New Type of Generalized Difference Sequence Space of Non-Absolute Type and Some Matrix Transformations

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Abstract. In the present paper, we introduce a new difference sequence space $r^p_B(u, p)$ by using the Riesz mean and the $B$-difference matrix. We show $r^p_B(u, p)$ is a complete linear metric space and is linearly isomorphic to the space $l(p)$. We have also computed its $\alpha$-, $\beta$- and $\gamma$-duals. Furthermore, we have constructed the basis of $r^p_B(u, p)$ and characterize a matrix class $(r^p_B(u, p), l_\infty)$.

1. Introduction, Background and Notation

We denote the set of all sequences (real or complex) by $\omega$. Any subspace of $\omega$ is called the sequence space. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of non-negative integers, of real numbers and of complex numbers, respectively. Let $l_\infty$, $c$ and $c_0$ denote the space of all bounded, convergent and null sequences, respectively. Also, by $c_0$, $l_1$ and $l(p)$ we denote the spaces of all convergent, absolutely and $p$-absolutely convergent series, respectively.

Let $X$ be a real or complex linear space, $h$ be a function from $X$ to the set $\mathbb{R}$ of real numbers. Then, the pair $(X, h)$ is called a paranormed space and $h$ is a paranorm for $X$, if the following axioms are satisfied:

\begin{itemize}
  \item[(pn.1)] $h(\theta) = 0$,
  \item[(pn.2)] $h(-x) = h(x)$, \quad (\text{linearity})
  \item[(pn.3)] $h(x + y) \leq h(x) + h(y)$, \quad (\text{subadditivity})
  \item[(pn.4)] scalar multiplication is continuous, that is, $|\alpha_n - \alpha| \to 0$ and $h(x_n - x) \to 0$ imply $h(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha$'s in $\mathbb{R}$ and $x$'s in $X$, \quad (\text{continuity})
\end{itemize}

Assume here and after that $(p_k)$ be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H$ and $M = \max\{1, H\}$. Then, the linear space $l(p)$ was defined by Maddox [10] as follows

$$l(p) = \{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\}$$

which is complete space paranormed by

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Let $X, Y$ be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$. Then, the matrix $A$ defines the $A$-transformation from $X$ into $Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = (Ax_k) = \{(Ax)_k\}$, the $A$-transform of $x$ exists and is in $Y$, where $(Ax)_k = \sum_n a_{nk}x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to $\infty$. Let $(q_k)$ be a sequence of positive numbers and let us write, $Q_n = \sum_{k=0}^{n} q_k$ for $n \in \mathbb{N}$. Then, the matrix $R^{q} = (r_{nk}^{q})$ of the Riesz mean $(R, q_n)$ is given by

$$r_{nk}^{q} = \begin{cases} \frac{q_n}{Q_n}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

The Riesz mean $(R, q_n)$ is regular if and only if $Q_n \to \infty$ as $n \to \infty$ [17].

Recently, Neyaz and Hamid [18] introduced the sequence space $r^{q}(u, p)$ as

$$r^{q}(u, p) = \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \right|^p < \infty \right\}, \quad (0 < p_k \leq H < \infty).$$

Kizmaz [8] defined the difference sequence spaces $Z(\Delta)$ as follows

$$Z(\Delta) = \{ x = (x_k) \in \omega : (\Delta x_k) \in Z \},$$

where $Z \in \{l_\infty, c, c_0 \}$ and $\Delta x_k = x_k - x_{k-1}$.

Altay and Başar [2] defined the sequence space of $p$-bounded variation $bv_{p}$, which is defined as

$$bv_{p} = \left\{ x = (x_k) \in \omega : \sum_k |x_k - x_{k-1}|^p < \infty \right\}, \quad 1 \leq p < \infty.$$ 

With the notation of (1), the space $bv_{p}$ can be re-defined as

$$bv_{p} = (l_{p})_{\Delta}, \quad 1 \leq p < \infty$$

where, $\Delta$ denotes the matrix $\Delta = (\Delta_{nk})$ and is defined as

$$\Delta_{nk} = \begin{cases} (-1)^{n-k}, & \text{if } n - 1 \leq k \leq n, \\ 0, & \text{if } k < n - 1 \text{ or } k > n. \end{cases}$$
Neyaz and Hamid [19] introduced the space $r^q(\triangle^p u)$ as:

$$r^q(\triangle^p u) = \left\{ x = (x_k) \in \ell : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_k q_j \triangle x_j \right|^{p_k} < \infty \right\},$$

where $(0 < p_k \leq H < \infty)$.

