0 = k_0(\varepsilon) \in \mathbb{N}\). In this case we write

\[ \lim_{m,n,s,t \to \infty} \|x_{mn} - x_{st}, z\| = 0. \]

Now, we begin with quoting the following lemmas due to Sarabadan et al. [24] and Dündar, Sever [5] which are needed throughout the paper.

**Lemma 1.1.** [24, Theorem 4.3] Let \(I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}\) be a strongly admissible ideal with property (AP2) and \((X, \| \cdot, \|)\) be a finite dimensional 2-normed space, then for a double sequence \(x = (x_{mn})\) of \(X\), \(I_2 - \lim_{m,n \to \infty} x_{mn} = L\) implies \(I_2 - \lim_{m,n \to \infty} x_{mn} = L\).

**Lemma 1.2.** [5, Theorem 3.2] Let \((X, \| \cdot, \|)\) be a linear 2-normed space and \(I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}\) be a strongly admissible ideal. If \(x = (x_{mn})\) in \(X\) is \(I_2\)-convergent then \(x = (x_{mn})\) is \(I_2\)-Cauchy double sequence.

**Lemma 1.3.** [5, Theorem 3.4] Let \((X, \| \cdot, \|)\) be a linear 2-normed space and \(I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}\) be a strongly admissible ideal. If \(x = (x_{mn})\) in \(X\) is \(I_2^*\)-Cauchy double sequence then \(x = (x_{mn})\) is \(I_2^*\)-Cauchy double sequence.
2. Main Results

The proof of the following lemma is similar to the proof of [2, Theorem 1], so we omit it.

**Lemma 2.1.** Let \((X, \| \cdot \|)\) be a linear 2-normed space and \(I_2, I_2^* \subset 2^{N \times N}\) be a strongly admissible ideal. Then for \(x = (x_{mn})\) be a double sequence of \(X, I_2^* - \lim_{m,n \to \infty} x_{mn} = I_2 - \lim_{m,n \to \infty} x_{mn} = I_2^*\).

**Lemma 2.2.** Let \((X, \| \cdot \|)\) be a linear 2-normed space and \(I_2, I_2^* \subset 2^{N \times N}\) be a strongly admissible ideal. Then for \(x = (x_{mn})\) be a double sequence of \(X, L \in X\) and for each nonzero \(z \in X,\)

\[
P - \lim_{m,n \to \infty} \|x_{mn} - L, z\| = 0 \text{ implies } I_2 - \lim_{m,n \to \infty} \|x_{mn} - L, z\| = 0.
\]

**Proof.** Let

\[
P - \lim_{m,n \to \infty} \|x_{mn} - L, z\| = 0.
\]

For each \(\varepsilon > 0\) and nonzero \(z \in X\) there exists \(k_0 = k_0(\varepsilon) \in N\) such that \(\|x_{mn} - L, z\| < \varepsilon\) for all \(m, n \geq k_0\). Then,

\[
A(\varepsilon) = \{(m, n) \in N \times N : \|x_{mn} - L, z\| \geq \varepsilon\} \\
\subseteq (N \times \{1, 2, \ldots, (k_0 - 1)\} \cup \{1, 2, \ldots, (k_0 - 1)\} \times N).
\]

Since \(I_2\) is a strongly admissible ideal we have \((N \times \{1, 2, \ldots, (k_0 - 1)\} \cup \{1, 2, \ldots, (k_0 - 1)\} \times N) \in I_2\) and so \(A(\varepsilon) \in I_2\). Hence, this completes the proof. \(\square\)

Now, we study certain properties of regularly convergence, regularly \((I_2, I)\)-convergence and regularly \((I_2, I)\)-Cauchy double sequences in 2-normed spaces.

