Some KKM Theorems in Modular Function Spaces

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Abstract. In this paper, a coincidence theorem is obtained which is generalization of Ky Fan’s fixed point theorem in modular function spaces. A modular version of Fan’s minimax inequality is proved. Moreover, some best approximation theorems are presented for multi-valued mappings.

1. Introduction

Modular function spaces are natural generalization of spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Calderon-Lozanovskii and many others. The theory of mappings defined on convex subsets of modular function spaces generalized by Khamsi et al. (see e.g. [3–5]). There is a large set of modular space applications in various parts of analysis, probability and mathematical statistics (see e.g. [11–13]).

We need the following definitions in sequel, from [6, 7]:

Let $\Omega$ be a nonempty set and $\Sigma$ be a nontrivial $\sigma$-algebra of subsets of $\Omega$. Let $\mathcal{P}$ be a $\sigma$-ring of subsets of $\Omega$, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$, we denote the linear space of all simple functions with supports in $\mathcal{P}$. By $M_\infty$, we will denote the space of all extended measurable functions, i.e. all functions $f : \Omega \to [-\infty, +\infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(w) \to f(w)$ for all $w \in \Omega$. By $1_A$, we denote the characteristic function of the set $A$.

Definition 1.1. Let $\rho : M_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that $\rho$ is a regular convex function pseudomodular if

(i) $\rho(0) = 0$;

(ii) $\rho$ is monotone, i.e. $|f(w)| \leq |g(w)|$ for all $w \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in M_\infty$;

(iii) $\rho$ is orthogonally subadditive, i.e. $\rho(f 1_{A \cup B}) \leq \rho(f 1_A) + \rho(f 1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in M_\infty$;

(iv) $\rho$ has the Fatou property, i.e. $|f_n(w)| \uparrow |f(w)|$ for all $w \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in M_\infty$;

(v) $\rho$ is order continuous in $\mathcal{E}$, i.e. $g_n \in \mathcal{E}$ and $|g_n(w)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.
We say that $A \in \Sigma$ is $\rho$-null if $\rho(gA) = 0$ for every $g \in E$. A property holds $\rho$-almost everywhere if the exceptional set is $\rho$-null, we define 

$$M(\Omega, \Sigma; \mathcal{P}, \rho) = \{f \in M; |f(w)| < \infty \rho-a.e.\}.$$ 

We will write $M$ instead of $M(\Omega, \Sigma; \mathcal{P}, \rho)$.

**Definition 1.2.** Let $\rho$ be a regular convex function pseudomodular. We say that $\rho$ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ $\rho$-a.e.

The class of all nonzero regular convex function modulars defined on $\Omega$ will be denoted by $\mathcal{R}$.

**Definition 1.3.** Let $\rho$ be a convex function modular. A modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly $L_\rho$, defined by 

$$L_\rho = \{f \in M; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$ 

The the formula 

$$\|f\|_\rho = \inf\{\alpha > 0; \rho(f/\alpha) \leq 1\}.$$ 

defines a norm in $L_\rho$ which is frequently called the Luxemburg norm.

The $\|\|_\rho$-distance, from an $f$ to a set $Y \subset L_\rho$, to be the quantity 

$$\text{dist}_{\|\|_\rho}(f, Y) = \inf\{\|f - g\|_\rho : g \in Y\}.$$ 

From [7], $(L_\rho, \|\|_\rho)$ is a complete metric space and the norm $\|\|_\rho$ is monotone with respect to the natural order in $M$. Therefore we can define the $\|\|_\rho$-Hausdorff distance by 

$$H_{\|\|_\rho}(X, Y) = \max\{\sup\{\text{dist}_{\|\|_\rho}(f, Y) : f \in X\}, \sup\{\text{dist}_{\|\|_\rho}(g, X) : g \in Y\}\},$$ 

for each $X, Y \subseteq L_\rho$.

**Definition 1.4.** Let $\rho \in \mathcal{R}$.

(i) We say $(f_n)$ is $\rho$-convergent to $f$ and write $f_n \to f$ ($\rho$) if and only if $\rho(f_n - f) \to 0$.

(ii) A subset $B \subset L_\rho$ is called $\rho$-closed if for any sequence of $f_n \in B$, the convergence $f_n \to f$ ($\rho$) implies that $f$ belong to $B$.

