Reverse Order Law \((ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger\) in Rings with Involution

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Abstract. In this paper we study several equivalent conditions for the reverse order law \((ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger\) in rings with involution. We extend some well-known results to more general settings.

1. Introduction

Let \(\mathcal{R}\) be an associative ring with the unit 1. If \(a, b \in \mathcal{R}\) are invertible, then \(ab\) is invertible too and the inverse of the product \(ab\) satisfied the reverse order law \((ab)^{-1} = b^{-1}a^{-1}\). This formula cannot trivially be extended to the Moore–Penrose inverse of the product \(ab\). Many authors studied this problem and proved some equivalent conditions for \((ab)^\dagger = b^\dagger a^\dagger\) in setting of matrices, operators or rings [1–6, 8, 9, 11, 12, 15, 20–22]. Because the reverse order law \((ab)^\dagger = b^\dagger a^\dagger\) does not always holds, it is not easy to simplify various expressions that involve the Moore-Penrose inverse of product. In addition to \((ab)^\dagger = b^\dagger a^\dagger\), \((ab)^\dagger\) may be expressed as \((ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger\), \((ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger\), \((ab)^\dagger = b^\dagger a^\dagger - b^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^\dagger\) etc. These equalities are called mixed-type reverse order laws for the Moore-Penrose inverse of a product. When investigating various reverse order laws for \((ab)^\dagger\), we notice that some of them are in fact equivalent (see [15, 19, 20]). In this paper we investigate necessary and sufficient conditions for the reverse order law \((ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger\) in the setting of rings with involution.

An involution \(a \mapsto a^\dagger\) in a ring \(\mathcal{R}\) is an anti-isomorphism of degree 2, that is,

\[(a^\dagger)^\dagger = a, \quad (a+b)^\dagger = a^\dagger + b^\dagger, \quad (ab)^\dagger = b^\dagger a^\dagger.\]

An element \(a \in \mathcal{R}\) is self-adjoint if \(a^\dagger = a\).

The Moore–Penrose inverse (or MP-inverse) of \(a \in \mathcal{R}\) is the element \(b \in \mathcal{R}\), if the following equations hold [16–18]:

\[(1) \ aba = a, \quad (2) \ bab = b, \quad (3) \ (ab)^\dagger = ab, \quad (4) \ (ba)^\dagger = ba.\]
There is at most one $b$ such that above conditions hold (see [17]), and such $b$ is denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$. If $a$ is invertible, then $a^\dagger$ coincides with the ordinary inverse of $a$.

If $\delta \subset \{1,2,3,4\}$ and $b$ satisfies the equations $(i)$ for all $i \in \delta$, then $b$ is an $\delta$–inverse of $a$. The set of all $\delta$–inverse of $a$ is denote by $a\{\delta\}$. Notice that $a\{1,2,3,4\} = \{a^\dagger\}$.

The following result is well-known and frequently used in the rest of the paper.

**Theorem 1.1.** [7, 14] For any $a \in \mathcal{R}^\dagger$, the following is satisfied:

(a) $(a^\dagger)^\dagger = a$;

(b) $(a^\dagger)^\dagger = (a^\dagger)^\ast$;

(c) $(a^\dagger a)^\dagger = a^\dagger (a^\dagger)^\ast$;

(d) $(aa^\dagger)^\dagger = (a^\dagger)^\ast a^\dagger$;

(f) $a^\ast = a^\dagger aa^\ast = a^\ast aa^\dagger$;

(g) $a^\dagger = (a^\ast a)^\dagger a^\ast = a^\ast (aa^\ast)^\dagger$;

(h) $(a^\dagger)^\ast = a(a^\ast a)^\dagger = (aa^\ast)^\dagger a$.

From the last theorem we see that the following chain of equivalences hold:

$$a \in \mathcal{R}^\dagger \Leftrightarrow a^\ast \in \mathcal{R}^\dagger \Leftrightarrow aa^\ast \in \mathcal{R}^\dagger \Leftrightarrow a^\ast a \in \mathcal{R}^\dagger.$$

Let $\mathcal{A}$ be a unital $C^*$-algebra with the unit 1. An element $a \in \mathcal{A}$ is regular if there exists some $b \in \mathcal{A}$ satisfying $aba = a$.

**Theorem 1.2.** [10] In a unital $C^*$-algebra $\mathcal{A}$, $a \in \mathcal{A}$ is MP-invertible if and only if $a$ is regular.

An element $p \in \mathcal{A}$ is a projection if $p = p^2 = p^*$. Set $\mathcal{P}(\mathcal{A}) = \{p \in \mathcal{A} : p^2 = p = p^*\}$. In [13], Li proved the following important results which consider some equivalent conditions for $pq$, $(p, q \in \mathcal{P}(\mathcal{A}))$, to be Moore-Penrose invertible and formula for Moore-Penrose inverse of product of projection in a $C^*$-algebra.

**Lemma 1.3.** [13] Let $p, q \in \mathcal{P}(\mathcal{A})$. Then the following statements are equivalent:

(a) $pq$ is Moore-Penrose invertible;

(b) $qp$ is Moore-Penrose invertible;

(c) $(1 – p)(1 – q)$ is Moore-Penrose invertible;

(d) $(1 – q)(1 – p)$ is Moore-Penrose invertible.
Theorem 1.4. [13] Let $p, q \in \mathcal{P}(\mathbb{A})$. If $pq$ is Moore-Penrose invertible, then:

$$(qp)^\dagger = pq - p((1 - p)(1 - q))^\dagger q.$$ 

The reverse order law for the Moore-Penrose inverse is an useful computational tool in applications (solving linear equations in linear algebra or numerical analysis), and it is also interesting from the theoretical point of view.

The reverse-order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ was first studied by Galperin and Waksman [8]. A Hilbert space version of their result was given by Isumino [11]. They proved that $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ holds if and only if $\mathcal{R}((a^\dagger)b) = \mathcal{R}(ab)$ and $\mathcal{R}(b^\dagger a^\dagger) = \mathcal{R}((ab)^\dagger)$, for linear operators $a$ and $b$, where $\mathcal{R}(\cdot)$ denotes the range of an operator. Many results concerning the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ for complex matrices appeared in Tian’s papers [19] and [20], where the author used finite dimensional methods (mostly properties of the rank of a complex matrices). Moreover, the operator analogues of these results are proved in [4] for linear operators on Hilbert spaces, using the operator matrices. In [15], a set of equivalent conditions for this reverse order rule for the Moore-Penrose inverse in the setting of $C^*$-algebra is presented, extending the results for complex matrices from [20]. This result can be formulate for elements in ring with involution in the following way.

Theorem 1.5. [15] Let $\mathcal{R}$ be a ring with involution and let $a, b \in \mathcal{R}^\dagger$. Then the following statements are equivalent:

(a) $ab, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$;

(b) $ab, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(a^\dagger abb^\dagger)^\dagger = b(ab)^\dagger a$;

(c) $ab, a^\dagger ab, abb^\dagger \in \mathcal{R}^\dagger$ and $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger = b^\dagger(abb^\dagger)^\dagger$;

(d) $ab, a^\dagger ab, abb^\dagger \in \mathcal{R}^\dagger$, $(a^\dagger ab)^\dagger = (ab)^\dagger a$ and $(abb^\dagger)^\dagger = b(ab)^\dagger$;

(e) $a^\dagger ab, abb^\dagger, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$, $(a^\dagger ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger$ and $(abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger$;

(f) $ab, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(ab)^\dagger = b(a^\dagger abb^\dagger)^\dagger a^\dagger$;

(g) $ab, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(a^\dagger abb^\dagger)^\dagger = (b^\dagger)(ab)^\dagger(a^\dagger)^\dagger$.

In this paper we present new results for the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution. Thus, we extend the known results for matrices [19] and for Hilbert space operators [4] to more general settings. The most important properties of the MP-inverse will be used in proving various equivalent conditions such that the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ holds. Although these results are known, we use different methods, depending on algebraic properties of rings with involution.
2. Reverse Order Law in Rings

In this section we present necessary and sufficient conditions such that the reverse order law \((ab)^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger\) holds.

**Theorem 2.1.** Let \(\mathcal{R}\) be a ring with involution and let \(a, b \in \mathcal{R}^t\). Then the following statements are equivalent:

(a) \(ab, ab^\dagger abb^\dagger \in \mathcal{R}^t\) and \((ab)^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger;\)

(b) \(ab, ab^\dagger abb^\dagger \in \mathcal{R}^t\) and \((ab)^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger;\)

(b) \((a^\dagger)^\dagger b, ab^\dagger abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger;\)

(c) \((a^\dagger)^\dagger b, (a^\dagger)abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger [(a^\dagger)abb^\dagger]^\dagger a^\dagger;\)

(c) \((a^\dagger)^\dagger b, ab^\dagger abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger [(a^\dagger)abb^\dagger]^\dagger a^\dagger;\)

(d) \(b^\dagger a^\dagger, bb^\dagger a^\dagger a^\dagger \in \mathcal{R}^t\) and \((b^\dagger a^\dagger)^\dagger = a^\dagger (bb^\dagger a^\dagger)^\dagger b;\)

(d) \(b^\dagger a^\dagger, bb^\dagger a^\dagger a^\dagger \in \mathcal{R}^t\) and \((b^\dagger a^\dagger)^\dagger = a^\dagger (bb^\dagger a^\dagger)^\dagger b;\)

(e) \(a^\dagger ab, ab^\dagger \in \mathcal{R}^t\) and \((a^\dagger ab)^\dagger a^\dagger = b^\dagger (abb^\dagger)^\dagger;\)

(e) \(a^\dagger ab, ab^\dagger \in \mathcal{R}^t\) and \((a^\dagger ab)^\dagger a^\dagger = b^\dagger (abb^\dagger)^\dagger;\)

(f) \(a^\dagger ab, ab^\dagger \in \mathcal{R}^t\) and \((a^\dagger ab)^\dagger a^\dagger = b^\dagger (abb^\dagger)^\dagger;\)

(f) \(a^\dagger ab, ab^\dagger \in \mathcal{R}^t\) and \((a^\dagger ab)^\dagger a^\dagger = b^\dagger (abb^\dagger)^\dagger;\)

(g) \(a^\dagger ab, ab^\dagger \in \mathcal{R}^t\) and \((a^\dagger ab)^\dagger a^\dagger = b^\dagger (abb^\dagger)^\dagger;\)

(g) \(a^\dagger ab, ab^\dagger \in \mathcal{R}^t\) and \((a^\dagger ab)^\dagger a^\dagger = b^\dagger (abb^\dagger)^\dagger;\)