**Definition 2.3.** Let \((X, \| \cdot \|)\) be a linear 2-normed space. A double sequence \((x_{mn})\) in \(X\) is said to be regularly convergent, if it is convergent in Pringsheim’s sense and the limits

\[
\lim_{m \to \infty} x_{mn}, \quad (n \in N) \quad \text{and} \quad \lim_{n \to \infty} x_{mn}, \quad (m \in N),
\]

exist for each fixed \(n \in N\) and \(m \in N\), respectively. Note that if \((x_{mn})\) is regularly convergent to \(L\) in \(X\), then the limits

\[
\lim_{n \to \infty} \lim_{m \to \infty} x_{mn} \quad \text{and} \quad \lim_{m \to \infty} \lim_{n \to \infty} x_{mn}
\]

exist and are equal to \(L\). In this case we write

\[
r - \lim_{m,n \to \infty} x_{mn} = L \quad \text{or} \quad x_{mn} \xrightarrow{r} L.
\]

**Definition 2.4.** Let \(I_2 \subset 2^{N \times N}\) be a strongly admissible ideal, \(I \subset 2^N\) be an admissible ideal and \((X, \| \cdot \|)\) be a linear 2-normed space. A double sequence \((x_{mn})\) in \(X\) is said to be regularly \((I_2, I)\)-convergent \((r(I_2, I)\)-convergent\), if it is \(I_2\)-convergent in Pringsheim’s sense and for each \(\varepsilon > 0\) and nonzero \(z \in X\), the following statements hold:

\[
\{m \in N : \|x_{mn} - L, z\| \geq \varepsilon\} \in I
\]

for some \(L \in X\), for each \(n \in N\) and

\[
\{n \in N : \|x_{mn} - K, z\| \geq \varepsilon\} \in I
\]

for some \(K \in X\), for each \(m \in N\).

If \((x_{mn})\) is regularly \((I_2, I)\)-convergent \((r(I_2, I)\)-convergent\) to \(L \in X\), then the limits \(I - \lim_{n \to \infty} \lim_{m \to \infty} x_{mn}\) and \(I - \lim_{m \to \infty} \lim_{n \to \infty} x_{mn}\) exist and are equal to \(L\).

**Theorem 2.5.** Let \(I_2 \subset 2^{N \times N}\) be a strongly admissible ideal, \(I \subset 2^N\) be an admissible ideal and \((X, \| \cdot \|)\) be a linear 2-normed space. If a double sequence \((x_{mn})\) in \(X\) is regularly convergent, then \((x_{mn})\) is \((I_2, I)\)-convergent.
Proof. Let \((x_{mn})\) be regularly convergent. Then \((x_{mn})\) is convergent in Pringsheim’s sense and the limits \(\lim_{m \to \infty} x_{mn} (n \in \mathbb{N})\) and \(\lim_{m \to \infty} x_{mn} (m \in \mathbb{N})\) exist. By Lemma 2.2, \((x_{mn})\) is \(I_2\)-convergent. Also, for each \(\varepsilon > 0\) and nonzero \(z \in X\), there exist \(m = m_0(\varepsilon)\) and \(n = n_0(\varepsilon)\) such that
\[
\|x_{mn} - L_n, z\| < \varepsilon
\]
for some \(L_n\) and each fixed \(n \in \mathbb{N}\) for every \(m \geq m_0\) and
\[
\|x_{mn} - K_m, z\| < \varepsilon
\]
for some \(K_m\) and each fixed \(m \in \mathbb{N}\) for every \(n \geq n_0\). Then, since \(I\) is an admissible ideal so for each \(\varepsilon > 0\) and nonzero \(z \in X\), we have
\[
\{m \in \mathbb{N} : \|x_{mn} - L_n, z\| \geq \varepsilon\} \subset \{1, 2, \ldots, m_0 - 1\} \in I,
\]
\[
\{n \in \mathbb{N} : \|x_{mn} - K_m, z\| \geq \varepsilon\} \subset \{1, 2, \ldots, n_0 - 1\} \in I.
\]
Hence, \((x_{mn})\) is \(r(I_2, I)\)-convergent in \(X\). \(\square\)

**Definition 2.6.** Let \(I_2\) be a strongly admissible ideal of \(\mathbb{N} \times \mathbb{N}\), \(I\) be an admissible ideal of \(\mathbb{N}\) and \((X, \|\cdot, \cdot\|)\) be a linear 2-normed space. A double sequence \((x_{mn})\) in \(X\) is said to be \(r(I_2', I')\)-convergent, if there exist the sets \(M \in \mathcal{F}(I_2)\) (i.e., \(\mathbb{N} \times \mathbb{N} \setminus M \in I_2\)), \(M_1 \in \mathcal{F}(I)\) and \(M_2 \in \mathcal{F}(I)\) (i.e., \(\mathbb{N} \setminus M_1 \in I\) and \(\mathbb{N} \setminus M_2 \in I\)) such that the limits
\[
\lim_{m,n \to \infty} x_{mn}, \lim_{m \to \infty} x_{mn} \text{ and } \lim_{n \to \infty} x_{mn}
\]
exist for each fixed \(n \in \mathbb{N}\) and \(m \in \mathbb{N}\), respectively.

