The Expression for the Group Inverse of the Anti–triangular Block Matrix

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Abstract. In this paper, we present the explicit expression for the group inverse of the sum of two matrices. As an application, the explicit expression for the group inverse of the anti-triangular block matrix \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) and \( \begin{pmatrix} 0 & B \\ C & D \end{pmatrix} \) are obtained without any conditions on sub-blocks.

1. Introduction

Let \( M_n(\mathbb{C}) \) be the set of all \( n \times n \) matrix on complex field \( \mathbb{C} \) and let \( I \) denote the unit of \( M_n(\mathbb{C}) \). For an element \( A \in M_n(\mathbb{C}) \), if there is an element \( B \in M_n(\mathbb{C}) \) which satisfies \( ABA = A \), then \( B \) is called a \([1]\)-inverse of \( A \). If \( ABA = A \) and \( BAB = B \) hold, then \( B \) is called a \([2]\)-inverse of \( A \), denoted by \( A^+ \). An element \( B \) is called the Drazin inverse of \( A \), if \( B \) satisfies

\[ A^kBA = A^k, \quad BAB = B, \quad AB = BA \quad \text{for some integer } k. \]

\( B \) is denoted by \( A^D \). The least such integer \( k \) is called the index of \( A \), denoted by \( \text{ind}(A) \). We denote by \( A^* = I - AA^D \) the spectral idempotent of \( A \). In the case \( \text{ind}(A) = 1 \), \( A^D \) reduces to the group inverse of \( A \), denoted by \( A^# \).

The Drazin inverse has various applications in singular differential equations and singular difference equations, Markov chains, and iterative methods (see [4–7, 9–11, 15, 16, 20]). In 1979, S. Campbell and C. Meyer proposed an open problem to find an explicit representation for the Drazin inverse of a \( 2 \times 2 \) block matrix \( \begin{pmatrix} A & C \\ B & D \end{pmatrix} \) in terms of its sub-blocks, where \( A \) and \( D \) are supposed to be square matrices (see [4]). A simplified problem to find an explicit representation for the Drazin inverse of \( \begin{pmatrix} A & C \\ B & 0 \end{pmatrix} \) was proposed by S. Campbell in 1983 (see [7]). Until now, both problems have not been solved. However, many authors have considered the two problems under certain conditions on the sub-blocks (see [3, 9, 12, 13, 15, 17, 18, 25]). As

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a special case, the expression for the group inverse of $2 \times 2$ block matrix also has been studied under some conditions (see [1, 2, 8, 14, 18, 19, 23]).

In this paper, we give the explicit expression for the group inverse of the sum of two matrices. As an application, the expression for the group inverse of the anti-triangular matrix $egin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ and $egin{pmatrix} 0 & B \\ C & D \end{pmatrix}$ are presented without any conditions on sub-blocks.

2. Preliminaries

In this section, we present some important lemmas and investigate the expression of the group inverse in term of its generalized inverse. Let us begin with a familiar lemma.

**Lemma 2.1.** [19] Let $A, B \in M_n(\mathbb{C})$, then $I + AB$ is invertible if $I + BA$ is invertible and $(I + AB)^{-1} = I - A(I + BA)^{-1}B$.

**Lemma 2.2.** Let $P$ be an idempotent matrix in $M_n(\mathbb{C})$, then $I - P$ is invertible if $P = 0$.

**Proof.** Since $I - P$ is invertible, there is an $X \in M_n(\mathbb{C})$ such that $X(I - P) = I$, that is, $X - XP = I$. So, $0 = XP - XP = P$. \qed

**Lemma 2.3.** [24, Theorem 4.5.9] Let $A \in M_n(\mathbb{C})\setminus\{0\}$. Then the following conditions are equivalent:

1. $A^\#$ exists.
2. $AA^+ + A^+A - I$ is invertible for some $A^+$.
3. $A^2A^+ + I - AA^+$ is invertible for some $A^+$.
4. $A^2A^+ + I - AA^+$ is invertible for any $A^+$.
5. $A + I - AA^+$ is invertible for some $A^+$.
6. $A + I - AA^+$ is invertible for any $A^+$.
7. $A^+A^2 + I - A^+A$ is invertible for some $A^+$.
8. $A^+A^2 + I - A^+A$ is invertible for any $A^+$.
9. $A + I - A^+A$ is invertible for some $A^+$.
10. $A + I - A^+A$ is invertible for any $A^+$.