(h) \((a^\dagger)^\dagger b, (a^\dagger)abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger [(a^\dagger)abb^\dagger]^\dagger;\)

(h) \((a^\dagger)^\dagger b, (a^\dagger)abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger [(a^\dagger)abb^\dagger]^\dagger;\)

(i) \((a^\dagger)^\dagger b, (a^\dagger)abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger [(a^\dagger)abb^\dagger]^\dagger;\)

(i) \((a^\dagger)^\dagger b, (a^\dagger)abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger [(a^\dagger)abb^\dagger]^\dagger;\)

(j) \((a^\dagger)^\dagger b, (a^\dagger)abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger [(a^\dagger)abb^\dagger]^\dagger;\)

(j) \((a^\dagger)^\dagger b, (a^\dagger)abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger [(a^\dagger)abb^\dagger]^\dagger;\)

(k) \((a^\dagger)^\dagger b, (a^\dagger)abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger [(a^\dagger)abb^\dagger]^\dagger;\)

(k) \((a^\dagger)^\dagger b, (a^\dagger)abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger [(a^\dagger)abb^\dagger]^\dagger;\)

(l) \((a^\dagger)^\dagger b, (a^\dagger)abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger [(a^\dagger)abb^\dagger]^\dagger;\)

(l) \((a^\dagger)^\dagger b, (a^\dagger)abb^\dagger \in \mathcal{R}^t\) and \([(a^\dagger)^\dagger b]^\dagger = b^\dagger [(a^\dagger)abb^\dagger]^\dagger;\)
Proof. The equivalences (a1) ⇔ (a2) ⇔ (f1) follow from Theorem 1.5.

(a1) ⇒ (b1): Using the hypothesis $ab\dagger = b\dagger(a^\dagger abb\dagger)a^\dagger$ and Theorem 1.1, we get

\[
(a^\dagger)bb\dagger (a^\dagger abb\dagger) a^\dagger b = (a^\dagger) a^\dagger(ab\dagger(a^\dagger abb\dagger) a^\dagger ab) = (a^\dagger) a^\dagger ab = (a^\dagger) b,
\]

\[
b\dagger(a^\dagger abb\dagger)a^\dagger b(a^\dagger abb\dagger) a^\dagger = (b\dagger(a^\dagger abb\dagger)a^\dagger ab)(a^\dagger abb\dagger) a^\dagger = b\dagger(a^\dagger abb\dagger) a^\dagger aa^\dagger = b\dagger(a^\dagger abb\dagger) a^\dagger a^\dagger = b\dagger a\dagger, \]

\[
((a^\dagger)bb\dagger (a^\dagger abb\dagger) a^\dagger)^* = ((a^\dagger) a^\dagger abb\dagger(a^\dagger abb\dagger) a^\dagger)^* = aa^\dagger abb\dagger(a^\dagger abb\dagger) a^\dagger
\]

\[
= abb\dagger(a^\dagger abb\dagger) a^\dagger = (ab\dagger(a^\dagger abb\dagger) a^\dagger)^*
\]

\[
= (aa^\dagger abb\dagger(a^\dagger abb\dagger) a^\dagger)^* = (a^\dagger) a^\dagger abb\dagger(a^\dagger abb\dagger) a^\dagger
\]

\[
= (a^\dagger) bb\dagger(a^\dagger abb\dagger) a^\dagger,
\]

\[
(b\dagger(a^\dagger abb\dagger) a^\dagger a^\dagger)b^\dagger = (b\dagger(a^\dagger abb\dagger) a^\dagger ab)^\dagger = b\dagger(a^\dagger abb\dagger) a^\dagger a^\dagger b = b\dagger(a^\dagger abb\dagger) a^\dagger (a^\dagger) b.
\]

Hence, by these four equalities and the definition of MP-inverse, we deduce that $(a^\dagger) b \in R^a$ and $[(a^\dagger) b]^\dagger = b\dagger(a^\dagger abb\dagger) a^\dagger$.

(b1) ⇒ (a1): Since

\[
abb\dagger(a^\dagger abb\dagger) a^\dagger ab = a(a^\dagger abb\dagger(a^\dagger abb\dagger) a^\dagger ab) b = aa^\dagger abb\dagger b = ab,
\]

\[
b\dagger(a^\dagger abb\dagger) a^\dagger a^\dagger abb\dagger(a^\dagger abb\dagger) a^\dagger = b\dagger(a^\dagger abb\dagger) a^\dagger a^\dagger,
\]

we conclude that $b\dagger(a^\dagger abb\dagger) a^\dagger \in (ab)[1, 2]$. From $[(a^\dagger) b]^\dagger = b\dagger(a^\dagger abb\dagger) a^\dagger$, we have that the elements $(a^\dagger) bb\dagger(a^\dagger abb\dagger) a^\dagger$, $b\dagger(a^\dagger abb\dagger) a^\dagger (a^\dagger) b$ are self-adjoint. Then

\[
(a^\dagger)bb\dagger(a^\dagger abb\dagger) a^\dagger = (aa^\dagger abb\dagger(a^\dagger abb\dagger) a^\dagger)^* = (a^\dagger) a^\dagger abb\dagger(a^\dagger abb\dagger) a^\dagger
\]

\[
= (a^\dagger) bb\dagger(a^\dagger abb\dagger) a^\dagger = ((a^\dagger) bb\dagger(a^\dagger abb\dagger) a^\dagger)^* = ((a^\dagger) a^\dagger abb\dagger(a^\dagger abb\dagger) a^\dagger)^*
\]

\[
= abb\dagger(a^\dagger abb\dagger) a^\dagger,
\]

\[
(b\dagger(a^\dagger abb\dagger) a^\dagger ab)^\dagger = (b\dagger(a^\dagger abb\dagger) a^\dagger a^\dagger b)^\dagger = b\dagger(a^\dagger abb\dagger) a^\dagger a^\dagger b = b\dagger(a^\dagger abb\dagger) a^\dagger ab,
\]

i.e. $abb\dagger(a^\dagger abb\dagger) a^\dagger$, $b\dagger(a^\dagger abb\dagger) a^\dagger ab$ are self-adjoint too. Therefore, $ab \in R^a$ and $(ab)^\dagger = b\dagger(a^\dagger abb\dagger) a^\dagger$.

(a1) ⇒ (c1): By the definition of MP-inverse and Theorem 1.1, we obtain

\[
a(b^\dagger)^\dagger b\dagger(a^\dagger abb\dagger) a^\dagger ab = a^\dagger abb\dagger(a^\dagger abb\dagger) a^\dagger ab(b^\dagger)^\dagger = aa^\dagger abb\dagger(b^\dagger)^\dagger = a(b^\dagger)^\dagger,
\]

\[
b\dagger(a^\dagger abb\dagger) a^\dagger a(b^\dagger)^\dagger b\dagger(a^\dagger abb\dagger) a^\dagger = b\dagger(a^\dagger abb\dagger) a^\dagger abb\dagger(a^\dagger abb\dagger) a^\dagger = b\dagger(a^\dagger abb\dagger) a^\dagger,
i.e. \( b'(a'ab^b)^\dagger a^+ \in [a(b')]\{1,2\} \). The condition \((ab)^\dagger = b'(a'ab^b)^\dagger a^+ \) give that the right hand side of the equality

\[
ab(b')^\dagger b'(a'ab^b)^\dagger a^+ = abb'(a'ab^b)^\dagger a^+
\]

is self-adjoint element. So, \( ab(b')^\dagger b'(a'ab^b)^\dagger a^+ \) is self-adjoint too. In the same way, from the equality

\[
(b'(a'ab^b)^\dagger a^+ (b')^\dagger)^* = (b'(a'ab^b)^\dagger a^+ (b')^\dagger)^* = b'(a'ab^b)^\dagger a^+ ab = b'(a'ab^b)^\dagger a^+ ab,
\]

we conclude that \( b'(a'ab^b)^\dagger a^+ (b')^\dagger \) is self-adjoint. Hence, \( a(b')^\dagger \in \mathbb{R}^I \) and \([a(b')^\dagger]^* = b'(a'ab^b)^\dagger a^+ \).

(c1) \(\Rightarrow\) (a1): By (1) and (2), we have \( b'(a'ab^b)^\dagger a^+ \in (ab]\{1,2\}. Since \([a(b')^\dagger]^* = b'(a'ab^b)^\dagger a^+ \), then \( a(b')^\dagger b'(a'ab^b)^\dagger a^+, b'(a'ab^b)^\dagger a^+ (b')^\dagger \) are self-adjoint. Thus, from

\[
abb'(a'ab^b)^\dagger a^+ = a(b')^\dagger b'(a'ab^b)^\dagger a^+,
\]

\[
(b'(a'ab^b)^\dagger a^+ (b')^\dagger)^* = (b'(a'ab^b)^\dagger a^+ (b')^\dagger)^* = b'(a'ab^b)^\dagger a^+ (b')^\dagger,
\]

we deduce that the elements \( abb'(a'ab^b)^\dagger a^+, b'(a'ab^b)^\dagger a^+ ab \) are self-adjoint too. So, we get that \( ab \in \mathbb{R}^I \) and \((ab)^\dagger = b'(a'ab^b)^\dagger a^+, \) i.e. the condition (a1) is satisfied.

(a1) \(\Rightarrow\) (d1): The condition \( a'ab^b \in \mathbb{R}^I \), by Theorem 1.1, implies \( bb^*a^+ a = (a'ab^b)^* \in \mathbb{R}^I \). Now we prove that \( a(bb^*a^+ a)b \in (bb^*a^+ a)^I \):

\[
b^*a^+ a(bb^*a^+ a)b = b^*(bb^*a^+ a)^I bb^*a^+ a = b^*bb^*a^+ aa^+ = b^*a^+ ,
\]

\[
a(bb^*a^+ a)^I bb^*a^+ a(bb^*a^+ a)b = a(bb^*a^+ a)^I b.
\]

Further, by (a1) \(\Leftrightarrow\) (c1) and the equality

\[
(b^*a^+ a(bb^*a^+ a)^I b)^* = b^*[(bb^*a^+ a)^I a^+ a(b')]^* = b^*(a'ab^b)^\dagger a^+ (b')^\dagger,
\]

it follows that the element \( b^*a^+ a(bb^*a^+ a)b \) is self-adjoint. To conclude that \( a(bb^*a^+ a)^I bb^*a^+ a \) is self-adjoint, we consider the equivalence (a1) \(\Leftrightarrow\) (b1) and the equality

\[
(a(bb^*a^+ a)^I bb^*a^+ a)^I = (a^+)bb^*[a(bb^*a^+ a)]^I a^* = (a^+)bb^*[a(bb^*a^+ a)]^I a^*.
\]

Therefore, \( b^*a^+ \in \mathbb{R}^I \) and \((bb^*a^+ a)^I = a(bb^*a^+ a)^I b\).