**Theorem 2.7.** Let \(I_2\) be a strongly admissible ideal of \(\mathbb{N} \times \mathbb{N}\), \(I\) be an admissible ideal of \(\mathbb{N}\) and \((X, \|\cdot, \cdot\|)\) be a linear 2-normed space. If a double sequence \((x_{mn})\) in \(X\) is \(r(I_2', I')\)-convergent, then it is \(r(I_2, I)\)-convergent.

**Proof.** Let \((x_{mn})\) in \(X\) be \(r(I_2', I')\)-convergent. Then, it is \(I_2\)-convergent and so, by Lemma 2.1, it is \(I_2\)-convergent. Also, there exist the sets \(M_1, M_2 \in \mathcal{F}(I)\) such that
\[
(\forall z \in X) (\forall \varepsilon > 0) (\exists m_0 \in \mathbb{N}) (\forall m \geq m_0) (m \in M_1) \|x_{mn} - L_n, z\| < \varepsilon, (n \in \mathbb{N})
\]
for some \(L_n \in X\) and
\[
(\forall z \in X) (\forall \varepsilon > 0) (\exists n_0 \in \mathbb{N}) (\forall n \geq n_0) (n \in M_2) \|x_{mn} - K_m, z\| < \varepsilon, (m \in \mathbb{N})
\]
for some \(K_m \in X\). Hence, for each \(\varepsilon > 0\) and nonzero \(z \in X\), we have
\[
A(\varepsilon) = \{m \in \mathbb{N} : \|x_{mn} - L_n, z\| \geq \varepsilon\} \subset H_1 \cup \{1, 2, \ldots, m_0 - 1\}, (n \in \mathbb{N}),
\]
\[
B(\varepsilon) = \{n \in \mathbb{N} : \|x_{mn} - K_m, z\| \geq \varepsilon\} \subset H_2 \cup \{1, 2, \ldots, n_0 - 1\}, (m \in \mathbb{N}),
\]
for \(H_1, H_2 \in I\). Since \(I\) is an admissible ideal we get
\[
H_1 \cup \{1, 2, \ldots, (m_0 - 1)\} \in I, \quad H_2 \cup \{1, 2, \ldots, (n_0 - 1)\} \in I
\]
and therefore \(A(\varepsilon), B(\varepsilon) \in I\). This shows that the double sequence \((x_{mn})\) is \(r(I_2, I)\)-convergent in \(X\). \(\square\)

**Theorem 2.8.** Let \(I_2 \subset 2^{\mathbb{N} \times \mathbb{N}}\) be a strongly admissible ideal with property (AP2), \(I \subset 2^\mathbb{N}\) be an admissible ideal with property (AP) and \((X, \|\cdot, \cdot\|)\) be a linear 2-normed space. If a double sequence \((x_{mn})\) is \(r(I_2, I)\)-convergent, then \((x_{mn})\) is \(r(I_2', I')\)-convergent in \(X\).
Proof. Let a double sequence \((x_{mn})\) in \(X\) be \(r(I_2, I)\)-convergent. Then \((x_{mn})\) is \(I_2\)-convergent and so \((x_{mn})\) is \(I_2^\ast\)-convergent, by Lemma 1.1. Also, for each \(\varepsilon > 0\) and nonzero \(z \in X\) we have

\[
A(\varepsilon) = \{m \in \mathbb{N} : \|x_{mn} - L_n, z\| \geq \varepsilon\} \in I
\]

for some \(L_n \in X\), for each \(n \in \mathbb{N}\) and

\[
C(\varepsilon) = \{n \in \mathbb{N} : \|x_{mn} - K_m, z\| \geq \varepsilon\} \in I
\]

for some \(K_m \in X\), for each \(m \in \mathbb{N}\).