**Proof.** The equivalence of (1) to (4) are presented in [24, Theorem 4.5.9]. Noting that $A^2A^+ + I - AA^+ = I + (A - I)AA^+$ and $A + I - AA^+ = I + A(I - A^+)$, by Lemma 2.1, we have (3) $\iff$ (5), (4) $\iff$ (6), (7) $\iff$ (9), (8) $\iff$ (10), (5) $\iff$ (9), (6) $\iff$ (10). \qed

**Lemma 2.4.** Let $A \in M_n(\mathbb{C})\setminus\{0\}$. If $A^\#$ exists then

$$A^\# = A(I + A - A^+A)^{-2} = (I + A - AA^+)^{-2}A = (I + A - AA^+)^{-1}A(I + A - A^+A)^{-1},$$

independent of the choice of $A^+$. 

Proof. Put \( W = A^2A^* + I - AA^* \). By Lemma 2.3, \( W \) is invertible. Set \( B = W^{-2}A \). Noting that \( AA^*W = WAA^* = A^2A^* \), we have
\[
AB = AW^{-2}A = AW^{-2}AA^*A = A^2A^*W^{-2}A = AA^*WW^{-2}A = W^{-1}A,
\]
\[
BA = W^{-2}A^2 = W^{-2}WA = W^{-1}A,
\]
\[
ABA = W^{-1}A^2 = W^{-1}WA = A,
\]
\[
BABB = W^{-1}AW^{-2}A = W^{-1}AAA^*W^{-2}A = W^{-1}AA^*W^{-1}A = W^{-2}A.
\]
The above indicate that \( B = A \) and independent of the choice of \( A^+ \).

Noting that
\[
(I + A - AA^*)^{-1}A = A(I + A - A^+A)^{-1},
\]
by Lemma 2.1, we have
\[
A^\# = (A^2A^* + I - AA^*)^{-2}A
\]
\[
= [I + (A - I)AA^*]^{-2}A
= [I - (A - I)(I + A - AA^*)^{-1}AA^*]^{-2}A
= [I - (A - I)(I + A - AA^*)^{-1}AA^*][2I - AA^*](I + A - AA^*)^{-1}A
= [I - (A - I)(I + A - AA^*)^{-1}AA^*]A(I + A - A^+A)^{-1}
= A(I + A - A^+A)^{-2}
= (I + A - AA^*)^{-2}A
= (I + A - AA^*)^{-1}A(I + A - A^+A)^{-1}.
\]

\[\square\]

Proposition 2.5. Let \( A, B \in M_n(\mathbb{C}) \). Then \( (AB)^\# \) exists iff \( I + AB - ABB^*(A^+ABB^*)^+A^+ \) is invertible iff \( I + AB - B^*(A^+ABB^*)^+A^+AB \) is invertible. In this case,
\[
(AB)^\# = (I + AB - ABB^*(A^+ABB^*)^+A^+)^{-2}AB
= AB[I + AB - B^*(A^+ABB^*)^+A^+AB]^{-2}.
\]

Proof. Noting that \( B^*(A^+ABB^*)^+A^+ \) is a \([1,2]\)–inverse of \( AB \). By Lemma 2.4, we get the results. \[\square\]

Corollary 2.6. Let \( A, B \in M_n(\mathbb{C}) \) with \( (AB)^\# \) exists.