In [3] the generalized difference matrix $B = (b_{nk})$ is defined as:

$$b_{nk} = \begin{cases} r, & \text{if } k = n, \\ s, & \text{if } k = n - 1, \\ 0, & \text{if } 0 \leq k < n - 1 \text{ or } k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$, $r, s \in \mathbb{R} - \{0\}$. The matrix $B$ can be reduced to difference matrix $\triangle$ incase $r = 1$, $s = -1$.

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors viz., [1, 4–7, 13, 16, 18, 19].

2. The Riesz Sequence Space $r^q_B(u, p)$ of Non-Absolute Type

In this section, we define the Riesz sequence space $r^q_B(u, p)$, and prove that the space $r^q_B(u, p)$ is a complete paranormed linear space and show it is linearly isomorphic to the space $l(p)$.

Define the sequence $y = (y_k)$, which will be frequently used, by the $R^q_B$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k(q) = \frac{1}{Q_k} \left\{ \sum_{j=0}^{k-1} u_k(q_j r + q_j s) x_j + u_k q_k r x_k \right\}, \quad (k \in \mathbb{N}). \quad (2)$$

Following Başar and Altay [1], Mursaleen et al [14, 15], Neyaz and Hamid [18, 19], Başarir and Öztürk [4], we define the sequence space $r^q_B(u, p)$ as the set of all sequences such that $R^q_B$ transform of it is in the space $l(p)$, that is,

$$r^q_B(u, p) = \{ x = (x_k) \in \ell : y_k(q) \in l(p) \}.$$

Note that if we take $r = 1$ and $s = -1$, the sequence spaces $r^q_B(u, p)$ reduces to $r^q(\triangle u)$, introduced by Neyaz and Hamid [19]. Also, if $(u_k) = c = (1, 1, ...)$, the sequence spaces $r^q_B(u, p)$ reduces to $r^q_B(p)$ Başarir [3].

With the notation of (1) that

$$r^q_B(u, p) = \{ l(p) \}^q_B.$$

Now, we prove the following theorem which is essential in the text.
Theorem 2.1. \( r^p_\mathcal{B}(u,p) \) is a complete linear metric space paranormed by \( h_B \), defined as

\[
h_B(x) = \left[ \sum_k \left\{ \frac{1}{Q_k} \left( \sum_{j=0}^{k-1} u_k(q_jr + q_{j+1}s)x_j + q_kr x_k \right) \right\}^{p_k} \right]^{\frac{1}{p_k}},
\]

where \( \text{sup}_k p_k = H \) and \( M = \max\{1,H\} \).

Proof. The linearity of \( r^p_\mathcal{B}(u,p) \) with respect to the co-ordinatewise addition and scalar multiplication follows from the inequalities which are satisfied for \( z, x \in r^p_\mathcal{B}(u,p) \) [11]

\[
\left[ \sum_k \left\{ \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k(q_jr + q_{j+1}s)(x_j + z_j) + q_kr x_k + z_k \right\}^{p_k} \right]^{\frac{1}{p_k}} 
\leq \sum_k \left\{ \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k(q_jr + q_{j+1}s)x_j + q_kr x_k \right\}^{p_k} \frac{1}{p_k} + \sum_k \left\{ \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k(q_jr + q_{j+1}s)z_j + q_kr z_k \right\}^{p_k} \frac{1}{p_k}
\]

(3)

and for any \( \alpha \in \mathbb{R} \) [12]

\[
|\alpha|^p \leq \max(1,|\alpha|^p).
\]

(4)

It is clear that, \( h_B(0) = 0 \) and \( h_B(x) = h_B(-x) \) for all \( x \in r^p_\mathcal{B}(u,p) \). Again the inequality (3) and (4), yield the subadditivity of \( h_B \) and

\[
h_B(\alpha x) \leq \max(1,|\alpha|)h_B(x).
\]

