(ψ, ϕ) -weak Contraction on Ordered Uniform Spaces

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Abstract. In this paper, we prove a fixed point theorem for (ψ, ϕ) –contractive mappings on ordered uniform space.

1. Introduction

We call a pair (X, δ) to be a uniform space which consists of a non-empty set X together with an uniformity δ of wherein the latter begins with a special kind of filter on X × X whose all elements contain the diagonal Δ = {(x, x) : x ∈ X}. If V ∈ δ and (x, y) ∈ V, (y, x) ∈ V then x and y are said to be V –close. Also a sequence {xₙ} in X, is said to be a Cauchy sequence with regard to uniformity δ if for any V ∈ δ, there exists N ≥ 1 such that xₙ and xₘ are V –close for m, n ≥ N. An uniformity δ defines a unique topology τ (δ) on X for which the neighborhoods of x ∈ X are the sets V(x) = {y ∈ X : (x, y) ∈ V} when V runs over δ.

A uniform space (X, δ) is said to be Hausdorff if and only if the intersection of all the V ∈ δ reduces to diagonal Δ of X i.e. (x, y) ∈ V for all V ∈ δ implies x = y. Notice that Hausdorffness of the topology induced by the uniformity guarantees the uniqueness of limit of a sequence in uniform space. An element V of uniformity δ is said to be symmetrical if V = V⁻¹ = {(y, x) : (x, y) ∈ V}. Since each V ∈ δ contains a symmetrical W ∈ δ and if (x, y) ∈ W then x and y are both W and V –close and then one may assume that each V ∈ δ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X, δ), they are naturally interpreted with respect to the topological space (X, τ (δ)).

Aamri and El Moutawakil [2] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an E –distance. Some other authors proved fixed point theorems using this concept ([4],[8],[10],[11],[16],[17]). In [5],[6] and [19] authors used the order relation on uniform space.

Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [18] and then by Nieto and Lopez [15]. Further results in this direction under weak contraction conditions were proved, e.g. ([3],[7],[9],[12],[14]).

In this paper, we establish a fixed point theorem satisfying (ψ, ϕ) –contractive condition on ordered uniform space. We also give an example.

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2. Preliminaries

**Definition 2.1.** ([2]) Let \((X, \mathcal{S})\) be a uniform space. A function \(p : X \times X \to \mathbb{R}^+\) is said to be an \(A\)-distance if for any \(V \in \mathcal{S}\), there exists \(\delta > 0\), such that \(p(z, x) \leq \delta\) and \(p(z, y) \leq \delta\) for some \(z \in X\) imply \((x, y) \in V\).

**Definition 2.2.** ([2]) Let \((X, \mathcal{S})\) be a uniform space. A function \(p : X \times X \to \mathbb{R}^+\) is said to be an \(E\)-distance if

1. \((p_1)\) \(p\) is an \(A\)-distance,
2. \((p_2)\) \(p(x, y) \leq p(x, z) + p(z, y)\) for all \(x, y, z \in X\).

**Example 2.3.** ([2]) Let \(X = [0, +\infty)\) and \(p(x, y) = \max\{x, y\}\). The function \(p\) is an \(A\)-distance. Also, \(p\) is an \(E\)-distance.

The following lemma embodies some useful properties of \(E\)-distance.

**Lemma 2.4.** ([1],[2]) Let \((X, \mathcal{S})\) be a Hausdorff uniform space and \(p\) be an \(E\)-distance on \(X\). Let \(\{x_n\}\) and \(\{y_n\}\) be arbitrary sequences in \(X\) and \(\{x_n\}, \{y_n\}\) be sequences in \(\mathbb{R}^+\) converging to 0. Then, for \(x, y, z \in X\), the following holds

1. \((a)\) If \(p(x_n, y) \leq \alpha_n\) and \(p(x_n, z) \leq \beta_n\) for all \(n \in \mathbb{N}\), then \(y = z\). In particular, if \(p(x, y) = 0\) and \(p(x, z) = 0\), then \(y = z\).
2. \((b)\) If \(p(x_n, y_n) \leq \alpha_n\) and \(p(x_n, z) \leq \beta_n\) for all \(n \in \mathbb{N}\), then \(\{y_n\}\) converges to \(z\).
3. \((c)\) If \(p(x_n, x_m) \leq \alpha_n\) for all \(m > n\), then \(\{x_n\}\) is a Cauchy sequence in \((X, \mathcal{S})\).

Let \((X, \mathcal{S})\) be a uniform space equipped with \(E\)-distance \(p\). A sequence in \(X\) is \(p\)-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting.