(iii) A nonempty subset $K$ of $L_\rho$ is said to be $\rho$-compact if for any family $\{A_\alpha ; A_\alpha \in 2^\Gamma, \alpha \in \Gamma\}$ of $\rho$-closed subsets with $K \cap A_{\alpha_1} \cap \cdots \cap A_{\alpha_n} \neq \emptyset$, for any $\alpha_1, \cdots, \alpha_n \in \Gamma$, we have 

$$K \cap \bigcap_{\alpha \in \Gamma} A_{\alpha} \neq \emptyset.$$ 

Let $\rho \in \mathcal{R}$. We have $\rho(f) \leq \lim \inf \rho(f_n)$, whenever $f_n \to f$ $\rho-a.e$. This property is equivalent to the Fatou property [6, Theorem 2.1].

The concept of KKM-mapping in modular function spaces, was introduced by Khamsi, Latif and Al-Sulami in 2011 [6]. They proved an analogue of Ky Fan’s fixed point theorem in these spaces:
Definition 1.5. Let $\rho \in \mathbb{R}$ and let $C \subseteq L_\rho$ be nonempty. A multi-valued mapping $G : C \rightrightarrows L_\rho$ is called a KKM mapping if

$$\text{conv}(\{f_1, \cdots, f_n\}) \subseteq \bigcup_{i \in \mathbb{N}} G(f_i)$$

for any $f_1, \cdots, f_n \in C$, where the notation conv$(A)$ describes the convex hull of $A$.

Theorem 1.6. [6, Theorem 3.2] Let $\rho \in \mathbb{R}$ and $C \subseteq L_\rho$ be nonempty and $G : C \rightrightarrows L_\rho$ be a KKM mapping such that for any $f \in C$, $G(f)$ is nonempty and $\rho$-closed. Assume there exists $f_0 \in C$ such that $G(f_0)$ is $\rho$-compact. Then, we have

$$\bigcap_{f \in C} G(f) \neq \emptyset.$$

Definition 1.7. Let $\rho \in \mathbb{R}$ and let $C$ be nonempty $\rho$-closed subset of $L_\rho$. Let $T : G \rightrightarrows L_\rho$ be a map. $T$ is called $\rho$-continuous if $(T(f_n))$ $\rho$-converges to $T(f)$ whenever $(f_n)$ $\rho$-converges to $f$. Also $T$ will be called strongly $\rho$-continuous if $T$ is $\rho$-continuous and

$$\liminf_{n \to \infty} \rho(g - T(f_n)) = \rho(g - T(f)),$$

for any sequence $(f_n) \subseteq C$ which $\rho$-converges to $f$ and for any $g \in C$.

In Section 2, we generalized some results of Khamsi et al. in [6]. In the next section, we proved a minimax inequality. Section 4 is devoted to some best approximation theorems for multi-valued mappings.

2. KKM-mapping and Coincidence Theorem

Here, we generalize the Ky Fan’s fixed point theorem which established in [6].

Lemma 2.1. Let $\rho \in \mathbb{R}$. Let $K \subseteq L_\rho$ be nonempty convex and $\rho$-compact. Let $T : K \rightrightarrows L_\rho$ be strongly $\rho$-continuous and $F : K \rightrightarrows K$ be $\rho$-continuous. Then, there exists $f_0 \in K$ such that

$$\rho(F(f_0) - T(f_0)) = \inf_{f \in K} \rho(F(f) - T(f_0)).$$

Proof. Consider the map $G : K \rightrightarrows L_\rho$ defined by

$$G(g) = \{f \in K; \rho(F(f) - T(f)) \leq \rho(F(g) - T(f))\}.$$

Clearly, for each $g \in K$, $G(g) \neq \emptyset$. For any sequence $(f_n) \subseteq G(g)$ which $\rho$-converges to $f$, by Fatou property, we have

$$\rho(F(f) - T(f)) \leq \liminf_{n \to \infty} \rho(F(f_n) - T(f_n)),$$

but $(f_n) \subseteq G(g)$, so

$$\liminf_{n \to \infty} \rho(F(f_n) - T(f_n)) \leq \liminf_{n \to \infty} \rho(F(g) - T(f_n)).$$