Let \( \{x^n\} \) be any sequence of points of the space \( r^p_\mathcal{B}(u,p) \) such that \( h_B(x^n - x) \to 0 \) and \( (\alpha_n) \) is a sequence of scalars such that \( \alpha_n \to \alpha \). Then, since the inequality

\[
h_B(x^n) \leq h_B(x) + h_B(x^n - x)
\]

holds by subadditivity of \( h_B \), \( |h_B(x^n)| \) is bounded and we thus have

\[
h_B(\alpha_n x^n - \alpha x) = \left[ \sum_k \left\{ \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k(q_jr + q_{j+1}s)(\alpha_n x^n_j - \alpha x_j) + u_kq_kr(\alpha_n x^n_k - \alpha x_k) \right\}^{p_k} \right]^{\frac{1}{p_k}} \leq |\alpha_n - \alpha|^\frac{p_k}{p_k} h_B(x^n) + |\alpha|^\frac{1}{p_k} h_B(x^n - x)
\]

which tends to zero as \( n \to \infty \). That is to say, that the scalar multiplication is continuous. Hence, \( h_B \) is paranorm on the space \( r^p_\mathcal{B}(u,p) \).

It remains to prove the completeness of the space \( r^p_\mathcal{B}(u,p) \). Let \( \{x^i\} \) be any Cauchy sequence in the space \( r^p_\mathcal{B}(u,p) \), where \( x^i = \{x^i_0, x^i_1, \ldots\} \). Then, for a given \( \varepsilon > 0 \) there exists a positive integer \( n_0(\varepsilon) \) such that
\[ h_B(x^i - x^j) < \epsilon \] (5)

for all \( i, j \geq n_0(\epsilon) \). Using definition of \( h_B \) and for each fixed \( k \in \mathbb{N} \) that

\[ \left| (R^m_b Bx^i)_k - (R^m_b Bx^j)_k \right| \leq \left[ \sum_k \left| (R^m_b Bx^i)_k - (R^m_b Bx^j)_k \right|^p \right]^{\frac{1}{p}} < \epsilon \]

for \( i, j \geq n_0(\epsilon) \), which leads us to the fact that \( \{ (R^m_b Bx^0)_k, (R^m_b Bx^1)_k, \ldots \} \) is a Cauchy sequence of real numbers for every fixed \( k \in \mathbb{N} \). Since \( \mathbb{R} \) is complete, it converges, say, \( (R^m_b Bx^i)_k \rightarrow ((R^m_b Bx)_k \text{ as } i \rightarrow \infty \). Using these infinitely many limits \( (R^0_b Bx)_0, (R^0_b Bx)_1, \ldots \), we define the sequence \( \{ (R^m_b Bx)_0, (R^m_b Bx)_1, \ldots \} \). From (5) for each \( m \in \mathbb{N} \) and \( i, j \geq n_0(\epsilon) \),

\[ \sum_{k=0}^{m} |(R^m_b Bx^i)_k - (R^m_b Bx^j)_k|^p \leq h_B(x^i - x^j)^m < \epsilon^m. \] (6)

Take any \( i, j \geq n_0(\epsilon) \). First, let \( j \rightarrow \infty \) in (6) and then \( m \rightarrow \infty \), we obtain

\[ h_B(x^i - x) \leq \epsilon. \]

Finally, taking \( \epsilon = 1 \) in (6) and letting \( i \geq n_0(1) \), we have by Minkowski’s inequality for each \( m \in \mathbb{N} \) that

\[ \left[ \sum_{k=0}^{m} |(R^m_b Bx)_k|^p \right]^{\frac{1}{p}} \leq h_B(x^i - x) + h_B(x^j) \leq 1 + h_B(x^i) \]

which implies that \( x \in r^0_b(u, p) \). Since \( h_B(x - x^i) \leq \epsilon \) for all \( i \geq n_0(\epsilon) \), it follows that \( x^i \rightarrow x \) as \( i \rightarrow \infty \), hence we have shown that \( r^0_b(u, p) \) is complete, hence the proof.

If we take \( r = 1, s = -1 \) in the theorem 2.1, then we have the following result which was proved by Neyaz and Hamid [19].

**Corollary 2.2.** \( r^s(\Delta^1_p) \) is a complete linear metric space paranormed by \( h_{\Delta^1} \), defined as

\[ h_{\Delta^1}(x) = \left[ \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k(q_j - q_{j+1})x_j + q_k u_k x_k \right|^p \right]^{\frac{1}{p}}, \]

where \( \sup_k p_k = \text{Hand} M = \max[1, H] \).