1. If \( B \) is invertible, then \( (AB)^\# \) exists iff \( I + AB - AA^+ \) is invertible. In this case,
\[
(AB)^\# = (I + AB - AA^+)^{-2}AB.
\]

2. If \( A \) is invertible, then \( (AB)^\# \) exists iff \( I + AB - B^+B \) is invertible. In this case,
\[
(AB)^\# = AB(I + AB - B^+B)^{-2}.
\]

3. Main Results

Let \( A, B, C, D \in M_n(\mathbb{C}) \). Throughout of this paper, we denote \( E_A = I - AA^+, F_A = I - A^+A \).

Lemma 3.1. Let \( A, X, Y \in M_n(\mathbb{C}) \) and \( Z = I - YA^+X, U = E_AX, V = YF_A, S = E_YZF_UL \). Let
\[
G = A^+ - F_AV^+YA^+ + (F_AV^+Z + A^+X)[F_US^+E_YYA^+ - (I - F_US^+E_YZ)]U^+E_A.
\]
Then \( G \) is a \([1]\)–inverse of \( A - XY \).
Proof.

\[(A - XY)G = AA^* + AA^*X[F_{uS}^*E_VYA^* - (I - F_{uS}^*E_VZ)U^*E_A]
= XE_YYA^* - (XXY^* + XXX^*X)[F_{uS}^*E_VYA^* - (I - F_{uS}^*E_VZ)U^*E_A]
= AA^* + X[F_{uS}^*E_YYA^* - (I - F_{uS}^*E_VZ)U^*E_A] + UU^*E_A
= XE_YYA^* - (XXY^* + X(I - Z)[F_{uS}^*E_VYA^* - (I - F_{uS}^*E_VZ)U^*E_A]
= AA^* + X[F_{uS}^*E_YYA^* - (I - F_{uS}^*E_VZ)U^*E_A] + UU^*E_A
= XE_YYA^* - (XXY^* + X(I - Z)[F_{uS}^*E_VYA^* - (I - F_{uS}^*E_VZ)U^*E_A]
= AA^* + UU^*E_A - XE_S^*E_VYA^* - XE_VZU^*E_A + XSS^*E_VZU^*E_A
= AA^* + UU^*E_A - XE_S^*E_VYA^* - XE_S^*E_U^*E_A
= I - EuE_A - XE^*_S^*E_V(YA^* + ZU^*E_A).
\]

+ F_AV^*YA^*XY - (F_AV^*Z + A^*X)[F_{uS}^*E_VYA^* - (I - F_{uS}^*E_VZ)U^*E_A]XY
= A^*A - F_AV^*V^*Y + F_AV^*V^* + (F_AV^* + A^*X)F_{uS}^*E_VY - A^*XY
+ F_AV^*YA^*XY - (F_AV^*Z + A^*X)[F_{uS}^*E_VYA^* - (I - F_{uS}^*E_VZ)U^*E_A]XY
= A^*A - F_AV^*V^* + (F_AV^* + A^*X)F_{uS}^*E_VY - A^*XY
+ F_AV^*(I - Z)Y - (F_AV^* + A^*X)[F_{uS}^*E_V(I - Z)Y - (I - F_{uS}^*E_VZ)U^*UY]
= A^*A + F_AV^*V^* - (A^*X + F_AV^*Z)Y
+ (F_AV^* + A^*X)[F_{uS}^*E_VZ + (I - F_{uS}^*E_VZ)U^*UY]
= A^*A + F_AV^*V^* + (F_AV^* + A^*X)[F_{uS}^*E_VZ + (I - F_{uS}^*E_VZ)U^*UY]
+ F_AV^*(I - Z)Y - (F_AV^* + A^*X)[F_{uS}^*E_V(I - Z)Y - (I - F_{uS}^*E_VZ)U^*UY]
= A^*A + F_AV^*V^* + (F_AV^* + A^*X)[F_{uS}^*E_VZ - F_{uS}^*E_VZU^*U - F_U^*]Y
= A^*A + F_AV^*V^* - (F_AV^* + A^*X)F_{uS}F_SY
= I - F_AV^* - (F_AV^* + A^*X)F_{uS}F_SY.
\]