Since $T$ is strongly $\rho$-continuous and $F$ is $\rho$-continuous

$$\liminf_{n \to \infty} \rho(F(f) - T(f_n)) = \rho(F(g) - T(f)).$$

Therefore

$$\rho(F(f) - T(f)) \leq \rho(F(g) - T(f)).$$
namely \( f \in G(g) \). Since for any sequence \( \{f_n\} \subset G(g) \) which \( \rho \)-converges to \( f \), we have \( f \in G(g) \), then \( G(g) \) is \( \rho \)-closed for any \( g \in K \). Now, we show that \( G \) is a KKM-mapping. If not, then there exists \( \{g_1, \ldots, g_n\} \subset K \) and \( f \in \text{conv}(\{g_i\}) \) such that \( f \notin \bigcup_{1 \leq i \leq n} G(g_i) \).

This implies
\[
\rho(F(g_i) - T(f)) \leq \rho(F(f) - T(f)), \quad \text{for } i = 1, \ldots, n
\]
Let \( \epsilon > 0 \) be such that \( \rho(F(g_i) - T(f)) \leq \rho(F(f) - T(f)) - \epsilon \), for \( i = 1, \ldots, n \). Since \( \rho \) is convex, for any \( g \in \text{conv}(\{g_i\}) \), we have
\[
\rho(F(g) - T(f)) \leq \rho(F(f) - T(f)) - \epsilon.
\]
On the other hand \( f \in \text{conv}(\{g_i\}) \), so we get
\[
\rho(F(f) - T(f)) \leq \rho(F(f) - T(f)) - \epsilon,
\]
which is a contradiction. Therefore, \( G \) is a KKM-mapping. By the \( \rho \)-compactness of \( K \), we deduce that \( G(g) \) is a compact for any \( g \in K \). Theorem 1.6 implies the existence of \( f_0 \in \bigcap_{g \in K} G(g) \). Hence, \( \rho(F(f_0) - T(f_0)) \leq \rho(F(g) - T(f_0)) \) for any \( g \in K \). So, we have \( \rho(F(f_0) - T(f_0)) = \inf_{g \in K} \rho(F(g) - T(f_0)) \). \( \square \)

**Theorem 2.2.** Let \( \rho \in \mathbb{R} \) and \( K \subset L_\rho \) be nonempty convex and \( \rho \)-compact. Let \( T : K \to L_\rho \) be strongly \( \rho \)-continuous, \( F : K \to K \) be \( \rho \)-continuous and \( F(K) \) is \( \rho \)-compact. Assume that for any \( f \in K \), with \( F(f) \neq T(f) \), there exists \( \alpha \in (0,1) \) such that
\[
F(K) \bigcap B_\rho(F(f), \alpha \rho(F(f) - T(f))) \bigcap B_\rho(T(f), (1 - \alpha)\rho(F(f) - T(f))) \neq \emptyset.
\]
Then, \( T(g) = F(g) \) for some \( g \in K \).

**Proof.** From the previous lemma, there exists \( f_0 \in K \) such that
\[
\rho(F(f_0) - T(f_0)) = \inf_{g \in K} \rho(F(g) - T(f_0)).
\]
We claim that \( T(f_0) = F(f_0) \). If \( T(f_0) \neq F(f_0) \), then by the \( \rho \)-compactness of \( F(K) \), there exists \( \alpha \in (0,1) \) such that
\[
K_0 = F(K) \bigcap B_\rho(F(f_0), \alpha \rho(F(f_0) - T(f_0))) \bigcap B_\rho(T(f_0), (1 - \alpha)\rho(F(f_0) - T(f_0))) \neq \emptyset.
\]
Let \( F(g) \in K_0 \). Then, \( \rho(F(g) - T(f_0)) \leq (1 - \alpha)\rho(F(f_0) - T(f_0)) \), which is a contradiction. \( \square \)

**Corollary 2.3.** Let \( \rho \in \mathbb{R} \) and \( K \subset L_\rho \) be nonempty convex and \( \rho \)-compact. Let \( F : K \to K \) be \( \rho \)-continuous and \( F(K) \) is \( \rho \)-compact. If \( T : K \to F(K) \) be strongly \( \rho \)-continuous, then \( T(g) = F(g) \) for some \( g \in K \).