Note that one can easily see the absolute property does not hold on the spaces \( r^0_b(u, p) \), that is \( h_B(x) \neq h_B(|x|) \) for at least one sequence in the space \( r^0_b(u, p) \) and this says that \( r^0_b(u, p) \) is a sequence space of non-absolute type.

**Theorem 2.3.** The sequence space \( r^0_b(u, p) \) of non-absolute type is linearly isomorphic to the space \( l(p) \), where \( 0 < p_k \leq H < \infty \).
Proof. To prove the theorem, we should show the existence of a linear bijection between the spaces $r_q^u(u,p)$ and $l(p)$, where $0 < p_k \leq H < \infty$. With the notation of (2), define the transformation $T$ from $r_q^u(u,p)$ to $l(p)$ by $x \rightarrow y = T(x)$. The linearity of $T$ is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence $T$ is injective.

Let $y \in l(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{n=0}^{k-1} (-1)^{k-n} \left( \frac{s^{k-n}}{p^{k-n} q_{n+1}} + \frac{s^{k-n}}{p^{k-n} q_n} \right) Q_n u_k^{-1} y_n + \frac{Q_k u_k^{-1} y_k}{r_k d_k}.$$

Then,

$$h_B(x) = \left[ \sum_k \left| \sum_{j=0}^{k-1} u_k(q,j,r+q_{j+1},s)x_j + u_k q_j r x_k \right|^{p_k} \right]^{\frac{1}{p_k}} = \left[ \sum_k \sum_{j=0}^{k} \delta_{kj} y_j \right]^{\frac{1}{p_k}} = h_1(y) < \infty,$$

where,

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j. \end{cases}$$

Thus, we have $x \in r_q^u(u,p)$. Consequently, $T$ is surjective and is paranorm preserving. Hence, $T$ is a linear bijection and this says us that the spaces $r_q^u(u,p)$ and $l(p)$ are linearly isomorphic, hence the proof.

3. Duals and Basis of $r_q^u(u,p)$

In this section, we compute $\alpha$-, $\beta$- and $\gamma$-duals of $r_q^u(u,p)$ and construct its basis.

**Theorem 3.1.** (i) Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Define the sets $D_1(u,p)$ and $D_2(u,p)$ as follows

$$D_1(u,p) = \bigcup_{B \geq 1} \left\{ a = (a_k) \in \omega : \sup_{k \in \mathbb{N}} \left| \sum_{n \in \mathbb{N}} \left( \varphi_n(u,k) a_n Q_n + \frac{a_n}{r_k d_n} u_n^{-1} Q_n \right) B^{-1} \right|^{p_k} < \infty \right\}$$

and
Then, \[ D_2(u, p) = \bigcup_{B > 1} \left\{ a = (a_k) \in \omega : \sum_k \left| \frac{a_k}{r_k u_{n+1}} + \nabla_a (n, k) \sum_{i=1}^n a_i Q_k \right| B^{-1} \right|^{p_k'} < \infty \} \]

where \[ \nabla_a(n, k) = (-1)^k - \frac{s_1^{k-n-1} + s_2^{k-n}}{r_k q_{n+1}} u_{n-1}^k. \]

Then,
\[
\left[ r_B^a(u, p) \right]^a = D_1(u, p), \quad \left[ r_B^a(u, p) \right]^b = D_2(u, p) \cap cs, \quad \text{and} \quad \left[ r_B^a(u, p) \right]^c = D_2(u, p).
\]

(ii) Let \( 0 < p_k \leq 1 \) for every \( k \in \mathbb{N} \). Define the sets \( D_3(u, p) \) and \( D_4(u, p) \) as follows
\[
D_3(u, p) = \left\{ a = (a_k) \in \omega : \sup_{k \in \mathbb{N}} \left\| \sum_k \nabla_a (n, k) a_n Q_k + \frac{a_n}{r_k q_{n+1}} u_{n-1}^k Q_{n+1} \right\| B^{-1} \right|^{p_k} < \infty \}
\]
and
\[
D_4(u, p) = \left\{ a = (a_k) \in \omega : \sup_k \left\| \frac{a_k}{r_k u_{k+1}} + \nabla_a (n, k) \sum_{i=1}^n a_i Q_k \right\| B^{-1} \right|^{p_k} < \infty \}.
\]

Then,
\[
\left[ r_B^a(u, p) \right]^a = D_3(u, p), \quad \left[ r_B^a(u, p) \right]^b = D_4(u, p) \cap cs, \quad \text{and} \quad \left[ r_B^a(u, p) \right]^c = D_4(u, p).
\]

For the proof of the Theorem 3.1, we need following lemmas.