It is easy to verify \((A - XY)G(A - XY) = A - XY\). This shows \(G\) is a \([1]^{-}\)-inverse of \(A - XY\). □

**Corollary 3.2.** Let \(A, X \in M_n(C)\) and \(Z = I + A^*X, U = E_AX, S = A^*AZF_U\). Let

\[
G = A^* + (F_A - A^*X)[F_{uS}^*A^* + (I - F_{uS}^*A^*AZ)U^*E_A],
\]

and

\[
(A + X)G = I - EuE_A + XE_S(A^* - A^*AZU^*E_A),
\]

\[
G(A + X) = I - F_UF_S.
\]

**Proof.** Replacing \(Y\) by \(-I\) in Lemma 3.1, we get easily the first part of the results. Noting that \(Z = I + A^*X\) and \(S = A^*AZF_U\), we have

\[
G(A + X) = I - (F_A - A^*X)F_{uS}F_S
= I - (I - A^*A - A^*X)F_{uS}F_S
= I - I - A^*A(I + A^*X)F_{uS}F_S
= I - (I - A^*AZ)F_{uS}F_S
= I - F_UF_S.
\]
Corollary 3.3. Let $A, X \in M_n(C)$ and $Z = I + XA^+, V = XF_A, S = E_VZAA^+$. Let
\[ G = A^* - F_AV^*XA^* + (F_AV^*Z + A^*X)[AA^*S^*E_VXA^* + (I - AA^*S^*E_VZ)E_A]. \]
Then $G$ is a $[1]$–inverse of $A + X$ and
\[
\begin{align*}
(A + X)G & = I - E_SE_V, \\
G(A + X) & = I - F_AF_V - (F_AV^*ZAA^* - A^*)F_SS.
\end{align*}
\]

Theorem 3.4. Let $A, X, Y \in M_n(C)$ and $Z = I - YA^+X, U = E_AX, V = YF_A, S = E_VZF_U$. Then $(A - XY)^\#$ exists iff
\[
A - XY + E_U(E_A + XE_SE_V(YA^* + ZU^*E_A))F_SF_Y
\]
is invertible iff
\[
A - XY + F_AF_V + (F_AV^*Z + A^*X)F_AF_SY
\]
is invertible and
\[
\]

\[
\begin{align*}
(A - XY)(A - XY)^+ & = (A - XY)G = I - E_UE_A - XE_SE_V(YA^* + ZU^*E_A), \\
(A - XY)^\dagger (A - XY) & = G(A - XY) = I - F_AF_V - (F_AV^*Z + A^*X)F_AF_SY.
\end{align*}
\]
So, the results follow by Lemma 2.3 and Lemma 2.4. \qed

Using Lemma 2.3, Lemma 2.4 and Corollary 3.2, Corollary 3.3, we have the following corollaries:

Corollary 3.5. Let $A, X \in M_n(C)$ and $Z = I + A^*X, U = E_AX, S = A^*AZF_U$. Then $(A + X)^\#$ exists iff
\[
A + X + E_U(E_A - XE_SE_V(A^* - A^*AZU^*E_A))F_SF_Y
\]
is invertible iff $A + X + F_SF_Y$ is invertible and
\[
(A + X)^\# = (A + X)(A + X + F_SF_Y)^{-1} = (A + X)
\]
\[
= (A + X)(A + X + F_AF_V + (F_AV^*ZAA^* - A^*)F_SS)^{-1} \].
\]

Corollary 3.6. Let $A, X \in M_n(C)$ and $Z = I + XA^+, V = XF_A, S = E_VZAA^+$. Then $(A + X)^\#$ exists iff
\[
A + X + F_AF_V + (F_AV^*ZAA^* - A^*)F_SS
\]
is invertible iff $A + X + E_SE_V$ is invertible and
\[
(A + X)^\# = (A + X + E_SE_V)^{-1}(A + X)
\]
\[
= (A + X)(A + X + F_AF_V + (F_AV^*ZAA^* - A^*)F_SS)^{-1} \].
\]