(d1) \(\Rightarrow\) (a1): We observe that by (1) and (2), \( b'(a'ab^b)^\dagger a^+ \in (ab]\{1,2\}. If the hypothesis \((bb^*a^+ a)^I = a(bb^*a^+ a)^I b \) holds, the elements \( b^*a^+ a(bb^*a^+ a)b \) and \( a(bb^*a^+ a)^I bb^*a^+ a \) are self-adjoint. Then, from

\[
abb'(a'ab^b)^\dagger a^+ = (a^+)[(a'ab^b)^\dagger a^+] = (a^+)[(bb^*a^+ a)^I bb^*a^+ a] = (a^+)bb^*[a(bb^*a^+ a)]^I a^+ = a(bb^*a^+ a)^I bb^*a^+
\]

and

\[
b^*a^+ a(bb^*a^+ a)b = b^*a^+ a(bb^*a^+ a)b = b^*a^+ a(bb^*a^+ a)^I b,
\]
we have \( b^t(a^t abb^\dagger) a^t \in (ab)^{\dagger} \) and \((ab)^{\dagger} = b^t(a^t abb^\dagger) a^t.\)

(b1) \(\Rightarrow\) (b2): First we will prove that \( (a^t) b b^t = a^t (a^t) b b^t \in \mathcal{R}^t \) and \((a^t) b b^t \in \mathcal{R}^t \) and \((a^t) b b^t = (b^t)[(a^t) b] a^t.\) Indeed, the equalities

\[
(a^t) b b^t (b^t) [(a^t) b] a^t (a^t) b b^t = a^t [(a^t) b] (a^t) b b^t = a^t (a^t) b b^t
\]

and

\[
(b^t) [(a^t) b] a^t (a^t) b b^t = (b^t) [(a^t) b] b [(a^t) b] a = (b^t) [(a^t) b] b [(a^t) b] a \\
= (b^t) [(a^t) b] a
\]

imply that \((b^t) [(a^t) b] a \in (a^t (a^t) b b^t) [1,2].\) The assumption \([(a^t) b] = b^t (a^t abb^t) a^t\) gives

\[
(a^t) b b^t (b^t) [(a^t) b] a^t = a^t [(a^t) b] [a^t] b = (a^t) (a^t) b ([a^t] b) \]

\[
= (a^t) b b^t (a^t abb^t) a^t
\]

and

\[
(b^t) [(a^t) b] a^t (a^t) b b^t = (b^t) [(a^t) b] b [(a^t) b] = (b^t) [(a^t) b] b [(a^t) b] a \\
= (b^t) [(a^t) b] a
\]

Since \(a^t abb^t (a^t abb^t) a^t abb^t\) and \(a^t abb^t a^t abb^t\) are self-adjoint, it follows that \(a^t (a^t) b b^t (b^t) [(a^t) b] a^t\) and

\[
(b^t) [(a^t) b] a^t (a^t) b b^t\]

are self-adjoint too. Hence, we see that \([(a^t) b] = b^t [(a^t) b] [1,2].\)

Now we check that \([(a^t) b] a^t = b^t (b^t) [(a^t) b] b [a^t + b^t b [(a^t) b] b a^t = b^t [(a^t) b] b a^t.\)

\[
(a^t) b b^t (b^t) [(a^t) b] a^t = b^t [(a^t) b] [(a^t) b] a = b^t [(a^t) b] [a^t b = (a^t) b,
\]


\[
([a^t] b) (a^t) b b^t = [(a^t) b] (a^t) b
\]

\[
= (a^t) b [(a^t) b] a = (a^t) b b^t (a^t) b
\]

\[
(b^t) [(a^t) b] a^t (a^t) b = b^t [(a^t) b] a = b^t [(a^t) b] [a^t b = (a^t) b.
\]
Finally, by the equality $[(a^t)a]b^t = (b^t)\Gamma [(a^t)b]a^t$, we have

\[
[(a^t)b]^t = b'(b^t)^\Gamma [(a^t)b]^t a^t = b'[(a^t)b]^t a^t.
\]

Thus, the condition (b2) is satisfied.

(b2) ⇒ (b1): To prove $a^t b b^t + b [(a^t)b]^t a^t$, notice that

\[
a^t b b^t [(a^t)b]^t (a^t) a^t b b^t = a^t (a^t) b b^t = a^t b b^t,
\]

(5)

\[
b [(a^t)b]^t (a^t) a^t b b^t [(a^t)b]^t (a^t) = b [(a^t)b]^t (a^t) b [(a^t)b]^t (a^t) = b [(a^t)b]^t (a^t),
\]

(6)

i.e. $b [(a^t)b]^t (a^t) \in (a^t b b^t)(1, 2)$. Using the assumption $[(a^t)b]^t = b'[(a^t)b b^t]^t a^t$, we get

\[
a^t b b^t [(a^t)b]^t (a^t) = a^t (a^t) b [(a^t)b]^t (a^t) = a^t (a^t) b [(a^t)b]^t a^t
\]

\[
= (a^t a)^t b b^t [(a^t)^t b b^t]^t a^t
\]

\[
= a^t a (a^t)^t b b^t [(a^t)^t b b^t]^t = a^t (a^t)^t b b^t [(a^t)^t b b^t]^t
\]

\[
= (a^t a)^t b b^t [(a^t)^t b b^t]^t
\]

and

\[
b [(a^t)b]^t (a^t) a^t b b^t = b [(a^t)b]^t (a^t) b b^t = (b^t) [((a^t)b)^t (a^t)] b b^t
\]

\[
= (b^t) [((a^t)^t b b^t)^t (a^t)^t] b b^t = (b^t) [((a^t)^t b b^t)^t (a^t)^t] b b^t
\]

\[
= [(a^t)^t b b^t]^t (a^t)^t b b^t
\]

\[
= [(a^t)^t b b^t]^t (a^t)^t b b^t
\]

i.e. $a^t b b^t [(a^t)b]^t (a^t)$ and $b [(a^t)b]^t (a^t) a^t b b^t$ are self-adjoint elements. Consequently, $a^t b b^t + b [(a^t)b]^t a^t$. Then, we will show that $[(a^t)b]^t = b^t (a^t b b^t)^t a^t$. The equalities

\[
(a^t b b^t) a^t b b^t (a^t)^t b = (a^t)^t (a^t b b^t) (a^t)^t b = (a^t)^t a^t b b^t b = (a^t)^t b,
\]

\[
b [(a^t) b b^t] a^t b b^t (a^t)^t b = b [(a^t) b b^t] a^t b b^t b = b [(a^t) b b^t] a^t = b [(a^t) b b^t] a^t,
\]

yield $b [(a^t) b b^t] a^t \in [(a^t)^t b][1, 2]$. By $a^t b b^t = b [(a^t)^t b]^t (a^t)$, we get that

\[
(a^t)^t b b^t (a^t) b b^t a^t = (a^t)^t [b [(a^t)^t b]^t (a^t)^t a^t = (a^t)^t b [(a^t)^t b]^t
\]

\[
= (a^t)^t b [(a^t)^t b]^t + (a^t)^t b [(a^t)^t b]^t a^t = (a^t)^t b [(a^t)^t b]^t
\]

\[
= b [(a^t)^t b]^t (a^t) b b^t + b [(a^t)^t b]^t (a^t) b b^t = b [(a^t)^t b]^t (a^t) b b^t
\]

\[
= [(a^t)^t b]^t (a^t) b b^t = [(a^t)^t b]^t (a^t) b b^t,
\]
implying $b^i(a^iabb^i)^{+}a^i \in [(a^i)^{+}]b[3,4]$. Hence, the statement (b1) holds.

(c1) $\Rightarrow$ (c2): By definition, we check that $a^i(a^i) [a(b^i)^{+}] = a^i(a^i) [a(b^i)] = [a(a^i)][a(b^i)]^i(a^i)^{+}$. From

$$a^i(a^j)^{+} b^i [a(b^i)^{+}] (a^i)^{+} a^i(a^i)^{+} b^i = a^i([a(b^i)]^i(a^i)^{+})$$  \hspace{1cm} (7)

and

$$b[a(b^i)^{+}]^i(a^i)^{+} a^i(a^j)^{+} b^i [a(b^i)^{+}]^i(a^i)^{+} \hspace{0.5cm} = b[a(b^i)]^i(a^i)^{+} [a(b^i)]^i(a^i)^{+} \hspace{0.5cm} = b[a(b^i)]^i(a^i)^{+}$$ \hspace{1cm} (8)

we deduce that $b[a(b^i)]^i(a^i)^{+} \in (a^i(a^i)^{+})[1,2]$. The condition $[a(b^i)]^i = b^i[a^iabb^i]^i a^i$ gives

$$a^i(a^j)^{+} b^i [a(b^i)^{+}] (a^i)^{+} a^i(a^j)^{+} b^i = a^i([a(b^i)]^i(a^i)^{+}) \hspace{0.5cm} = (a^i(a^i)^{+}) b^i (a^i)^{+} a^i(a^i)^{+}$$

$$\hspace{0.5cm} = (a^i(a^i)^{+}) b^i (a^i)^{+} a^i(a^i)^{+} = (a^i abb^i)(a^i a)^{+} \hspace{0.5cm} = a^i(abb^i)(a^i a)^{+} \hspace{0.5cm}$$

and

$$b[a(b^i)]^i(a^i)^{+} a^i(a^j)^{+} b^i = b[a(b^i)]^i(a^i)^{+} b^i = ((b^i)^{+} [a(b^i)]^i(a^i)^{+})$$

$$\hspace{0.5cm} = ((b^i)^{+} b^i (a^i)^{+} a^i(a^i)^{+}) = (b^i)(a^i)^{+} a^i(a^i)^{+}$$

$$\hspace{0.5cm} = (a^i a^i)^{+} b^i b^i = (a^i)^{+} b^i b^i$$

Thus, $a^i(a^i)^{+} \in R^i$ and $[a^i(a^i)^{+}]^i = b[a(b^i)]^i(a^i)^{+}$. To obtain the equality $[a(b^i)]^i = b^i[a^iabb^i]^i a^i$ it is enough to prove that $[a(b^i)]^i = b^i[a(b^i)]^i(a^i)^{+} a^i = b^i b[a(b^i)]^i(a^i)^{+} a^i$. Since

$$a(b^i)^{+} b^i [a(b^i)]^i(a^i)^{+} = a(b^i)^{+} [a(b^i)]^i(a^i)^{+} = a(b^i)^{+}$$

$$\hspace{0.5cm} = b^i b[a(b^i)]^i(a^i)^{+} a^i = b^i b[a(b^i)]^i(a^i)^{+} a^i = b^i b[i(a^i)]^i$$

$$(a^i)^{+} [a(b^i)]^i = [a(b^i)]^i(a^i)^{+} = aa^i a(b^i) [a(b^i)]^i$$

$$\hspace{0.5cm} = a(b^i)^{+} [a(b^i)]^i$$

is self–adjoint,

$$(b^i)^{+} [a(b^i)]^i = (b^i)^{+} [a(b^i)]^i = a(a^i) b^i (b^i)^{+}$$

$$\hspace{0.5cm} = [a(b^i)]^i(a^i)^{+}$$

then $[a(b^i)]^i = b^i [a(b^i)]^i a^i = b^i [a^i a(a^i)]^i a^i$ and (c2) is satisfied.