**Definition 2.5.** ([1],[2]) Let \((X, \mathcal{S})\) be a uniform space and \(p\) be an \(E\)-distance on \(X\). Then

1. \((i)\) \(X\) is said to be \(S\)-complete if for every \(p\)-Cauchy sequence \(\{x_n\}\) there exists \(x \in X\) with \(\lim_{n \to \infty} p(x_n, x) = 0\),
2. \((ii)\) \(X\) is said to be \(p\)-Cauchy complete if for every \(p\)-Cauchy sequence \(\{x_n\}\) there exists \(x \in X\) with \(\lim_{n \to \infty} x_n = x\) with respect to \(\mathcal{S}\),
3. \((iii)\) \(f : X \to X\) is \(p\)-continuous if \(\lim_{n \to \infty} p(x_n, x) = 0\) implies \(\lim_{n \to \infty} p(f x_n, f x) = 0\),
4. \((iv)\) \(f : X \to X\) is \(\mathcal{S}\)-continuous if \(\lim_{n \to \infty} x_n = x\) with respect to \(\mathcal{S}\) implies \(\lim_{n \to \infty} f x_n = f x\) with respect to \(\mathcal{S}\).

**Remark 2.6.** ([2]) Let \((X, \mathcal{S})\) be a Hausdorff uniform space and let \(\{x_n\}\) be a \(p\)-Cauchy sequence. Suppose that \(X\) is \(S\)-complete, then there exists \(x \in X\) such that \(\lim_{n \to \infty} p(x_n, x) = 0\). Lemma 2.4 \((b)\) then gives \(\lim_{n \to \infty} x_n = x\) with respect to the topology \(\mathcal{S}\). Therefore \(S\)-completeness implies \(p\)-Cauchy completeness.

We shall also state the following definition of altering distance function which is required in the sequel to establish a fixed point theorem in uniform space.

**Definition 2.7.** ([6]) A function \(\psi : [0, \infty) \to [0, \infty)\) is called an altering distance function if the following properties are satisfied:

1. \((i)\) \(\psi(0) = 0\),
2. \((ii)\) \(\psi\) is continuous and monotonically nondecreasing.

3. Fixed Point Result

**Theorem 3.1.** Let \((X, \mathcal{S})\) be a Hausdorff uniform space, \(\leq\) be a partial order on \(X\). Suppose \(p\) be an \(E\)-distance on \(S\)-complete space \(X\). Let \(T : X \to X\) be a \(p\)-continuous or \(\mathcal{S}\)-continuous nondecreasing mapping such that for all comparable \(x, y \in X\) with

\[
\psi(p(Tx, Ty)) \leq \psi(p(x, y)) - \varphi(p(x, y)),
\]

where \(\psi, \varphi : [0, \infty) \to [0, \infty)\) are altering distance functions.

If there exists \(x_0 \in X\) with \(x_0 \leq T(x_0)\) then \(T\) has a fixed point.
Proof. If \( T(x_0) = x_0 \) then the proof is finished. Suppose that \( T(x_0) \neq x_0 \). Since \( x_0 \leq T(x_0) \) and \( T \) is nondecreasing, we obtain by induction that

\[
x_0 \leq T(x_0) \leq T^2(x_0) \leq T^3(x_0) \leq \cdots \leq T^n(x_0) \leq T^{n+1}(x_0) \leq \cdots .
\]

Put \( x_{n+1} = T x_n \), for all \( n \geq 1 \). If there exists a positive integer \( N \) such that \( x_N = x_{N+1} \), then \( x_N \) is a fixed point of \( T \). Now, we may assume that \( x_n \neq x_{n+1} \), for all \( n \geq 0 \).

From (1), we have for all \( n \geq 0 \),

\[
\psi(p(x_{n+2}, x_n)) \leq |\psi(p(x_{n+1}, x_n))| \leq \psi(p(x_{n+1}, x_n) - \psi(p(x_{n+1}, x_n)) \leq \psi(p(x_{n+1}, x_n)).
\]

Together with that \( \psi \) is nondecreasing implies that the sequence \( \{p(x_{n+1}, x_n)\} \) is monotone decreasing and hence there exists an \( r \geq 0 \) such that

\[
\lim_{n \to \infty} p(x_{n+1}, x_n) = r .
\]

Letting \( n \to \infty \) in (2) and using the continuity of \( \psi \) and \( \phi \), we obtain

\[
\psi(r) \leq \psi(r) - \phi(r)
\]

which is a contradiction unless \( r = 0 \). Hence,

\[
\lim_{n \to \infty} p(x_{n+1}, x_n) = 0.
\]

Similarly, we can show \( \lim_{n \to \infty} p(x_n, x_{n+1}) = 0 \).