Now put for each nonzero \(z \in X\)

\[
A_1 = \{m \in \mathbb{N} : \|x_{mn} - L_n, z\| \geq 1\},
\]

\[
A_k = \left\{ m \in \mathbb{N} : \frac{1}{k} \leq \|x_{mn} - L_n, z\| < \frac{1}{k-1} \right\}
\]

for \(k \geq 2\), for some \(L_n \in X\) and for each \(n \in \mathbb{N}\). It is clear that \(A_1 \cap A_j = \emptyset\) for \(i \neq j\) and \(A_j \in I\) for each \(i \in \mathbb{N}\).

By the property \(\text{(AP)}\) there is a countable family of sets \(\{B_1, B_2, \ldots\}\) in \(I\) such that \(A_j \triangle B_j\) is a finite set for each \(j \in \mathbb{N}\) and \(B = \bigcup_{j=1}^{\infty} B_j \in I\).

We prove that

\[
\lim_{m \to \infty \atop m \in M} \|x_{mn} - L_n, z\| = 0, \quad \text{for some} \quad L_n \quad \text{and for each} \quad n \in \mathbb{N}
\]

for each nonzero \(z \in X\) and for \(M = \mathbb{N} \setminus B \in \mathcal{F}(I)\). Let \(\delta > 0\) be given. Choose \(k \in \mathbb{N}\) such that \(1/k < \delta\). Then, for each nonzero \(z \in X\) we have

\[
\{ m \in \mathbb{N} : \|x_{mn} - L_n, z\| \geq \delta \} \subseteq \bigcup_{j=1}^{k} A_j \quad \text{for some} \quad L_n \quad \text{and for each} \quad n \in \mathbb{N}.
\]

Since \(A_j \triangle B_j\) is a finite set for \(j \in \{1, 2, \ldots, k\}\), there exists \(m_0 \in \mathbb{N}\) such that

\[
\left( \bigcup_{j=1}^{k} B_j \right) \cap \{ m : m \geq m_0 \} = \left( \bigcup_{j=1}^{k} A_j \right) \cap \{ m : m \geq m_0 \}.
\]

If \(m \geq m_0\) and \(m \notin B\) then

\[
m \notin \bigcup_{j=1}^{k} B_j \quad \text{and so} \quad m \notin \bigcup_{j=1}^{k} A_j.
\]

Thus, for each nonzero \(z \in X\) we have \(\|x_{mn} - L_n, z\| < \frac{1}{k} < \delta\) for some \(L_n\) and for each \(n \in \mathbb{N}\). This implies that

\[
\lim_{m \to \infty \atop m \in M} \|x_{mn} - L_n, z\| = 0.
\]

Hence, for each nonzero \(z \in X\) we have

\[
I^\ast - \lim_{m \to \infty \atop m \in M} \|x_{mn} - L_n, z\| = 0
\]

for some \(L_n\) and for each \(n \in \mathbb{N}\).

Similarly, for the set \(C(\varepsilon) = \{ n \in \mathbb{N} : \|x_{mn} - K_m, z\| \geq \varepsilon\} \in I\), for each nonzero \(z \in X\) we have

\[
I^\ast - \lim_{m \to \infty \atop m \in M} \|x_{mn} - K_m, z\| = 0
\]

for \(K_m\) and for each \(m \in \mathbb{N}\). Hence, a double sequence \((x_{mn})\) is \(r(I_2^\ast, I^\ast)\)-convergent.

Now, we give the definitions of \(r(I_2, I)\)-Cauchy sequence and \(r(I_2^\ast, I^\ast)\)-Cauchy sequence.
Definition 2.9. Let $I_2$ be a strongly admissible ideal of $\mathbb{N} \times \mathbb{N}$, $I$ be an admissible ideal of $\mathbb{N}$ and $(\mathbb{N}, \| \cdot \|)$ be a linear 2-normed space. A double sequence $(x_{mn})$ in $X$ is said to be regularly $((I_2, I), r(I))$-Cauchy if it is $I_2$-Cauchy in Pringsheim’s sense and for each $\varepsilon > 0$ and nonzero $z \in X$ there exist $k_n = k_n(\varepsilon, z) \in \mathbb{N}$ and $l_m = l_m(\varepsilon, z) \in \mathbb{N}$ such that the following statements hold:

\[
A_1(\varepsilon) = \{m \in \mathbb{N} : \|x_{mn} - x_{k_n n}, z\| \geq \varepsilon \} \in I, \quad (m \in \mathbb{N}),
\]
\[
A_2(\varepsilon) = \{n \in \mathbb{N} : \|x_{mn} - x_{ml_m n}, z\| \geq \varepsilon \} \in I, \quad (m \in \mathbb{N}).
\]