(c2) $\Rightarrow$ (c1): First we will prove that $a^i abb^i \in R^i$ and $(a^i abb^i)^{+} = (b^i)[a(b^i)]^i a$. The equalities

$$a^i abb^i (b^i)^{+} [a(b^i)]^i a^i abb^i = a^i (a^i) [a(b^i)]^i a^i abb^i = a^i (a^i) b^i (b^i)^{+} = a^i abb^i,$$

$$(b^i)^{+} [a(b^i)]^i a^i abb^i (b^i)^{+} [a(b^i)]^i a^i abb^i (b^i)^{+} [a(b^i)]^i a^i abb^i (b^i)^{+} [a(b^i)]^i a^i abb^i (b^i)^{+} [a(b^i)]^i a^i abb^i (b^i)^{+} [a(b^i)]^i a^i abb^i (b^i)^{+} [a(b^i)]^i a^i abb^i (b^i)^{+} [a(b^i)]^i a^i$$
imply that $(b^+)^*[a(b^+)][a(a^+)]^*a \in (a^+a^+)1,2]$. Using the hypothesis $[a(b^+)]^* = b^*[a^+(a^+)^+]a^+$, we get that
\[
a^+a^+aa^+a(b^+)^*[a(b^+)][a(a^+)]^*a = (a^+(a^+)^+)\{a^+(a^+)^+\} (a^+) \vdash
\]
\[
= (a^+(a^+)^+)b^*[a^+(a^+)^+]a^+(a^+) \vdash
\]
\[
= (a^+(a^+)^+)a^+(a^+)^+]a^+(a^+) \vdash
\]
\[
= a^+a^+aa^+a(b^+)^*[a^+(a^+)^+] \vdash
\]
\[
= a^+(a^+)^+]a^+(a^+)^+] \text{ is self–adjoint}
\]
and
\[
(b^+)^*[a(b^+)][a(a^+)]^*aa^+a(b^+)^* = (b^+)^*[a(b^+)][a(a^+)]^*aa^+a(b^+)^* = (b^+)^*[a(b^+)][a(a^+)]^*aa^+a(b^+)^*
\]
\[
= (b^+)^*[a(b^+)][a(a^+)]^*aa^+a(b^+)^* = (b^+)^*[a(b^+)][a(a^+)]^*aa^+a(b^+)^*
\]
\[
= (b^+)^*[a(b^+)][a(a^+)]^*aa^+a(b^+)^* = (b^+)^*[a(b^+)][a(a^+)]^*aa^+a(b^+)^*
\]
\[
= a^+(a^+)^+]a^+(a^+)^+] \text{ is self–adjoint}
\]

Hence, we conclude that $(a^+a^+) \vdash (b^+)^*[a(b^+)][a(a^+)]^*a$. Now in order to show that the equality $[a(b^+)]^* = b^\dagger(a^+a^+)a^\dagger$ holds, we prove that $[a(b^+)]^* = b^\dagger(a^+a^+)a^\dagger = b^\dagger a^\dagger a^\dagger$. Indeed, by definition and
\[
a(b^+)b^\dagger[\{a^+(a^+)^+]a^\dagger = (b^+)b^\dagger[\{a^+(a^+)^+]a^\dagger = a^\dagger a^\dagger = a^\dagger a^\dagger = a^\dagger a^\dagger = a^\dagger a^\dagger
\]
\[
= (b^+)b^\dagger[\{a^+(a^+)^+]a^\dagger = (b^+)b^\dagger[\{a^+(a^+)^+]a^\dagger = a^\dagger a^\dagger = a^\dagger a^\dagger = a^\dagger a^\dagger
\]
\[
= (b^+)b^\dagger[\{a^+(a^+)^+]a^\dagger = (b^+)b^\dagger[\{a^+(a^+)^+]a^\dagger = a^\dagger a^\dagger = a^\dagger a^\dagger = a^\dagger a^\dagger
\]
we have $[a(b^+)]^* = b^\dagger[\{a^+(a^+)^+]a^\dagger = b^\dagger(a^+a^+)a^\dagger$. So, the condition (c1) is satisfied.
\[(d1) \Rightarrow (d2): \text{Let us check that } (b^+)b^\dagger(a^+) = (b^+)b^\dagger(a^+) \in R^2 \text{ and } [(b^+)b^\dagger(a^+)][a^+] = a^\dagger(a^+)b^\dagger. \text{ By}
\]
\[
(b^+)b^\dagger(a^+)a^\dagger(a^+)b^\dagger(a^+)b^\dagger(a^+) = (b^+)b^\dagger(a^+)a^\dagger(a^+)b^\dagger(a^+)
\]
\[
= (b^+)b^\dagger(a^+)b^\dagger(a^+)
\]
\[
= a^\dagger(a^+)b^\dagger(a^+)b^\dagger
\]
\[
= a^\dagger(a^+)b^\dagger(a^+)
\]
\[
\text{obviously, } a^\dagger(a^+)b^\dagger(a^+) \in [(b^+)b^\dagger(a^+)][1,2]. \text{ Further, from the condition } (b^+)a^\dagger = a(bb^\dagger a^+)^\dagger b, \text{ we get}
\]
\[
(b^+)b^\dagger(a^+)a^\dagger(b^+)a^\dagger = (b^+)b^\dagger(a^+)a^\dagger(b^+)a^\dagger = (b^+)b^\dagger(a^+)a^\dagger(b^+)a^\dagger
\]
\[
= (b^+)b^\dagger(a^+)a^\dagger(b^+)a^\dagger = (b^+)b^\dagger(a^+)a^\dagger(b^+)a^\dagger
\]
\[
= (b^+)b^\dagger(a^+)a^\dagger(b^+)a^\dagger = (b^+)b^\dagger(a^+)a^\dagger(b^+)a^\dagger
\]
and
\[
a^\dagger(a^+)b^\dagger(a^+)a^\dagger = a^\dagger(a^+)b^\dagger(a^+)a^\dagger = a^\dagger(a^+)b^\dagger(a^+)a^\dagger
\]
\[
= (a^\dagger(a^+)b^\dagger(a^+)a^\dagger = (a^\dagger(a^+)b^\dagger(a^+)a^\dagger
\]
\[
= (a^\dagger(a^+)b^\dagger(a^+)a^\dagger = (a^\dagger(a^+)b^\dagger(a^+)a^\dagger
\]
\[
= (b^+)b^\dagger(a^+)a^\dagger(b^+)a^\dagger = (b^+)b^\dagger(a^+)a^\dagger(b^+)a^\dagger
\]
i.e. \(a'(bb^a)\) \(b' \in \{b(b_b)^\dagger)^a\} \{3,4\}\). Thus, \([b(b_b)^\dagger)^a\] = \(a'(bb^a)^\dagger\). Then, a direct computation shows that \(b(b_a)^\dagger\) = \((a')a'(bb^a)^\dagger b(b_b)^\dagger = aa'(bb^a)^\dagger b(b_b)^\dagger b\):

\[
b^\dagger b^a = \text{is self–adjoint},
\]

Therefore, \((bb^a)^\dagger = (a')a'(bb^a)^\dagger b(b_b)^\dagger = (a')[(b(b_b)^\dagger)^a] \{b(b_b)^\dagger\}, \text{by } \{b(b_b)^\dagger)^a\} \dagger = (b(b_b)^\dagger)^a\).

\((d2) \Rightarrow (d1): \text{To prove } b(b_b)^a \in \mathbb{R}^d \text{ and } (bb^a)^\dagger = (b(b_b)^\dagger)^\dagger b_b^a, \text{first we have } a'(bb^a)^\dagger b_b^a \in (bb^a)^\dagger b_b^a[1,2],\text{ from}
\]

We use the hypothesis \((b(b_b)^\dagger)^\dagger = (a')[(b(b_b)^\dagger)^a] \{b(b_b)^\dagger\} \text{ to obtain that } a'(b(b_b)^\dagger)^\dagger b_b^a \in (bb^a)^\dagger b_b^a[3,4] \text{ in the following way:}
\]

So, \((b(b_b)^\dagger)^\dagger = a'(b(b_b)^\dagger)^\dagger b_b^a \text{ and then to obtain } (b(b_a)^\dagger = a(bb^a)^\dagger b \text{ it is enough to check that } (b(b_a)^\dagger = aa'(b(b_a)^\dagger)^\dagger b_b^a:
\]

\[
b^\dagger b^a = \text{is self–adjoint},
\]

\(b^\dagger a = \text{is self–adjoint}.
\]
Thus, the condition (d1) is satisfied.

(a1) ⇒ (e1): This implication follows from Theorem 1.5.