Similarly, we have

Corollary 3.3. Let $A, X \in M_n(C)$ and $Z = I + XA^+, V = XF_A, S = E_VZAA^+$. Let
\[ G = A^* - F_AV^*XA^* + (F_AV^*Z + A^*X)[AA^*S^*E_VXA^* + (I - AA^*S^*E_VZ)E_A]. \]
Then $G$ is a $[1]$–inverse of $A + X$ and
\[
\begin{align*}
(A + X)G & = I - E_SE_V, \\
G(A + X) & = I - F_AF_V - (F_AV^*ZAA^* - A^*)F_SS.
\end{align*}
\]
In the following, we investigate the expression for the group inverse of anti-triangular matrix. First, we cite a lemma which comes from [21].

**Lemma 3.7.** [21] Suppose $M, X \in M_{n}(C)$. Then $N = M - MXM$ has a $(1)$–inverse $Y$ iff $M$ has a $(1)$–inverse $X + (I - XM)Y(I - MX)$. 

**Lemma 3.8.** Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $Z = D - CA^{*}B, P = E_{A}B, Q = CF_{A}, R = ZF_{P}, W = E_{R}Q^{*}$. Then there exists a $(1)$–inverse $G$ of $M$ such that

$$I - MG = \begin{pmatrix} E_{P}E_{A} & 0 \\ -E_{W}E_{R}(ZP^{*}E_{A} + CA^{*}) & E_{W}E_{R} \end{pmatrix}.$$ 

**Proof.** Taking $X = \begin{pmatrix} A^{*} & 0 \\ 0 & 0 \end{pmatrix}$ and $N = M - MXM$. Let $Y$ be a $(1)$–inverse of $N$. Then by Lemma 3.7, $G = X + (I - XM)Y(I - MX)$ be a $(1)$–inverse of $M$ and


(1)

Let $N_{1} = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}, N_{2} = \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}$. Then $N = N_{1} + N_{2} = \begin{pmatrix} 0 & E_{A}B \\ CF_{A} & D - CA^{*}B \end{pmatrix}$.

Note that

$$N_{1}^{*} = \begin{pmatrix} 0 & Q^{*} \\ P^{*} & 0 \end{pmatrix}, \quad T = I + N_{2}N_{1}^{*} = \begin{pmatrix} I & 0 \\ ZP^{*} & 0 \end{pmatrix},$$

$$N_{1}N_{1}^{*} = \begin{pmatrix} PP^{*} & 0 \\ 0 & QQ^{*} \end{pmatrix}, \quad N_{1}^{*}N_{1} = \begin{pmatrix} Q^{*}Q & 0 \\ 0 & PP^{*} \end{pmatrix},$$

$$V = N_{2}F_{N_{1}} = \begin{pmatrix} 0 & 0 \\ 0 & ZF_{P} \end{pmatrix}, \quad S = E_{V}TN_{1}N_{1}^{*} = \begin{pmatrix} PP^{*} & 0 \\ 0 & E_{R}Q^{*}ZP^{*} \end{pmatrix}$$

and $S^{*} = \begin{pmatrix} PP^{*} & 0 \\ 0 & W^{*} \end{pmatrix}$. Thus, by Corollary 3.3, we have

$$I - NY = E_{S}E_{V} = \begin{pmatrix} E_{P} & 0 \\ -E_{W}E_{R}ZP^{*} & E_{W}E_{R} \end{pmatrix}.$$

Hence, by Eq.(1), we have

$$I - MG = (I - NY)(I - MX)$$

$$= \begin{pmatrix} E_{P} & 0 \\ -E_{W}E_{R}ZP^{*} & E_{W}E_{R} \end{pmatrix} \begin{pmatrix} E_{A} & 0 \\ -CA^{*} & I \end{pmatrix}$$

$$= \begin{pmatrix} E_{P}E_{A} & 0 \\ -E_{W}E_{R}(ZP^{*}E_{A} + CA^{*}) & E_{W}E_{R} \end{pmatrix}.$$