Definition 2.10. Let $I_2$ be a strongly admissible ideal of $\mathbb{N} \times \mathbb{N}$, $I$ be an admissible ideal of $\mathbb{N}$ and $(\mathbb{N}, \| \cdot \|)$ be a linear 2-normed space. A double sequence $(x_{mn})$ is said to be regularly $((I_2, I'), r(I_2, I'))$-Cauchy, if there exist the sets $M \subseteq \mathbb{N} \times \mathbb{N}$, $M_1 \subseteq \mathbb{N} \times \mathbb{N}$, $M_2 \subseteq \mathbb{N} \times \mathbb{N}$ such that $(I_2, I')$-Cauchy and by Lemma 1.3, also, since the double sequence $(x_{mn})$ is $r(I_2, I')$-Cauchy so the existence of the sets $M_1, M_2 \subseteq \mathbb{N} \times \mathbb{N}$ such that for each $\varepsilon > 0$ and nonzero $z \in X$ there exist $k_n = k_n(\varepsilon) \in \mathbb{N}$ and $l_m = l_m(\varepsilon) \in \mathbb{N}$ such that

\[
\|x_{mn} - x_{k_n n}, z\| < \varepsilon, \quad \text{for each } m \in M_1 \text{ and for each } n \in \mathbb{N},
\]
\[
\|x_{mn} - x_{ml_m n}, z\| < \varepsilon, \quad \text{for each } n \in M_2 \text{ and for each } m \in \mathbb{N},
\]

whenever $m, n, s, t, k_n, l_m \geq N$.

Theorem 2.11. Let $I_2$ be a strongly admissible ideal of $\mathbb{N} \times \mathbb{N}$ and $I$ be an admissible ideal of $\mathbb{N}$ and $(\mathbb{N}, \| \cdot \|)$ be a linear 2-normed space. If a double sequence $(x_{mn})$ in $X$ is $r(I_2, I')$-Cauchy, then it is $r(I_2, I)$-Cauchy.

Proof. Since a double sequence $(x_{mn})$ in $X$ is $r(I_2, I')$-Cauchy, it is $I_2'$-Cauchy. We know that $I_2'$-Cauchy implies $I_2$-Cauchy by Lemma 1.3. Also, since the double sequence $(x_{mn})$ is $r(I_2, I')$-Cauchy so there exist the sets $M_1, M_2 \subseteq \mathbb{N} \times \mathbb{N}$ such that for each $\varepsilon > 0$ and nonzero $z \in X$ there exist $k_n = k_n(\varepsilon) \in \mathbb{N}$ and $l_m = l_m(\varepsilon) \in \mathbb{N}$ such that

\[
\|x_{mn} - x_{k_n n}, z\| < \varepsilon, \quad \text{for each } m \in M_1 \text{ and for each } n \in \mathbb{N},
\]
\[
\|x_{mn} - x_{ml_m n}, z\| < \varepsilon, \quad \text{for each } n \in M_2 \text{ and for each } m \in \mathbb{N},
\]

for $N = N(\varepsilon) \in \mathbb{N}$ and $m, n, k_n, l_m \geq N$. Therefore, for $H_1 = \mathbb{N} \setminus M_1 \subset I, H_2 = \mathbb{N} \setminus M_2 \subset I$ we have

\[
A_1(\varepsilon) = \{m \in \mathbb{N} : \|x_{mn} - x_{k_n n}, z\| \geq \varepsilon \} \subset H_1 \cup \{1, 2, \ldots, N - 1\}, \quad (n \in \mathbb{N})
\]

for $m \in M_1$ and

\[
A_2(\varepsilon) = \{n \in \mathbb{N} : \|x_{mn} - x_{ml_m n}, z\| \geq \varepsilon \} \subset H_2 \cup \{1, 2, \ldots, N - 1\}, \quad (m \in \mathbb{N})
\]

for $n \in M_2$. Since $I$ is an admissible ideal,

\[
H_1 \cup \{1, 2, \ldots, N - 1\} \in I \quad \text{and} \quad H_2 \cup \{1, 2, \ldots, N - 1\} \in I.
\]

Hence, we have $A_1(\varepsilon), A_2(\varepsilon) \in I$ and $(x_{mn})$ is $r(I_2, I)$-Cauchy double sequence.