(e1) ⇒ (a1): We will verify that $a^t abb^t \in \mathcal{R}^+$ and $(a^t abb^t)^t = b(a^t ab)^t$. Obviously,

\begin{align*}
    a^t abb^t b(a^t ab)^t a^t abb^t &= (a^t ab)(a^t ab)^t a^t ab) b^t = a^t abb^t, \\
    b(a^t ab)^t a^t abb^t b(a^t ab)^t &= b(a^t ab)^t a^t ab(a^t ab)^t = b(a^t ab)^t, \\
    a^t abb^t b(a^t ab)^t &= a^t ab(a^t ab)^t \text{ is self-adjoint.} (13)
\end{align*}

From $(a^t ab)^t a^t = b^t (abb^t)^t$, we have

\[ b(a^t ab)^t a^t abb^t = b b^t (abb^t)^t abb^t = ((abb^t)^t abb^t)^t = (abb^t)^t abb^t, \]

which implies that element $b(a^t ab)^t a^t abb^t$ is self-adjoint. Thus, the conditions $a^t abb^t \in \mathcal{R}^+$ and $(a^t abb^t)^t = b(a^t ab)^t$ hold. By this equality and (e1), we obtain

\[ b^t (a^t abb^t)^t a^t = b^t b(a^t ab)^t a^t = b^t bb^t (abb^t)^t = b^t (abb^t)^t. \] (14)

From

\[ \begin{align*}
    abb^t (abb^t)^t ab &= (abb^t (abb^t)^t abb^t)b = abb^t b = ab, \\
    b^t (abb^t)^t abb^t (abb^t)^t &= b^t (abb^t)^t,
\end{align*} \]

we conclude that $b^t (abb^t)^t \in (ab)[1, 2]$. Next, $abb^t (abb^t)^t$ is self-adjoint and, by (e1), $b^t (abb^t)^t ab = (a^t ab)^t a^t ab$ is self-adjoint too. Consequently, $ab \in \mathcal{R}^+$ and $(ab)^t = b^t (abb^t)^t$. Then, by (14), we observe that $(ab)^t = b^t (a^t abb^t)^t a^t$. Hence, the statement (a1) is satisfied. Notice that from (e1) follows $(ab)^t = b^t (abb^t)^t = (a^t ab)^t a^t$.

(b1) ⇒ (e2): Let us remark that $b^t (a^t abb^t)^t \in (a^t ab)[1, 2, 3]$ follows from

\[ a^t abb^t (a^t abb^t)^t a^t ab = (a^t ab b (a^t abb^t)^t a^t abb^t)b = a^t abb^t b = a^t ab, \]

\[ b^t (a^t abb^t)^t a^t abb^t (a^t abb^t)^t = b^t (a^t abb^t)^t, \]

\[ a^t abb^t (a^t abb^t)^t \text{ is self-adjoint.} \]

Similarly, $(a^t abb^t)^t a^* \in [(a^t)^t bb^t][1, 2, 4]$ follows from

\[ (a^t)^t bb^t (a^t abb^t)^t a^* (a^t)^t bb^t = (a^t)^t (a^t ab b (a^t abb^t)^t a^t abb^t) = (a^t)^t a^t abb^t = (a^t)^t bb^t, \]

\[ (a^t abb^t)^t a^* (a^t)^t bb^t (a^t abb^t)^t a^* = (a^t ab b (a^t abb^t)^t a^t abb^t)^t a^* = (a^t abb^t)^t a^*, \]

\[ (a^t abb^t)^t a^* (a^t)^t bb^t = (a^t abb^t)^t a^* \text{ is self-adjoint.} \]

The assumption $[(a^t)^t b]^t = b^t (a^t abb^t)^t a^*$ gives that

\[ b^t (a^t abb^t)^t a^* ab = b^t (a^t abb^t)^t a^* b \text{ is self-adjoint,} \]
\((a^\dagger)^* b b^\dagger (a^\dagger a b b^\dagger)^* a\) is self – adjoint, i.e. \(b^\dagger (a^\dagger a b b^\dagger)^* a \in (a^\dagger a b b^\dagger)[4]\) and \((a^\dagger a b b^\dagger)^* a \in [(a^\dagger)^* b b^\dagger][3]\). Therefore, \(a^\dagger a b, (a^\dagger)^* b b^\dagger \in \mathcal{R}^1, (a^\dagger a b)^* = b^\dagger (a^\dagger a b b^\dagger)^*\) and \([(a^\dagger)^* b b^\dagger)^* = (a^\dagger a b b^\dagger)^* a\). Now, \((a^\dagger a b)^* a^\dagger = b^\dagger (a^\dagger a b b^\dagger)^* a^\dagger = [(a^\dagger)^* b] = b^\dagger [(a^\dagger)^* b b^\dagger]^*\), i.e. the condition (e2) is satisfied.

(e2) ⇒ (b1): Notice that, by (11), (12) and (13), we have \(b(a^\dagger a b)^* \in (a^\dagger a b b^\dagger)[1,2,3]\). The condition \((a^\dagger a b)^* a^\dagger = b^\dagger [(a^\dagger)^* b b^\dagger]^*\) implies

\[
\begin{align*}
    b(a^\dagger a b)^* a^\dagger abb^\dagger &= b(a^\dagger a b)^* a^\dagger b b^\dagger[(a^\dagger)^* b b^\dagger]^* (a^\dagger)^* b b^\dagger \\
    &= [(a^\dagger)^* b b^\dagger]^* (a^\dagger)^* b b^\dagger)^* (a^\dagger)^* b b^\dagger \\
    &= [(a^\dagger)^* b b^\dagger]^* (a^\dagger)^* b b^\dagger)^* \text{ is self – adjoint.}
\end{align*}
\]

So, \(a^\dagger a b b^\dagger \in \mathcal{R}^1\) and \((a^\dagger a b)^* = b(a^\dagger a b)^*\). Then

\[
\begin{align*}
    b^\dagger (a^\dagger a b b^\dagger)^* a^\dagger &= b^\dagger b(a^\dagger a b)^* a^\dagger = b^\dagger b(b^\dagger[(a^\dagger)^* b b^\dagger]^* (a^\dagger)^* b b^\dagger) \\
    &= b^\dagger [(a^\dagger)^* b b^\dagger]^* (a^\dagger)^* b b^\dagger)^* = b^\dagger [(a^\dagger)^* b b^\dagger]^*. \\
    (a^\dagger)^* b b^\dagger[(a^\dagger)^* b b^\dagger]^* \text{ is self – adjoint,}
\end{align*}
\]

we see that \(b^\dagger [(a^\dagger)^* b b^\dagger]^* \in [(a^\dagger)^* b][1,2,3]\). Using (e2), we have

\[
\begin{align*}
    b^\dagger [(a^\dagger)^* b b^\dagger]^* (a^\dagger)^* b &= (a^\dagger a b)^* a^\dagger (a^\dagger)^* b = (a^\dagger a b)^* a^\dagger a^\dagger a b,
\end{align*}
\]

i.e. \(b^\dagger [(a^\dagger)^* b b^\dagger]^* \in [(a^\dagger)^* b][4]\). Thus, \((a^\dagger)^* b \in \mathcal{R}^1, [(a^\dagger)^* b]^* = b^\dagger [(a^\dagger)^* b b^\dagger]^*\) and, by (15), \([(a^\dagger)^* b]^* = b^\dagger (a^\dagger a b b^\dagger)^* a^\dagger\).

(c1) ⇒ (c3): By elementary computations, we obtain

\[
\begin{align*}
    a^\dagger a(b^\dagger) b^\dagger (a^\dagger a b b^\dagger)^* a^\dagger a(b^\dagger) &= b^\dagger (a^\dagger a b b^\dagger)^* (a^\dagger a b b^\dagger)^* (a^\dagger a b b^\dagger) (b^\dagger) = (a^\dagger a b b^\dagger)^* (a^\dagger a b b^\dagger)^* (b^\dagger) = a^\dagger a(b^\dagger)^*, \\
    b^\dagger (a^\dagger a b b^\dagger)^* (a^\dagger a b b^\dagger)^* = b^\dagger (a^\dagger a b b^\dagger)^* (a^\dagger a b b^\dagger)^* = b^\dagger (a^\dagger a b b^\dagger)^*, \\
    a^\dagger a(b^\dagger) b^\dagger (a^\dagger a b b^\dagger)^* &= a^\dagger a(b^\dagger) b^\dagger (a^\dagger a b b^\dagger)^* \text{ is self – adjoint,}
\end{align*}
\]

that is \(b^\dagger (a^\dagger a b b^\dagger)^* \in [a^\dagger a(b^\dagger)^*][1,2,3]\). We easy check that \((a^\dagger a b b^\dagger)^* a^\dagger \in (a b b^\dagger)[1,2,4]\):

\[
\begin{align*}
    abb^\dagger (a^\dagger a b b^\dagger)^* a^\dagger abb^\dagger &= a(a^\dagger a b b^\dagger)^* a^\dagger abb^\dagger = a a^\dagger abb^\dagger = abb^\dagger, \\
    (a^\dagger a b b^\dagger)^* a^\dagger a^\dagger abb^\dagger &= (a^\dagger a b b^\dagger)^* a^\dagger = (a^\dagger a b b^\dagger)^* a^\dagger, \\
    (a^\dagger a b b^\dagger)^* a^\dagger a^\dagger abb^\dagger = (a^\dagger a b b^\dagger)^* a^\dagger = (a^\dagger a b b^\dagger)^* a^\dagger, \\
    (a^\dagger a b b^\dagger)^* a^\dagger abb^\dagger = a^\dagger abb^\dagger \text{ is self – adjoint.}
\end{align*}
\]

The hypothesis \([a(b^\dagger)]^* = b^\dagger (a^\dagger a b b^\dagger)^* a^\dagger\) implies

\[
\begin{align*}
    b^\dagger (a^\dagger a b b^\dagger)^* a^\dagger a(b^\dagger)^* \text{ is self – adjoint}
\end{align*}
\]
and
\[ abb^i(a^i abb^i)^a = a(b^i)^j b^i(a^i abb^i)^a \]
is self-adjoint.

Consequently, the statements \( a^i a(b^j)^i \), \( abb^i \in R^j \), \( [a^i a(b^j)^i]^j = b^i(a^i abb^i)^j \) and \( (abb^i)^j = (a^i abb^i)^j a \) hold.

Finale, we get the equality in (e3), from \( [a^i a(b^j)^i]^j a = b^i(a^i abb^i)^j a = (a^i abb^i)^j a \).