Next we show that \( \{x_n\} \) is a \( p \)-Cauchy sequence. Assume \( \{x_n\} \) is not \( p \)-Cauchy. Then there exists an \( \varepsilon > 0 \) for which we can find subsequences \( \{x_{m(k)}\} \) and \( \{x_{n(k)}\} \) of \( \{x_n\} \) with \( m(k) > n(k) > k \) such that

\[
p(x_{m(k)}, x_{m(k)}) \geq \varepsilon.
\]

Further, corresponding to \( n(k) \), we can choose \( m(k) \) in such a way that it is the smallest integer with \( m(k) > n(k) \) and satisfying (3). Hence,

\[
p(x_{n(k)}, x_{m(k)}) < \varepsilon.
\]

Then we have

\[
\varepsilon \leq p(x_{m(k)}, x_{m(k)}) \leq p(x_{n(k)}, x_{m(k)}) + p(x_{m(k)}, x_{m(k)}) ,
\]

that is

\[
\varepsilon \leq p(x_{n(k)}, x_{m(k)}) < \varepsilon + p(x_{n(k)}, x_{m(k)}) .
\]

Taking the limit as \( k \to \infty \), we have

\[
\lim_{k \to \infty} p(x_{n(k)}, x_{m(k)}) = \varepsilon.
\]

From (2),

\[
p(x_{n(k)}, x_{m(k)}) \leq p(x_{n(k)}, x_{n(k)} + 1) + p(x_{n(k)} + 1, x_{m(k)}) + p(x_{m(k)}, x_{m(k)})
\]

and

\[
p(x_{n(k)} + 1, x_{m(k)} + 1) \leq p(x_{n(k)} + 1, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) + p(x_{m(k)} + 1, x_{m(k)} + 1).
\]
Taking the limit as $k \to \infty$ we have

$$\lim_{k \to \infty} p\left(x_n(k+1), x_m(k+1)\right) = \epsilon. \quad (5)$$

From (1),

$$\psi\left(p\left(x_n(k+1), x_m(k+1)\right)\right) \leq \psi\left(p\left(x_n(k), x_m(k)\right)\right) - \phi\left(p\left(x_n(k), x_m(k)\right)\right).$$

Letting $k \to \infty$ in the above inequality, using (4), (5) and the continuities of $\psi$ and $\phi$, we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

which is a contradiction by virtue of a property of $\phi$.

Hence $\{x_n\}$ is a $p$-Cauchy sequence. Since $S$–completeness of $X$, there exists a $z \in X$ such that

$$\lim_{n \to \infty} p(x_n, z) = 0$$

Moreover, the $p$–continuity of $T$ implies that $\lim_{n \to \infty} p(Tx_n, Tz) = 0$. So, by Lemma 2.4 (a), $z = Tz$. Using Remark 2.6, the proof is similar when $T$ is $\tau(\delta)$–continuous.

**Example 3.2.** Let $X = [0, 1]$ equipped with usual metric $d(x, y) = |x - y|$ and a partial order be defined as $x \leq y$ whenever $y \leq x$ and suppose

$$\delta = \{V \subset X \times X : \Delta \subset V\}.$$

Define the function $p$ as $p(x, y) = y$ for all $x, y$ in $X$ and $T : X \to X$ defined by $T(t) = \frac{t^2}{1+t}$. Consider the functions $\phi$ and $\psi$ defined as follows

$$\phi(t) = \frac{t}{1+t} \quad \text{and} \quad \psi(t) = t.$$

**Definition of $\delta$, $\cap_{V \in \delta} V = \Delta$ and this show that the uniform space $(X, \delta)$ is Hausdorff uniform space. And also $X$ is $S$–complete.** On the other hand, $\rho$ is an $E$–distance. $T$ is $p$–continuous and $\phi$ and $\psi$ are continuous, monotone nondecreasing. For $x = 0.5$ and $y = 0.3$, using usual metric, (1) does not hold. However, we have that for all $x, y \in X$

$$\psi\left(p\left(Tx, Ty\right)\right) \leq \psi\left(p\left(x, y\right)\right) - \psi\left(p\left(x, y\right)\right).$$

And 0 is the fixed point of $T$.

**References**

[7] S.C. Binayak, A. Kundu, ($\psi, \alpha, \beta$) –