**Lemma 3.2.** \([7](i) \) Let \( 1 < p_k \leq H < \infty \). Then \( A \in (l(p) : l_1) \) if and only if there exists an integer \( B > 1 \) such that
\[
\sup_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} \left| a_{nk} B^{-1} \right|^{p_k} < \infty.
\]

(ii) Let \( 0 < p_k \leq 1 \). Then \( A \in (l(p) : l_1) \) if and only if
\[
\sup_{k \in \mathbb{N}} \sup_{n \in \mathbb{N}} \left| a_{nk} B^{-1} \right|^{p_k} < \infty.
\]

**Lemma 3.3.** \([9](i) \) Let \( 1 < p_k \leq H < \infty \). Then, \( A \in (l(p) : l_\infty) \) if and only if there exists an integer \( B > 1 \) such that
\[
\sup_n \sum_k |a_{nk}B^{-1}|p_k' < \infty. \quad (7)
\]

(ii) Let \(0 < p_k \leq 1\) for every \(k \in \mathbb{N}\). Then \(A \in (l(p) : l_\infty)\) if and only if
\[
\sup_{n,k \in \mathbb{N}} |a_{nk}|p_k < \infty. \quad (8)
\]

**Lemma 3.4.** [9] Let \(0 < p_k \leq H < \infty\) for every \(k \in \mathbb{N}\). Then \(A \in (l_p) : c)\) if and only if (7) and (8) hold along with
\[
\lim_n a_{nk} = \beta_k \text{ for } k \in \mathbb{N}. \quad (9)
\]

**Proof of Theorem 3.1.** We consider the case \(1 < p_k \leq H < \infty\) for every \(k \in \mathbb{N}\). Let us take any \(a = (a_n) \in \omega\). From (2) we can easily see that
\[
a_n x_n = \sum_{k=0}^{n-1} \nabla u(n,k)a_n Q_k y_k + \frac{a_n Q_n y_n}{r_d n} u_k^{-1} = \sum_{k=0}^{n} c_{nk} y_k = (Cy)_n, \quad (10)
\]

where \(n \in \mathbb{N}\) and \(C = (c_{nk})\) is defined by
\[
c_{nk} = \begin{cases} 
\nabla u(n,k)a_n Q_k, & \text{if } 0 \leq k \leq n-1, \\
\frac{a_n Q_n}{r_d n} u_k^{-1}, & \text{if } k = n, \\
0, & \text{if } k > n,
\end{cases}
\]

where \(k, n \in \mathbb{N}\). Thus, we deduce from (10) with Lemma 3.2 that \(ax = (a_n x_n) \in l_1\) whenever \(x = (x_n) \in r_B^h(u,p)\) if and only if \(Cy \in l_1\) whenever \(y \in l(p)\). This shows that \([r_B^h(u,p)]^\prime = D_1(u,p)\).

Further, consider the equation
\[
\sum_{k=0}^{n} a_n x_n = \sum_{k=0}^{n} \left( \frac{a_k}{r_d k} u_k^{-1} + \nabla u(n,k) \sum_{i=k+1}^{n} a_i \right) Q_k y_k = (Dy)_n, \quad (11)
\]

where \(n \in \mathbb{N}\) and \(D = (d_{nk})\) is defined by
\[
d_{nk} = \begin{cases} 
\left( \frac{a_k}{r_d k} u_k^{-1} + \nabla u(n,k) \sum_{i=k+1}^{n} a_i \right) Q_k, & \text{if } 0 \leq k \leq n, \\
0, & \text{if } k > n,
\end{cases}
\]
for all \( k, n \in \mathbb{N} \). Thus we deduce from (11) with Lemma 3.3 that \( ax = (a_n x_n) \in cs \) whenever \( x = (x_n) \in r^q_B(u, p) \) if and only if \( Dy \in c \) whenever \( y \in l(p) \). Therefore, we derive from (11) that

\[
\sum_k \left\| \left( \frac{a_k}{r_{d/k}} u_k^{-1} + \nabla_a(n, k) \sum_{i=k+1}^n a_i u_i^{-1} Q_i \right) B^{-1} \right\|_{p_k} < \infty,
\]

and \( \lim_{n \to \infty} d_{nk} \) exists and hence shows that \( [r^q_B(u, p)]^\beta_2 = D^2(u, p) \cap cs \).