(e3) \( \Rightarrow \) (c1): First, we verify that \( a^i abb^i \in R^j \) and \( (a^i abb^i)^j = (a^i abb^i)^j a \). Indeed,
\[
\begin{align*}
a^i abb^i(a^i abb^i)^a & = a^i (a^i abb^i)^j abb^i = a^i abb^i, \quad (16) \\
(abb^i)^j a & = (abb^i)^j a (abb^i)^a = (abb^i)^j a, \quad (17) \\
(abb^i)^j a & = (a^i abb^i)^j a = (a^i abb^i)^j a \quad \text{is self-adjoint.} \quad (18)
\end{align*}
\]

By the assumption \( [a^i a(b^j)^i]^j a = b^i(a^i abb^i)^j \), we have
\[
\begin{align*}
a^i abb^i(a^i abb^i)^a & = a^i a(b^i)^j b^i(a^i abb^i)^j a = a^i a(b^i)^j [a^i a(b^i)^j]^j a^j \\
& = (a^j a a(b^j)^j [a^j a(b^j)^j] a^j)^j = (a^j a a(b^j)^j [a^j a(b^j)^j] a^j)^j \\
& = a^i a(b^i)^j [a^i a(b^i)^j] a^j \quad \text{is self-adjoint.} \quad (19)
\end{align*}
\]

Hence, by (16)-(19), \( a^i abb^i \in R^j \) and \( (a^i abb^i)^j = (a^i abb^i)^j a \). Further, we obtain \( [a^i a(b^j)^i]^j a \in [a(b^j)^i][1, 2] \) as a simple consequence of the equalities
\[
\begin{align*}
a(b^j)^j [a^i a(b^j)^i]^j a = a(a^i a(b^j)^i [a^i a(b^j)^i]^j a^j) = a^j a b^j = a(b^j)^j,
\end{align*}
\]
\[
\begin{align*}
[a^i a(b^j)^i]^j [a^i a(b^j)^i]^j a^j = \left[a^i a(b^j)^i\right]^j a^j.
\end{align*}
\]

From (e3), we get
\[
\begin{align*}
a(b^j)^j [a^i a(b^j)^i]^j a = a(b^j)^j b^i(a^j abb^i)^j = abb^i(a^i abb^i)^j
\end{align*}
\]
which implies \( [a^i a(b^j)^i]^j a \in [a(b^j)^i][3] \). Obviously, \( [a^i a(b^j)^i]^j a = a(b^j)^j \) is self-adjoint and therefore, \( a(b^j)^j \in R^j \) and \( [a(b^j)^i]^j a \). Now, by \( (a^i abb^i)^j = (a^i abb^i)^j a \) and (e3),
\[
\begin{align*}
b^i(a^i abb^i)^j a = b^i(a^i abb^i)^j a = [a^i a(b^j)^i]^j a^j a = [a^i a(b^j)^i]^j a^j = [a(b^j)^i]^j.
\end{align*}
\]

(d1) \( \Rightarrow \) (e4): Since
\[
\begin{align*}
bb^j a^i a(bb^j a) a bb^j a^j = (bb^j a^i a(bb^j a) a bb^j a^i a) a^j = bb^j a^i a a^j = bb^j a^i,
\end{align*}
\]
\[
\begin{align*}
a(bb^j a^i a) bb^j a^i a(bb^j a) a^j = a(bb^j a) a^j,
\end{align*}
\]
and \( bb^j a^i a(bb^j a) a^j \) is self-adjoint, we have that \( a(bb^j a) a^j \in (bb^j a^j)[1, 2, 3] \). The statement \( (bb^j a) a^j b \in (bb^j a^j)[1, 2, 4] \) holds because
\[
\begin{align*}
b^i a^i a(bb^j a) a bb^j a^j = b^i (bb^j a^i a(bb^j a) a bb^j a^i a) a^{(i)} = b^i bb^j a^i a = b^i a^i a,
\end{align*}
\]
\[
\begin{align*}
(bb^j a^i a) bb^j a^i a(bb^j a) a^j b = (bb^j a) a^j b,
\end{align*}
\]
and the element \((bb^t a^t)bb^t a^t\) is self-adjoint. From \((b^t a^t)^t = a(bb^t a^t)b\), we conclude that the elements \(a(bb^t a^t)bb^t a^t\), \(b^t a^t a(bb^t a^t)bb^t a^t\) are self-adjoint. Hence, \(bb^t a^t, b^t a^t a \in \mathcal{R}^t\), \((bb^t a^t)^t = a(bb^t a^t)b\) and \((b^t a^t)^t = (bb^t a^t)b\). Then, we get \((bb^t a^t)^t b = a(bb^t a^t)b(= (b^t a^t)^t) = a(b^t a^t)^t\).

(e4) \(\Rightarrow\) (d1): Because

\[
bb^t a^t aa^t (bb^t a^t)^t bb^t a^t = (bb^t a^t(bb^t a^t)^t bb^t a^t)a = bb^t a^t a,
\]

then

\[
a^t(bb^t a^t)^t bb^t a^t aa^t (bb^t a^t)^t = a^t(bb^t a^t)^t bb^t a^t(bb^t a^t)^t = a(a(bb^t a^t)^t,\]

and

\[
bb^t a^t aa^t (bb^t a^t)^t = bb^t a^t(bb^t a^t)^t is self-adjoint,
\]

we deduce that \(a^t(bb^t a^t)^t \in (bb^t a^t)[1, 2, 3]\). The condition \((bb^t a^t)^t b = a(b^t a^t)a\) gives

\[
(a^t(bb^t a^t)^t bb^t a^t)^t = (a^t(a(bb^t a^t)^t b^t a^t)^t b^t a^t a)^t = (b^t a^t)^t b^t a^t a = bb^t a^t a
\]

Thus, \(bb^t a^t a \in \mathcal{R}^t\) and \((bb^t a^t)^t = a(b^t a^t)^t\). By this equality and (e4), we have

\[
a(bb^t a^t)^t b = aa^t(bb^t a^t)^t b = aa^t a(b^t a^t)^t = a(b^t a^t)^t.
\]

So, to obtain \((b^t a^t)^t = a(bb^t a^t)^t b\) it is enough to prove that \((b^t a^t)^t = a(b^t a^t)^t\). We can easily check that \(a(b^t a^t)^t \in (b^t a^t)[1, 2, 3]\):

\[
b^t a^t a(bb^t a^t)bb^t a^t = (b^t a^t a(bb^t a^t)bb^t a^t)bb^t a^t = b^t a^t a bb^t a^t = b^t a^t,
\]

\[
a(bb^t a^t)^t b^t a^t a(bb^t a^t)bb^t a^t = a(bb^t a^t)^t b^t a^t a = b^t a^t a,\]

\[
b^t a^t a(bb^t a^t)^t is self-adjoint
\]

and, by (e4), the element \(a(bb^t a^t)^t bb^t a^t = (bb^t a^t)^t bb^t a^t\) is self-adjoint. Therefore, \(b^t a^t \in \mathcal{R}^t\) and \((b^t a^t)^t = a(bb^t a^t)^t b\), i.e. the condition (d1) holds.

(a2) \(\Rightarrow\) (e5): The elementary computations show that \(b^t(a'abb')^t \in (a'ab)[1, 2, 3]\) and \((a'abb')^t a^t \in (abb')[1, 2, 4]\) follow from

\[
a'abb'(a'abb')^t a'ab = (a'abb'(a'abb')^t a'abb')(b^t)^t = a'abb' = a'ab,
\]

\[
b^t(a'abb')^t a'abb'(a'abb')^t = b^t(a'abb')^t,\]

\[
a'abb'(a'abb')^t is self-adjoint
\]

and

\[
abb'(a'abb')^t a'abb' = (a^t)^t (a'abb'(a'abb')^t a'abb') = (a^t)^t a'abb' = abb',\]

\[
(a'abb')^t a'abb'(a'abb')^t a^t = (a'abb')^t a^t,
\]

\[
(a'abb')^t a'abb' is self-adjoint.
By the hypothesis \((ab)^\dagger = b'(a'abb')^\dagger a'\), we observe that the elements \(b'(a'abb')^\dagger a'ab\), \(a'abb'(a'abb')^\dagger a'\) are self-adjoint, i.e. \(b'(a'abb')^\dagger \in (a'ab)[4]\) and \((a'abb')^\dagger a' \in (a'bb')[3]\). Hence, \(a'ab, a'bb \in R^\dagger\), \((a'ab)^\dagger = b'(a'abb')^\dagger\) and \((a'bb)^\dagger = (a'abb')^\dagger a'\). Then \((a'ab)^\dagger a' = b'(a'abb')^\dagger a' (= (ab)^\dagger) = b'(ab)^\dagger\).

\((e5) \Rightarrow (a2):\) In order to prove that \(a'abb' \in R^\dagger\), we get first that \((b^\dagger)'(a'ab)^\dagger \in (a'abb')[1, 2]\), by

\[ a'abb'(b^\dagger)'(a'ab)^\dagger a'ab' = (a'ab(a'ab)^\dagger a'ab)b' = a'abb', \]

\[ (b^\dagger)'(a'ab)^\dagger a'ab'(b^\dagger)'(a'ab)^\dagger = (b^\dagger)'(a'ab)^\dagger a'ab(a'ab)^\dagger = (b^\dagger)'(a'ab)^\dagger. \]

The equality \(a'abb'(b^\dagger)'(a'ab)^\dagger = a'ab(ab)^\dagger\) implies that \((b^\dagger)'(a'ab)^\dagger \in (a'abb')[3]\). From the condition \((a'ab)^\dagger a' = b'(ab)^\dagger\), it follows

\[ ((b^\dagger)'(a'ab)^\dagger a'abb')' = ((b^\dagger)'b'(ab)^\dagger a'abb')' = (ab)^\dagger a'ab'bb^\dagger = (ab)^\dagger abb'\]

implying that \((b^\dagger)'(a'ab)^\dagger \in (a'abb')[4]\). Therefore, we have \(a'abb' \in R^\dagger\) and \((a'abb)^\dagger = (b^\dagger)'(a'ab)^\dagger\). This equality and \((e5)\) give

\[ b'(a'abb')^\dagger a' = b'(b^\dagger)'(a'ab)^\dagger a' = b'(b^\dagger)'b'(a'ab)^\dagger = b'(ab)^\dagger. \]

To complete the proof we will show that \((ab)^\dagger = b'(ab)^\dagger\). Notice that, by

\[ ab'bb'(a'ab)^\dagger ab = (ab'bb'(a'ab)^\dagger ab)' = ab'bb'(b^\dagger)' = ab, \]

\[ b'(ab)^\dagger ab'bb'(a'ab)^\dagger = b'(ab)^\dagger, \]

we get \(b'(ab)^\dagger \in (ab)[1, 2]\). Since \(ab'bb'(a'ab)^\dagger\) is self-adjoint, and, by \((e5)\),

\[ b'(ab)^\dagger ab = (a'ab)^\dagger a'ab \]

is self-adjoint too, we obtain that \(ab \in R^\dagger\) and \((ab)^\dagger = b'(ab)^\dagger\). Then, from \((20)\), \((ab)^\dagger = b'(a'abb')^\dagger a' (= b'(ab)^\dagger = (a'ab)^\dagger a')\).