3. A Minimax Inequality

In this section, a modular version of Fan’s minimax inequality [2] is obtained.

**Definition 3.1.** Let \( \rho \in \mathbb{R}, L_\rho \) be a modular function space and \( C \) be a convex subset of \( L_\rho \). A function \( f : C \to \mathbb{R} \) is said to be metrically quasi-concave (resp., metrically quasi-convex) if for each \( \lambda \in \mathbb{R} \), the set \( \{g \in C : f(g) > \lambda\} \) (resp., \( \{g \in C : f(g) < \lambda\} \) is convex.

**Lemma 3.2.** Let \( \rho \in \mathbb{R} \). Suppose \( C \) is a convex subset of a modular function space \( L_\rho \), and the function \( f : C \times C \to \mathbb{R} \) satisfies the following conditions:

...
1) for each \( g \in C \), the function \( f(., g) : C \to \mathbb{R} \) is metrically quasi-concave (resp., metrically quasi-convex) and
2) there exists \( \gamma \in \mathbb{R} \) such that \( f(g, g) \leq \gamma \) (resp., \( f(g, g) \geq \gamma \)) for each \( g \in C \).

Then, the mapping \( G : C \to L_\rho \), which is defined by

\[ G(g) = \{ h \in C : f(g, h) \leq \gamma \} \text{ (resp., } G(g) = \{ h \in C : f(g, h) \geq \gamma \} \),

is a KKM-mapping.

**Proof.** The conclusion is proved for the concave case, the convex case is completely similar. Assume that \( G \) is not a KKM-mapping. Then there exists a finite subset \( A = \{ g_1, \cdots , g_n \} \) of \( C \) and a point \( g_0 \in conv(A) \) such that \( g_0 \notin G(g) \) for each \( i = 1, \cdots , n \). We set

\[ \lambda = \min \{ f(g_i, g_0) : i = 1, \cdots , n \} > \gamma, \]

and \( B = \{ e \in C : f(e, g_0) > \lambda \} \), where \( \lambda > \lambda_0 > \gamma \). For each \( i \), we have \( g_i \in B \). By hypothesis 1), \( B \) is convex and hence \( conv(A) \subseteq B \). So, \( g_0 \in B \), and we have \( f(g_0, g_0) > \lambda_0 > \gamma \), which is a contradiction by assumption 2). Thus, \( G \) is a KKM-mapping.

**Definition 3.3.** Let \( \rho \in \mathbb{R} \). A real-valued function \( f : L_\rho \times L_\rho \to \mathbb{R} \) is said to be \( \rho \)-generally lower (resp., upper) semi continuous on \( L_\rho \) whenever, for each \( g \in L_\rho \), \( \{ h \in L_\rho : f(g, h) \leq \lambda \} \) (resp., \( \{ h \in L_\rho : f(g, h) \geq \lambda \} \)) is \( \rho \)-closed for each \( \lambda \in \mathbb{R} \).

The following is the analogue of Fan’s minimax inequality in modular function spaces.

**Theorem 3.4.** Let \( \rho \in \mathbb{R} \). Suppose \( C \) is a nonempty, \( \rho \)-compact and convex subset of a complete modular function space \( L_\rho \) and \( f : C \times C \to \mathbb{R} \) satisfies the following

1) \( f \) is a \( \rho \)-generally lower (resp., upper) semi continuous ;
2) for each \( h \in C \), the function \( f(., h) : C \to \mathbb{R} \) is metrically quasi-concave (resp., metrically quasi-convex) and
3) there exists \( \gamma \in \mathbb{R} \) such that \( f(g, g) \leq \gamma \) (resp., \( f(g, g) \geq \gamma \)) for each \( g \in C \).

Then, there exists an \( h_0 \in C \) such that

\[
\sup_{g \in C} f(g, h_0) \leq \sup_{g \in C} f(g, g),
\]

\[ (\text{resp., } \inf_{g \in C} f(g, h_0) \geq \inf_{g \in C} f(g, g)). \]

for each \( g \in C \).