As proved above, from Lemma 3.4 together with (12) that \( ax = (a_k x_k) \in bs \) whenever \( x = (x_n) \in r^q_B(u, p) \) if and only if \( Dy \in l_\infty \) whenever \( y = (y_k) \in l(p) \). Therefore, we again obtain the condition (12) which means that \( [r^q_B(u, p)]^\gamma = D^2(u, p) \) and this completes the proof.

**Theorem 3.5.** Define the sequence \( b^{(k)}(q) = \{ b^{(k)}(q) \} \) of the elements of the space \( r^q_B(u, p) \) for every fixed \( k \in \mathbb{N} \) by

\[
b^{(k)}_n(q) = \begin{cases} 
\frac{Q_k}{r_{q/k}} u_k^{-1} + \nabla_a(n, k) Q_k, & \text{if } 0 \leq n \leq k, \\
0, & \text{if } n > k.
\end{cases}
\]

Then, the sequence \( \{b^{(k)}(q)\} \) is a basis for the space \( r^q_B(u, p) \) and for any \( x \in r^q_B(u, p) \) has a unique representation of the form

\[
x = \sum_k \lambda_k(q) b^{(k)}(q)
\]

where, \( \lambda_k(q) = (R^q_B x)_k \) for all \( k \in \mathbb{N} \) and \( 0 < p_k \leq H < \infty \).

**Proof.** It is clear that \( \{b^{(k)}(q)\} \subset r^q_B(u, p) \), since

\[
R^q_B b^{(k)}(q) = e^{(k)} \in l(p) \text{ for } k \in \mathbb{N}
\]

and \( 0 < p_k \leq H < \infty \), where \( e^{(k)} \) is the sequence whose only non-zero term is 1 in \( k^{th} \) place for each \( k \in \mathbb{N} \).

Let \( x \in r^q_B(u, p) \) be given. For every non-negative integer \( m \), we put

\[
x^{[m]} = \sum_{k=0}^m \lambda_k(q) b^{(k)}(q).
\]

Then, by applying \( R^q_B \) to (15) and using (14), we obtain
\[ R^0_B (x^{[m]}) = \sum_{k=0}^{m} \lambda_k(q) R^0_B [t^k(q)] = \sum_{k=0}^{m} (R^0_B x)_k e^k \]

and

\[
(R^0_B (x^{[m]}))_i = \begin{cases} 
0, & \text{if } 0 \leq i \leq m \\
(R^0_B x)_i, & \text{if } i > m
\end{cases}
\]

where \( i, m \in \mathbb{N} \). Given \( \varepsilon > 0 \), there exists an integer \( m_0 \) such that

\[
\left( \sum_{i=m}^{\infty} |(R^0_B x)_i|^p \right)^{1/p} < \frac{\varepsilon}{2},
\]

for all \( m \geq m_0 \). Hence,

\[
l_B (x^{[m]}) = \left( \sum_{i=m}^{\infty} |(R^0_B x)_i|^p \right)^{1/p} \leq \left( \sum_{i=m_0}^{\infty} |(R^0_B x)_i|^p \right)^{1/p} < \frac{\varepsilon}{2} < \varepsilon
\]

for all \( m \geq m_0 \), which proves that \( x \in r^0_B (u, p) \) is represented as (14).

Let us show the uniqueness of the representation for \( x \in r^0_B (u, p) \) given by (13). Suppose, on the contrary; that there exists a representation \( x = \sum_k \mu_k(q) b^k(q) \). Since the linear transformation \( T \) from \( r^0_B (u, p) \) to \( l^\infty(p) \) used in the Theorem 3 is continuous we have

\[
(R^0_B x)_n = \sum_k \mu_k(q) (R^0_B t^k(q))_n = \sum_k \mu_k(q) e^k_n = \mu_n(q)
\]

for \( n \in \mathbb{N} \), which contradicts the fact that \( (R^0_B)_n = \lambda_n(q) \) for all \( n \in \mathbb{N} \). Hence, the representation (13) is unique. This completes the proof.