\((b2) \Rightarrow (e6):\) To show that \((a'a)^\dagger b, (a'a)^\dagger bb' \in R^\dagger\), let us remark that from

\[ (a'a)^\dagger bb'[(a'a)^\dagger bb']^\dagger (a'a)^\dagger b = ((a'a)^\dagger bb'[(a'a)^\dagger bb']^\dagger (a'a)^\dagger bb')(b^\dagger)' = (a'a)^\dagger bb'(b^\dagger)' = (a'a)^\dagger b, \]

\[ b'[((a'a)^\dagger bb']^\dagger (a'a)^\dagger bb']^\dagger = b'[(a'a)^\dagger bb']^\dagger, \]

and

\[ (a'a)^\dagger bb'[(a'a)^\dagger bb']^\dagger a^\dagger bb' = a(a'a)^\dagger bb'[(a'a)^\dagger bb']^\dagger (a'a)^\dagger bb' = a(a'a)^\dagger bb' = (a'a)^\dagger bb', \]

\[ [(a'a)^\dagger bb']^\dagger a^\dagger [(a'a)^\dagger bb']^\dagger a^\dagger = [(a'a)^\dagger bb']^\dagger (a'a)^\dagger bb'[(a'a)^\dagger bb']^\dagger a^\dagger = [(a'a)^\dagger bb']^\dagger a^\dagger, \]

we get \(b'[(a'a)^\dagger bb']^\dagger \in [(a'a)^\dagger b][1, 2]\) and \([(a'a)^\dagger bb']^\dagger a^\dagger \in [(a'a)^\dagger bb'][1, 2]\). Obviously, the elements \((a'a)^\dagger bb'[(a'a)^\dagger bb']^\dagger\) and \([(a'a)^\dagger bb']^\dagger a^\dagger bb' = [(a'a)^\dagger bb']^\dagger (a'a)^\dagger bb'\) are self-adjoint. From the hypothesis \((a'a)^\dagger b^\dagger = b'[(a'a)^\dagger bb']^\dagger a^\dagger b\) we have that \(b'[[(a'a)^\dagger bb']^\dagger (a'a)^\dagger b = b'[(a'a)^\dagger bb']^\dagger a^\dagger b\) and \((a'a)^\dagger bb'[(a'a)^\dagger bb']^\dagger a^\dagger b\)
are self-adjoint elements. Thus, \((a^\dagger a)b, (a^\dagger a)^*bb\) \(\in \mathcal{R}^2\), \([(a^\dagger a)b]^\dagger = b'[(a^\dagger a)^*bb]^\dagger\) and \([(a^\dagger a)b]^\dagger = [(a^\dagger a)^*bb]^\dagger a^\dagger\). Now we deduce that \([(a^\dagger a)b]^\dagger a^\dagger = b'[a^\dagger a)^*bb]^\dagger a^\dagger = [(a^\dagger a)^*bb]^\dagger a^\dagger\).

\((e6) \Rightarrow (b2)\): To prove the condition \(a^\dagger b^* \in \mathcal{R}^2\) we observe that \((b^\dagger)'[(a^\dagger a)b]^\dagger \in [(a^\dagger a)^*bb']^1[1, 2, 3] \) by

\[
(a^\dagger a)^*bb'(b'^\dagger)[(a^\dagger a)^*bb'] = (a^\dagger a)^*b[(a^\dagger a)^*bb'] = (a^\dagger a)^*bb',
\]

\[
(b^\dagger)'[(a^\dagger a)^*bb'][(a^\dagger a)^*bb'] = (b^\dagger)'[(a^\dagger a)^*bb'][(a^\dagger a)^*bb'] = (b^\dagger)'[(a^\dagger a)^*bb'],
\]

\[
(a^\dagger a)^*bb'(b'^\dagger)[(a^\dagger a)^*bb'] = (a^\dagger a)^*b[(a^\dagger a)^*bb'] \quad \text{is self-adjoint.}
\]

Using the equality \([(a^\dagger a)b]^\dagger a^\dagger = b'[a^\dagger a)^*bb']^\dagger\), we obtain

\[
((b^\dagger)'[(a^\dagger a)^*bb']^\dagger(a^\dagger a)^*bb')^\dagger = (b^\dagger)'[(a^\dagger a)^*bb']^\dagger(a^\dagger a)^*bb')^\dagger = (a^\dagger a)^*bb'(b'^\dagger)[(a^\dagger a)^*bb']^\dagger
\]

that is \((b^\dagger)'[(a^\dagger a)b]^\dagger \in [(a^\dagger a)^*bb']^1[4]\). So, we get \((a^\dagger a)^*bb^* \in \mathcal{R}^4\) and \([(a^\dagger a)^*bb']^1 = b'[a^\dagger a)^*bb']^\dagger\). By this equality and \((e6)\),

\[
b'[a^\dagger a)^*bb']^\dagger a^\dagger = b'[a^\dagger a)^*bb']^\dagger a^\dagger = b'[a^\dagger a)^*bb']^\dagger a^\dagger = b'[a^\dagger a)^*bb']^\dagger a^\dagger = b'[a^\dagger a)^*bb']^\dagger a^\dagger.
\]

If we show that \((a^\dagger a)^*bb']^1 \in [a^\dagger a)^*bb']^1[1, 2, 3]\), by

\[
(a^\dagger a)^*bb'][a^\dagger a)^*bb']^1 = [(a^\dagger a)^*bb']^1(a^\dagger a)^*bb'](b^\dagger)'a^\dagger = (a^\dagger a)^*bb'](b^\dagger)'a^\dagger = (a^\dagger a)^*bb'](b^\dagger)'a^\dagger
\]

\[
(b'[a^\dagger a)^*bb']^\dagger(a^\dagger a)^*bb'] = b'[a^\dagger a)^*bb']^\dagger(a^\dagger a)^*bb'] = b'[a^\dagger a)^*bb']^\dagger
\]

\[
(a^\dagger a)^*bb']^\dagger(a^\dagger a)^*bb'] = b'[a^\dagger a)^*bb']^\dagger \quad \text{is self-adjoint.}
\]

The condition \(b'[a^\dagger a)^*bb']^\dagger \in [a^\dagger a)^*bb']^1[4]\) holds, because \((e6)\) gives

\[
b'[a^\dagger a)^*bb']^\dagger(a^\dagger a)^*bb'] \in [a^\dagger a)^*bb']^1[1, 2, 3] \quad \text{and} \quad (a^\dagger a)^*bb']^1 \in [a^\dagger a)^*bb']^1[1, 2, 3].
\]

Hence, \((a^\dagger a)^*bb']^1 \in [a^\dagger a)^*bb']^1[1, 2, 3] \quad \text{and} \quad (a^\dagger a)^*bb']^1 \in [a^\dagger a)^*bb']^1[1, 2, 3], \quad \text{from}

\[
a^\dagger a(b^\dagger)'a' = (a^\dagger a)^*bb']^1 a^\dagger a(b^\dagger)'a' = (a^\dagger a)^*bb']^1 a^\dagger = a^\dagger a(b^\dagger)'a' = (a^\dagger a)^*bb']^1
\]

\[
b'[a^\dagger a)^*bb']^1 a^\dagger a(b^\dagger)'a' = b'[a^\dagger a)^*bb']^1 a^\dagger a(b^\dagger)'a' = b'[a^\dagger a)^*bb']^1 a^\dagger a(b^\dagger)'a' = b'[a^\dagger a)^*bb']^1
\]

\[
a^\dagger a(b^\dagger)'a' = a^\dagger a(b^\dagger)'a' = a^\dagger a(b^\dagger)'a' \quad \text{is self-adjoint.}
\]

\[
a^\dagger a(b^\dagger)'a' = a^\dagger a(b^\dagger)'a' = a^\dagger a(b^\dagger)'a' = a^\dagger a(b^\dagger)'a' = a^\dagger a(b^\dagger)'a' = a^\dagger a(b^\dagger)'a'
\]

\[
a^\dagger a(b^\dagger)'a' = a^\dagger a(b^\dagger)'a' = a^\dagger a(b^\dagger)'a' \quad \text{is self-adjoint.}
\]
The assumption \([a(b^*)]^\dagger = b^*[a'(a(bb^*))]^\dagger a^*\) implies that
\[
b^*[a'(a(bb^*))]^\dagger a^* a(b^*)^* \text{ is self-adjoint}
\]
and
\[
a(bb^*)^t[a'(a(bb^*))]^t a^* = a(b^*)^t b^*[a'(a(bb^*))]^t a^* \text{ is self-adjoint},
\]
i.e. \(b^*[a'(a(bb^*))]^t \in [a'(a(bb^*))]^t[4]\) and \([a'(a(bb^*))]^t a^* \in [a(bb^*)]^t[3]\). Therefore, we conclude \(a'(b^*), a(bb^*)^t \in \mathcal{R}^t, [a'(a(bb^*))]^t = b^*[a'(a(bb^*))]^t\) and \([a(bb^*)]^t = [a(a(bb^*))]^t a^\) . Now, we have \([a'(a(bb^*))]^t a^* = b^*[a'(a(bb^*))]^t a^* = [a(b^*)]^t = b^*[a(bb^*)]^t\).

(c7) \Rightarrow (c2): It is easy to check that \([a(bb^*)]^t(a^*) \in [a'(a(bb^*))]^t[1, 2, 4]\):
\[
a'(a(bb^*))^t[a(bb^*)]^t(a^*)^* a'(a(bb^*))^t = a'(a(bb^*))^t[1, 2, 4]
\]
Now, we have \([a'(a(bb^*))]^t a^* = b^*[a'(a(bb^*))]^t\),
\[
[a(bb^*)]^t(a^*)^* a'(a(bb^*))^t = [a'(a(bb^*))]^t a'(a(bb^*))^t \text{ is self-adjoint}.
\]
Using \([a'(a(bb^*))]^t a^* = b^*[a'(a(bb^*))]^t\), we obtain
\[
(a'(a(bb^*))^t[a(bb^*)]^t(a^*)^* = (a'(a(bb^*))^t[1, 2, 4]
\]
Hence, we have \(a'(a(bb^*))^t \in \mathcal{R}^t\) and \([a'(a(bb^*))]^t = [a(bb^*)]^t(a^*)^* . Since, by this equality and (c7),
\[
b^*[a'(a(bb^*))]^t a^* = b^*[a'(a(bb^*))]^t(a^*)^* a^* = [a'(a(bb^*))]^t a'(a(bb^*))^t a^* = [a(a(bb^*))]^t a^*,
\]
in order to show that \(a(b^*)^t \in \mathcal{R}^t\) and \([a(b^*)]^t = b^*[a'(a(bb^*))]^t a^*\), we will prove that \([a(b^*)]^t = [a'(a(bb^*))]^t a^*\).