**Proof.** By hypothesis 3), \( \lambda = \sup_{g \in C} f(g, g) < \infty \). For each \( g \in C \), we define the mapping \( G : C \to C \) by

\[ G(g) = \{ h \in C : f(g, h) \leq \lambda \}, \]

which is \( \rho \)-closed by assumption 1). By Lemma 3.2, \( G \) is a KKM-mapping. So by using Theorem 1.6, we have

\[ \bigcap_{g \in C} G(g) \neq \emptyset. \]

Therefore, there exists an \( h_0 \in \bigcap_{g \in C} G(g) \). Thus, \( f(g, h_0) \leq \lambda \) for every \( g \in C \). Hence,

\[ \sup_{g \in C} f(g, h_0) \leq \sup_{g \in C} f(g, g). \]

This completes the proof.
4. Some Best Approximation Theorems

In this section, we prove some best approximation theorems for multi-valued mappings in modular function spaces.

**Definition 4.1.** Let $X, Y \subseteq L_p$.

(i) A map $F : X \to Y$ is said to be $\rho$-upper semi continuous if for each $\rho$-closed set $B \subseteq Y$, $F^{-1}(B)$ is $\rho$-closed in $X$.

(ii) A map $G : D \subseteq X \to X$ is called quasi-convex if the set $G^{-1}(C)$ is convex for each convex subset $C$ of $X$.

First, note that the $\|\|_\rho$-Hausdorff distance can be rewritten as follows

$$H_{\|\|_\rho}(X, Y) = \inf\{e > 0 : X \subseteq O_e(Y) \text{ and } Y \subseteq O_e(X)\},$$

where, for each $A \subseteq L_p$, $O_e(A) = \{f \in L_p : \text{dist}_{\|\|_\rho}(f, A) < e\}$.

Also, by definitions of $\rho$-closed and $\rho$-compact sets in modular function spaces with $\|\|_\rho$-Hausdorff distance and by [8, Proposition 14.11] we conclude that, if $F(f)$ is $\rho$-compact for each $f \in X$, then $F$ is $\rho$-upper semi continuous if and only if for each $f \in X$ and $\epsilon > 0$, there exist $\delta > 0$ such that for each $f' \in B(f, \delta)$, we have $F(f') \subseteq B(f, \epsilon)$.

**Theorem 4.2.** Let $\rho \in \mathfrak{R}$. Suppose $X$ is a $\rho$-compact subset of $L_p$ and $F, G : X \to L_p$ are $\rho$-upper semi continuous maps with nonempty $\rho$-compact convex values and $G$ is quasi-convex. Then, there exists $f_0 \in X$ such that

$$H_{\|\|_\rho}(G(f_0), F(f_0)) = \inf_{f \in X} H_{\|\|_\rho}(G(f), F(f_0)).$$

**Proof.** Let $S : X \to X$ be defined by

$$S(g) = \{f \in X : H_{\|\|_\rho}(G(f), F(f)) \leq H_{\|\|_\rho}(G(g), F(f))\}.$$

For each $g \in X$, $S(g) \neq \emptyset$. We show that $S(g)$ is $\rho$-closed for each $g \in X$. Suppose that $\{g_n\}$ be a sequence in $S(g)$ such that $g_n \to g'(\rho)$. We claim that $g' \in S(g)$. Let $\epsilon > 0$ be arbitrary. Since $F$ is $\rho$-upper semi continuous with $\rho$-compact values, so there exists $N_1$ such that for each $n \geq N_1$, we have

$$F(g_n) \subseteq B(F(g'), \epsilon).$$

Similarly, there exists $N_2$ such that for each $n \geq N_2$, we have

$$G(g_n) \subseteq B(G(g'), \epsilon).$$

Let $N = \max\{N_1, N_2\}$. Then, we have

$$H_{\|\|_\rho}(G(g'), F(g')) \leq H_{\|\|_\rho}(G(g'), G(g_n)) + H_{\|\|_\rho}(G(g_n), F(g_n))$$