4. Matrix Mappings on the Space \( r^0_B (u, p) \)

In this section, we characterize the matrix mappings from the space \( r^0_B (u, p) \) to the space \( l^\infty \).

**Theorem 4.1.** (i) Let \( 1 < p_k \leq H < \infty \) for every \( k \in \mathbb{N} \). Then \( A \in \left( r^0_B (u, p) : l^\infty \right) \) if and only if there exists an integer \( B > 1 \) such that
Now, by combining (19) and the following inequality which holds for any fixed  \( n \in \mathbb{N} \):

\[
C(B) = \sup_{n} \sum_{k} \left\| \left( \frac{a_{nk}}{r, t \mathcal{U}_k} + \mathcal{V}_u(n, k) \sum_{i=k+1}^{n} a_{ni} \right) Q_k \right\|_{p_k} < \infty \tag{16}
\]

and \( |a_{nk}| \in \mathbb{C} \) for each \( n \in \mathbb{N} \).

(ii) Let \( 0 < p_k \leq 1 \) for every \( k \in \mathbb{N} \). Then \( A \in \left( r^\beta_B (u, p) : l_\infty \right) \) if and only if

\[
\sup_{k} \left\| \left( \frac{a_{nk}}{r, t \mathcal{U}_k} + \mathcal{V}_u(n, k) \sum_{i=k+1}^{n} a_{ni} \right) Q_k \right\|_{p_k} < \infty \tag{17}
\]

and \( |a_{nk}| \in \mathbb{C} \) for each \( n \in \mathbb{N} \).

**Proof.** We will prove (i) and (ii) can be proved in a similar fashion. So, let \( A \in \left( r^\beta_B (u, p) : l_\infty \right) \) and \( 1 < p_k \leq H < \infty \) for every \( k \in \mathbb{N} \). Then \( Ax \) exists for \( x \in r^\beta_B (u, p) \) and implies that \( |a_{nk}| \in \left( r^\beta_B (u, p) \right)^\beta \) for each \( n \in \mathbb{N} \). Hence necessity of (16) holds.

Conversely, suppose that the necessities (16) hold and \( x \in r^\beta_B (u, p) \), since \( |a_{nk}| \in \left( r^\beta_B (u, p) \right)^\beta \) for every fixed \( n \in \mathbb{N} \), so the \( A \)-transform of \( x \) exists. Consider the following equality obtained by using the relation (11) that

\[
\sum_{k=0}^{m} a_{nk} x_k = \sum_{k} \left( \left( \frac{a_{nk}}{r, t \mathcal{U}_k} + \mathcal{V}_u(n, k) \sum_{i=k+1}^{n} a_{ni} \right) Q_k \right) y_k. \tag{18}
\]

Taking into account the assumptions we derive from (18) as \( m \to \infty \) that

\[
\sum_{k} a_{nk} x_k = \sum_{k} \left( \left( \frac{a_{nk}}{r, t \mathcal{U}_k} + \mathcal{V}_u(n, k) \sum_{i=k+1}^{n} a_{ni} \right) Q_k \right) y_k \tag{19}
\]

Now, by combining (19) and the following inequality which holds for any \( B > 0 \) and any complex numbers \( a, b \)

\[
|ab| \leq B \left( |aB^{-1}|^p + |b|^p \right)
\]

with \( p^{-1} + p'^{-1} = 1 \) [10, 16], one can easily see that

\[
\sup_{n \in \mathbb{N}} \sum_{k} a_{nk} x_k \leq \sup_{n \in \mathbb{N}} \sum_{k} \left\| \left( \frac{a_{nk}}{r, t \mathcal{U}_k} + \mathcal{V}_u(n, k) \sum_{i=k+1}^{n} a_{ni} \right) Q_k \right\|_{y_k} |y_k| \leq B \left( C(B) + \| h^\beta_B (y) \| \right) < \infty.
\]

This shows that \( Ax \in l_\infty \) whenever \( x \in r^\beta_B (u, p) \).

This completes the proof.
References