□

**Theorem 3.9.** Let $M = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ with $A^{*}, D^{*}$ exist. Then $M^{\#}$ exists iff $D^{*}CA^{*} = 0$. In this case,

$$M^{\#} = \begin{pmatrix} A^{*} \\ (D^{*})^{2}CA^{*} + D^{*}C(A^{*})^{2} - D^{*}CA^{*} & 0 \end{pmatrix}.$$
Then by Lemma 2.3, we have
\[ M + I - MG = \begin{pmatrix} A + A^* & 0 \\ C - E_D D^* A^* & D + E_D D^* \end{pmatrix}. \]

Since \( A + A^* \) and \( D + D^* \) are invertible, \( M + I - MG \) is invertible iff \( I - W W^* D^* \) is invertible. Thus, by Lemma 2.2, \( I - WW^* D^* \) is invertible iff \( WW^* D^* = 0 \) iff \( W = 0 \) iff \( D^* = D^* = 0 \). Hence, if \( M^* \) exists, then
\[ (M + I - MG)^{-1} = \begin{pmatrix} A^* & 0 \\ D^* CA^* - C & D^* + D^* \end{pmatrix}. \]

By simple calculation, we get
\[ M^* = \begin{pmatrix} A^* \\ (D^* CA^* + D^* (A^*)^2 - D^* CA^*) \end{pmatrix}. \]

\[ \square \]

**Theorem 3.10.** Let \( M = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix} \) and \( R = DF_B, W = E_R C^+ \). Then \( M^* \) exists iff
\[ DF_B - CB + E_WE_R (F_B + DB^* B) \]
is invertible. In this case,
\[ M^* = \begin{pmatrix} I + B \xi \eta & -B \xi \\ B^* - F_B \xi \eta & F_B \xi \end{pmatrix}. \]

Here,
\[ \xi = (DF_B - CB + E_WE_R (F_B + DB^* B))^{-1}, \]
\[ \eta = C + DB^* + E_WE_R (I - D) B^*. \]

**Proof.** Take \( A = 0 \) in Lemma 3.8, we have \( Z = D, P = 0, Q = CA^+, R = D, W = D^* QQ^* \) and
\[ M + I - MG = \begin{pmatrix} E_B & B \\ C - E_D D^* B^+ & D + E_D E_R \end{pmatrix} \]
\[ = \begin{pmatrix} I & B \\ C + DB^* + E_WE_R (I - D) B^* & D + E_D E_R \end{pmatrix} \begin{pmatrix} I & 0 \\ -B^* & 1 \end{pmatrix} \]
\[ = \begin{pmatrix} I & 0 \\ C + DB^* + E_WE_R (I - D) B^* & D - CB + E_WE_R (F_B + DB^* B) \end{pmatrix} \begin{pmatrix} E_B & B \\ -B^* & 1 \end{pmatrix}. \]

By Lemma 2.3, we have \( M^* \) exists iff \( DF_B - CB + E_WE_R (F_B + DB^* B) \) is invertible. Put
\[ \xi = (DF_B - CB + E_WE_R (F_B + DB^* B))^{-1}, \]
\[ \eta = C + DB^* + E_WE_R (I - D) B^*. \]

Then
\[ (M + I - MG)^{-1} = \begin{pmatrix} I + B \xi \eta & -B \xi \\ B^* - F_B \xi \eta & F_B \xi \end{pmatrix}. \]

Thus, by Lemma 3.1, we have
\[ M^* = \begin{pmatrix} I + B \xi \eta \\ B^* - F_B \xi \eta \end{pmatrix} \begin{pmatrix} I & 0 \\ B & C \end{pmatrix}. \]

\[ \square \]
Thus, by Corollary 3.6, we have $M^# = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^*$ if take $D = 0$ in Lemma 3.8. So, we present another method as following:

**Theorem 3.11.** Let $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ and $P = AF_C, W = E_pBB^*$. Then $M^#$ exists iff

$$AF_C - BC + E_W E_p (F_C + AC^+ C)$$

is invertible. In this case,

$$M^# = \begin{pmatrix} F_C \xi & C^+ - F_C \xi \eta \\ -C \xi & I + C \xi \eta \end{pmatrix}^2 \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}.$$

Here,

$$\xi = \{AF_C - BC + E_W E_p (F_C + AC^+ C)\}^{-1},$$

$$\eta = B + AC^+ + E_W E_p(I - A)C^+.$$

**Proof.** Let $M_1 = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, M_2 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ Then $M = M_1 + M_2$. Let $I_2$ be the identity of $M_2(\mathbb{C})$ and $P = AF_C, W = E_pBB^*$. Noting that

$$M_1^* = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad V = M_2 F_{M_1} = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix},$$

$$E_V = I_2 - VV^* = \begin{pmatrix} E_p & 0 \\ 0 & 1 \end{pmatrix}, \quad Z = I_2 + M_2 M_1^* = \begin{pmatrix} I & AC^+ \\ 0 & I \end{pmatrix},$$

$$S = E_V Z M_1 M_1^* = \begin{pmatrix} W & E_W AC^+ \\ 0 & CC^+ \end{pmatrix}, \quad S^* = \begin{pmatrix} W^* & -W^* E_W AC^+ \\ 0 & CC^+ \end{pmatrix},$$

$$E_S = \begin{pmatrix} E_W & -E_W E_p AC^+ \\ 0 & E_C \end{pmatrix}, \quad M + E_S E_V = \begin{pmatrix} A + E_W E_p & B - E_W E_p AC^+ \\ C & E_C \end{pmatrix}$$

and

$$M + E_S E_V = \begin{pmatrix} A + E_W E_p & B - E_W E_p AC^+ \\ C & E_C \end{pmatrix}$$

$$= \begin{pmatrix} A + E_W E_p & B - E_W E_p AC^+ + AC^+ + E_W E_p C^+ \\ C & I \end{pmatrix} \begin{pmatrix} I & -C^+ \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} AF_C - BC + E_W E_p (F_C + AC^+ C) & B + AC^+ + E_W E_p (I - A) C^+ \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -C^+ \\ 0 & I \end{pmatrix}$$

Thus, by Corollary 3.6, we have $M^#$ exists iff $AF_C - BC + E_W E_p (F_C + AC^+ C)$ is invertible. Put

$$\xi = \{AF_C - BC + E_W E_p (F_C + AC^+ C)\}^{-1},$$

$$\eta = B + AC^+ + E_W E_p (I - A) C^+.$$

Then, by Simple calculation, we have

$$(M + E_S E_V)^{-1} = \begin{pmatrix} F_C & C^+ \\ -C & I \end{pmatrix} \begin{pmatrix} \xi & -\xi \eta \\ 0 & I \end{pmatrix} = \begin{pmatrix} F_C \xi & C^+ - F_C \xi \eta \\ -C \xi & I + C \xi \eta \end{pmatrix}.$$


Thus, by Corollary 3.6,
\[ M^\# = (M + E_SE_V)^{-2}M = \left( \begin{array}{cc} F_C \xi & C^+ - F_C \xi \eta \\ -C \xi & I + C \eta \end{array} \right)^2 \left( \begin{array}{cc} A & B \\ C & 0 \end{array} \right). \]

\[ \square \]

**Corollary 3.12.** Let \( M = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \) Then \( M^\# \) exists if \( E_B F_C - BC \) is invertible. Put \( \xi = (E_B F_C - BC)^{-1} \), then
\[ M^\# = \begin{pmatrix} 0 & 0 \\ -C \xi BB^* (I - F_C \xi) C^+ C & -(I - F_C \xi) C^+ C \xi B \\ 0 \\ \end{pmatrix}. \]