$$+ H_{\|\|_\rho}(F(g_n), F(g'))$$

$$\leq 2\epsilon + H_{\|\|_\rho}(G(g_n), F(g_n))$$

$$\leq 2\epsilon + H_{\|\|_\rho}(G(g), F(g_n))$$

$$\leq 2\epsilon + H_{\|\|_\rho}(G(g), F(g')) + H_{\|\|_\rho}(G(g'), F(g_n))$$

$$\leq 3\epsilon + H_{\|\|_\rho}(G(g), F(g')).$$

Since $\epsilon$ was arbitrary, so

$$H_{\|\|_\rho}(G(g'), F(g')) \leq H_{\|\|_\rho}(G(g), F(g')).$$
so \( g' \in S(g) \). Now, we show that for each \( \{f_1, \cdots, f_n\} \subset X, \text{co}(\{f_1, \cdots, f_n\}) \subset S(\{f_1, \cdots, f_n\}) \). Assume to the contrary that, if there exists \( h \in \text{co}(\{f_1, \cdots, f_n\}) \) such that \( h \notin S(f) \) for each \( f \in \{f_1, \cdots, f_n\} \), then \( H_{\|.\|} (G(f), F(h)) \subset H_{\|.\|} (G(h), F(h)) \), for some \( f \in \{f_1, \cdots, f_n\} \). Moreover

\[
G(f) \cap \bigg( \bigcup_{h' \in F(h)} B(h', \max_{f' \in \{f_1, \cdots, f_n\}} H_{\|.\|} (G(f'), F(h))) \bigg) \neq \emptyset,
\]

for each \( f \in \{f_1, \cdots, f_n\} \). Since \( F(h) \) is convex, so

\[
\bigcup_{h' \in F(h)} B(h', \max_{f' \in \{f_1, \cdots, f_n\}} H_{\|.\|} (G(f'), F(h)))
\]

is convex. Since \( G \) is quasi-convex, then

\[
G(h) \cap \bigg( \bigcup_{h' \in F(h)} B(h', \max_{f' \in \{f_1, \cdots, f_n\}} H_{\|.\|} (G(f'), F(h))) \bigg) \neq \emptyset,
\]

and so \( H_{\|.\|} (G(h), F(h)) \leq \max_{f' \in \{f_1, \cdots, f_n\}} H_{\|.\|} (G(f'), F(h)) < H_{\|.\|} (G(h), F(h)) \). This is a contradiction. Now, by Theorem 1.6, there exists \( f_0 \in X \) such that \( f_0 \in \bigcap_{f \in X} S(f) \). Hence, \( H_{\|.\|} (G(f_0), F(f_0)) = \inf_{f \in X} H_{\|.\|} (G(f), F(f_0)) \).

**Corollary 4.3.** Let \( \rho \in \mathfrak{R} \). Suppose \( X \) is a \( \rho \)-compact subset of \( L_\rho \) and \( G : X \to X \) is an onto, quasi-convex and \( \rho \)-upper semi continuous map with nonempty \( \rho \)-compact convex values and \( S : X \to X \) is a continuous single valued map. Then, there exists \( f_0 \in X \) such that \( S(f_0) \in G(f_0) \).

**Corollary 4.4.** Let \( \rho \in \mathfrak{R} \). Suppose \( X \) is a \( \rho \)-compact subset of \( L_\rho \) and \( G : X \to X \) is a quasi-convex and \( \rho \)-upper semi continuous map with nonempty \( \rho \)-compact convex values. Then, there exists \( f_0 \in X \) such that

\[
H_{\|.\|} (G(f_0), f_0) = \inf_{f \in X} H_{\|.\|} (G(f), f_0).
\]

**Corollary 4.5.** Let \( \rho \in \mathfrak{R} \). Suppose \( X \) is a \( \rho \)-compact subset of \( L_\rho \) and \( G : X \to X \) is a \( \rho \)-upper semi continuous map with nonempty \( \rho \)-compact convex values. If \( G(f) \cap X = \emptyset \) for all \( f \in \partial X \), then \( G \) has a fixed point.

**Proof.** If \( G \) does not have a fixed point then by Theorem 4.2, there exists \( f_0 \in \partial X \) such that

\[
0 < H_{\|.\|} (f_0, G(f_0)) \leq H_{\|.\|} (f, G(f_0)),
\]

for all \( f \in X \). Since \( f_0 \in \partial X \), we have \( G(f_0) \cap X \neq \emptyset \), which is a contradiction. \( \square \)

**References**