Theorem 2.12. Let $I_2$ be a strongly admissible ideal of $\mathbb{N} \times \mathbb{N}$ and $I$ be an admissible ideal of $\mathbb{N}$ and $(\mathbb{N}, \| \cdot \|)$ be a linear 2-normed space. If a double sequence $(x_{mn})$ in $X$ is $r(I_2, I)$-convergent, then $(x_{mn})$ is $r(I_2, I)$-Cauchy double sequence.

Proof. Let $(x_{mn})$ be a $r(I_2, I)$-convergent double sequence in $X$. Then $(x_{mn})$ is $I_2$-convergent and by Lemma 1.2, it is $I_2$-Cauchy double sequence. Also for each $\varepsilon > 0$ and nonzero $z \in X$, we have

\[
A_1(\varepsilon) = \{m \in \mathbb{N} : \|x_{mn} - L_m, z\| \geq \frac{\varepsilon}{2} \} \in I
\]

for some $L_m$, for each $n \in \mathbb{N}$ and

\[
A_2(\varepsilon) = \{n \in \mathbb{N} : \|x_{mn} - K_n, z\| \geq \frac{\varepsilon}{2} \} \in I
\]
for some $K_m$, for each $m \in \mathbb{N}$. Since $I$ is an admissible ideal, the sets
\[ A_1^I(\frac{\varepsilon}{2}) = \{ m \in \mathbb{N} : \|x_{mn} - L_n, z\| < \frac{\varepsilon}{2}, \ (n \in \mathbb{N}) \} \]
for some $L_n$ and
\[ A_2^I(\frac{\varepsilon}{2}) = \{ n \in \mathbb{N} : \|x_{mn} - K_m, z\| < \frac{\varepsilon}{2}, \ (m \in \mathbb{N}) \} \]
for some $K_m$, are nonempty and belong to $\mathcal{F}(I)$. For $k_n \in A_1^I(\frac{\varepsilon}{2}, (n \in \mathbb{N}$ and $k_n > 0$ we have
\[ \|x_{k_n} - L_n, z\| < \frac{\varepsilon}{2} \]
for some $L_n$. Now, for each $\varepsilon > 0$ and nonzero $z \in X$ we define the set
\[ B_1(\varepsilon) = \{ m \in \mathbb{N} : \|x_{mn} - x_{k_n}, z\| \geq \varepsilon, \ (n \in \mathbb{N}) \} \]
where $k_n = k_n(\varepsilon) \in \mathbb{N}$. Let $m \in B_1(\varepsilon)$. Then for $k_n \in A_1^I(\frac{\varepsilon}{2}, (n \in \mathbb{N}$ and $k_n > 0$ we have
\[ \varepsilon \leq \|x_{mn} - x_{k_n}, z\| \leq \|x_{mn} - L_n, z\| + \|x_{k_n} - L_n, z\| < \|x_{mn} - L_n, z\| + \frac{\varepsilon}{2} \]
for some $L_n$. This shows that
\[ \frac{\varepsilon}{2} < \|x_{mn} - L_n, z\| \text{ and so } m \in A_1^I(\frac{\varepsilon}{2}). \]
Hence, we have $B_1(\varepsilon) \subset A_1^I(\frac{\varepsilon}{2})$.

Similarly, for each $\varepsilon > 0$, nonzero $z \in X$ and for $l_m \in A_2^I(\frac{\varepsilon}{2}, (m \in \mathbb{N}$ and $l_m > 0$ we have
\[ \|x_{ml_m} - K_m, z\| < \frac{\varepsilon}{2}, \ (m \in \mathbb{N}) \]
for some $K_m$. Therefore, it can be seen that
\[ B_2(\varepsilon) = \{ m \in \mathbb{N} : \|x_{ml_m} - K_m, z\| \geq \varepsilon \} \subset A_2^I(\frac{\varepsilon}{2}). \]
Hence, we have $B_1(\varepsilon), B_2(\varepsilon) \in I$. This shows that $(x_{mn})$ is $r(I_2, I)$-Cauchy double sequence. \qed

References

[5] E. Dündar, Y. Sever, $I_2$-Cauchy Double Sequences In 2-Normed Spaces, (Submitted to journal)