**Proof.** Let \( \xi = (E_B F_C - BC)^{-1} \). Then
\[ E_B \xi^{-1} = E_B F_C - \xi^{-1} F_C, \xi B C^+ = -C^+, B^+ BC \xi = -B^+. \]

Thus,
\[ \xi E_B = F_C \xi, C \xi E_B = 0, F_C \xi B = 0, BC \xi = -B B^+ \]
and
\[ (F_C \xi - I) C^+ C = (F_C \xi - I) (I - F_C) \\ = (F_C \xi - I) (F_C \xi F_C - F_C) \\ = (F_C \xi - I) - (F_C \xi F_C - F_C) \\ = (F_C \xi - I). \]

Associate with Theorem 3.11, we get the results. \( \square \)

**Corollary 3.13.** [9] Let \( M = \begin{pmatrix} I & 0 \\ C & 0 \end{pmatrix} \) and \( C^\# \) exists. Then
\[ \begin{pmatrix} I & I \\ C & 0 \end{pmatrix}^\# = \begin{pmatrix} C^n & C^n + C^\# \\ CC^\# & -C^\# \end{pmatrix}. \]

Now, we consider the group inverse of \( M = \begin{pmatrix} B & A \\ 0 & C \end{pmatrix} \). From the proof of Theorem 3.11, we have
\[ I_2 - \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^+ = E_S E_V = \begin{pmatrix} E_W E_P & -E_W E_P A C^+ \\ 0 & E_C \end{pmatrix}. \]

Here, \( P = AF_C, W = E_B BB^\# \).

Since
\[ \begin{pmatrix} B & A \\ 0 & C \end{pmatrix} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \]

by Corollary 2.6, we have \( \begin{pmatrix} B & A \\ 0 & C \end{pmatrix}^\# \) exists iff
\[ \begin{pmatrix} B & A \\ 0 & C \end{pmatrix} + I_2 - \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^+ = \begin{pmatrix} B + E_W E_P & A - E_W E_P A C^+ \\ 0 & C + E_C \end{pmatrix} \]
is invertible.
Assume that $B^g, C^g$ exist. Then $P = AC^g$, $W = E_PBB^g$ and \[
\begin{pmatrix}
B + E_W E_P & A - E_W E_P AC^g \\
0 & C + E_C
\end{pmatrix}
\] is invertible iff $B + E_W E_P$ is invertible. Noting that $E_W E_P B^g = E_W E_P$ and $B^g + B^g$ is invertible, we have $B + E_W E_P$ is invertible iff
\[(B + E_W E_P B^g)(B^g + B^g) = BB^g + E_W E_P B^g = I - (I - E_W E_P) B^g\]
is invertible iff $I - B^g(I - E_W E_P)$ is invertible by Lemma 2.1. So, by Lemma 2.2, we have $I - B^g(I - E_W E_P)$ is invertible iff $B^g(I - E_W E_P) = 0$ iff $B^g P = 0$. Thus, we get the following theorem by Corollary 2.6:

**Theorem 3.14.** [4, 8, 22] Let $M = \begin{pmatrix} B & A \\ 0 & C \end{pmatrix}$ with $B^g, C^g$ exist. Then $M^g$ exists iff $B^g AC^g = 0$. In this case,

\[
M^g = \begin{pmatrix} B^g & (B^g)^2 AC^g + B^g A(C^g)^2 - B^g A C^g \\ 0 & C^g \end{pmatrix}
\]

**Example 3.15.** Let $M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $M = \begin{pmatrix} 0 & B \\ I & D \end{pmatrix}$ Taking $B^g = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Then $R = DB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, W = E_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $DF_B - CB + E_W E_R (F_B + DB^g B) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ is invertible. Thus, by Theorem 3.10, we have $M^g$ exists and $M^g = M^2$.

**References**