Indeed, \([a'(a(bb^*))]^t a^* \in [a(b^*)]^t[1, 2, 4]\) follows from
\[
a(b^*)^t[a'(a(bb^*))]^t a'(a(bb^*))^t = (a^*)^t(a'(a(bb^*))^t a'(a(bb^*))^t = (a^*)^t a'(a(bb^*))^t = a(b^*)^t,
\]
\[
[a'(a(bb^*))]^t a'(a(bb^*))^t a^* = [a'(a(bb^*))]^t a^*,
\]
[a'(a(bb^*))]^t a'(a(bb^*))^t a^* is self-adjoint.

By (c7),
\[
a(b^*)^t[a'(a(bb^*))]^t a^* = a(b^*)^t b^*[a'(a(bb^*))]^t = a(bb^*)^t [a'(a(bb^*))]^t \text{ is self-adjoint}.
\]
So, \([a(b^*)]^t \in \mathcal{R}^t\) and \([a'(a(bb^*))]^t a^* = b^*[a'(a(bb^*))]^t a^*\), that is (c2) is satisfied.

(d2) \Rightarrow (e8): First, let us show that \((bb^*)^t a^* \in \mathcal{R}^t\). From
\[
(bb^*)^t a^* = ((bb^*)^t(a^*)^t) a^* = (b^*)^t a^* = (b^*)^t a^*.
\]
\[
[(bb^*)^t(a^*)^t]((bb^*)^t(a^*)^t) = [(bb^*)^t(bb^*)^t(a^*)^t] = [a^*]^t[(bb^*)^t(a^*)^t] = [a^*]^t[(bb^*)^t(a^*)^t] = [a^*]^t[(bb^*)^t(a^*)^t],
\]
By (e7),
\[
a(b^*)^t[a'(a(bb^*))]^t a^* = a(b^*)^t b^*[a'(a(bb^*))]^t = a(bb^*)^t [a'(a(bb^*))]^t \text{ is self-adjoint}.
\]
we deduce that \((a^\dagger)^*[(bb^\dagger)^*(a^\dagger)^\dagger] = (bb^\dagger)^*(a^\dagger)^*[(bb^\dagger)^*(a^\dagger)^\dagger]\) is self – adjoint.

The hypothesis \((bb^\dagger)^*(a^\dagger)^\dagger[(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(b^\dagger)^\dagger \in [b^\dagger(a^\dagger)^\dagger][1, 2, 3]\). The statement \([(bb^\dagger)^*(a^\dagger)^\dagger](b^\dagger)^\dagger \in [b^\dagger(a^\dagger)^\dagger][1, 2, 4]\) is a simple consequence of the equalities

\[
\begin{align*}
(b^\dagger(a^\dagger)^\dagger[(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(b^\dagger)^\dagger & = b^\dagger[(bb^\dagger)^*(a^\dagger)^\dagger[(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(a^\dagger)^\dagger) \\
& = b^\dagger(bb^\dagger)^*(a^\dagger)^\dagger = b^\dagger(a^\dagger)^\dagger,
\end{align*}
\]

\[
\begin{align*}
[(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(b^\dagger)^\dagger & = [(bb^\dagger)^*(a^\dagger)^\dagger][(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(b^\dagger)^\dagger \\
& = [(bb^\dagger)^*(a^\dagger)^\dagger]^\dagger(b^\dagger)^\dagger,
\end{align*}
\]

\([(bb^\dagger)^*(a^\dagger)^\dagger](b^\dagger)^\dagger b^\dagger(a^\dagger)^\dagger = [(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(a^\dagger)^\dagger \text{ is self – adjoint.}

The hypothesis \((b^\dagger a^\dagger)^\dagger = (a^\dagger)^\dagger[(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(b^\dagger)^\dagger\) gives that the elements

\[
(a^\dagger)^\dagger[(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger a^\dagger = (a^\dagger)^\dagger[(bb^\dagger)^*(a^\dagger)^\dagger](b^\dagger)^\dagger b^\dagger a^\dagger
\]

and

\[
b^\dagger(a^\dagger)^\dagger[(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(b^\dagger)^\dagger = b^\dagger a^\dagger[(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(b^\dagger)^\dagger
\]

are self-adjoint, i.e. we obtain that \((a^\dagger)^\dagger[(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(a^\dagger)^\dagger \in [(bb^\dagger)^*(a^\dagger)^\dagger][4]\) and \([(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(b^\dagger)^\dagger \in [b^\dagger(a^\dagger)^\dagger][3]\).

Consequently, \((bb^\dagger)^*a^\dagger, b^\dagger(a^\dagger)^\dagger \in \mathcal{R}^\dagger, [(bb^\dagger)^*a^\dagger] = (a^\dagger)^\dagger[(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(a^\dagger)^\dagger\) and \([b^\dagger(a^\dagger)^\dagger]^\dagger = [(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(b^\dagger)^\dagger\). Then

\[
[(bb^\dagger)^*a^\dagger] = (a^\dagger)^\dagger[(bb^\dagger)^*(a^\dagger)^\dagger][bb^\dagger]^\dagger(b^\dagger)^\dagger = (b^\dagger a^\dagger)^\dagger = (a^\dagger)^\dagger[b^\dagger(a^\dagger)^\dagger]^\dagger
\]

and, by Theorem 1.1, \((a^\dagger)^\dagger[(bb^\dagger)^*a^\dagger], (a^\dagger)^\dagger(b^\dagger)^\dagger = [b^\dagger(a^\dagger)^\dagger] \in \mathcal{R}^\dagger\). Applying involution to (21), we have \((b^\dagger[(a^\dagger)^\dagger][bb^\dagger]^\dagger] = [(a^\dagger)^\dagger(b^\dagger)^\dagger]^\dagger a^\dagger\) and the condition (e8) holds.

(e8) ⇒ (d2): By the elementary computations, we get

\[

(bb^\dagger)^*(a^\dagger)^\dagger a^\dagger[(bb^\dagger)^*a^\dagger] = ((bb^\dagger)^*a^\dagger[bb^\dagger]^\dagger(b^\dagger)^\dagger) = (bb^\dagger)^*a^\dagger(a^\dagger)^\dagger,
\]

\[
a^\dagger[(bb^\dagger)^*a^\dagger][bb^\dagger]^\dagger(a^\dagger)^\dagger = a^\dagger[(bb^\dagger)^*a^\dagger](bb^\dagger)^*a^\dagger[(bb^\dagger)^*a^\dagger]^\dagger = a^\dagger[(bb^\dagger)^*a^\dagger]^\dagger,
\]

\[
(bb^\dagger)^*(a^\dagger)^\dagger a^\dagger[(bb^\dagger)^*a^\dagger]^\dagger = (bb^\dagger)^*a^\dagger[(bb^\dagger)^*a^\dagger]^\dagger \text{ is self – adjoint,}
\]

which yield \(a^\dagger[(bb^\dagger)^*a^\dagger]^\dagger \in [(bb^\dagger)^*(a^\dagger)^\dagger][1, 2, 3]\). Applying involution to the condition \((b^\dagger[(a^\dagger)^\dagger][bb^\dagger]^\dagger) = [(a^\dagger)^\dagger(b^\dagger)^\dagger]^\dagger a^\dagger\), we obtain

\[
[(bb^\dagger)^*a^\dagger] = (a^\dagger)^\dagger[b^\dagger(a^\dagger)^\dagger]^\dagger
\]
and

\[
(a'[bb']^a)\dagger(bb')^a = (a'[bb']^a)^\dagger b'(a')^\dagger \\
= (a'[bb']^a)^\dagger \dagger (b'\dagger)^\dagger b'(a')^\dagger \\
= [b'(a')^\dagger]^\dagger b'(a')^\dagger a^\dagger a \\
= [b'(a')^\dagger]^\dagger b'a^\dagger a' \\
= [b'(a')^\dagger]^\dagger b'(a')^\dagger 
\]

Thus, \((bb')^a \in \mathcal{R}^1\) and \((bb')^a \in \mathcal{R}^1\). This equality and (22) give that

\[
(a')^\dagger[(bb')^a]^\dagger = (a')^\dagger[(bb')^a]^\dagger (b')^\dagger = \alpha^\dagger[a'(b')^\dagger] = (a')^\dagger[b'(a')^\dagger] \\
= (a')^\dagger[b'(a')^\dagger]^\dagger. 
\]

Now, to prove \((b^a)^\dagger = (a')^\dagger[(bb')^a]^\dagger (b')^\dagger\) it is enough to check that \((b^a)^\dagger = (a')^\dagger[b'(a')^\dagger]^\dagger\). We show that \((a')^\dagger[b'(a')^\dagger]^\dagger \in (b^a)^\dagger(1, 2, 3)\) by

\[
(b^a)^\dagger = (a')^\dagger[(bb')^a]^\dagger (b')^\dagger = (b^a)^\dagger b'(a')^\dagger a' \\
= (a')^\dagger[b'(a')^\dagger]^\dagger, \\
(b^a)^\dagger = (b^a)^\dagger \dagger (b')^\dagger \dagger = \alpha^\dagger[(bb')^a]^\dagger (b')^\dagger \\
(\alpha^\dagger[(bb')^a]^\dagger (b')^\dagger)^\dagger \dagger (b')^\dagger \dagger \dagger. 
\]

From (22),

\[
(a')^\dagger[b'(a')^\dagger]^\dagger b'(a')^\dagger = [(bb')^a]^\dagger (b')^\dagger b'(a')^\dagger = [(bb')^a]^\dagger (b')^\dagger a', 
\]

that is \((a')^\dagger[(bb')^a]^\dagger (b')^\dagger \in (b^a)^\dagger[4]\). So, we obtain that \((b^a)^\dagger \in \mathcal{R}^1\) and \((b^a)^\dagger = (a')^\dagger[(bb')^a]^\dagger = (a')^\dagger[(bb')^a]^\dagger (b')^\dagger\).

(a2) \implies (e9): From

\[
aa' abb' bb'(a' abb')^\dagger a^\dagger aa' abb' b = a(a' abb' (a' abb')^\dagger a^\dagger b) = aa' abb' b, \\
b^t(a' abb')^\dagger a^\dagger aa' abb' bb'(a' abb')^\dagger a^\dagger = b^t(a' abb')^\dagger a' abb'(a' abb')^\dagger a^\dagger = b^t(a' abb')^\dagger a^\dagger, 
\]

we conclude that \((a' abb')^\dagger a^\dagger \in (aa' abb'b)[1, 2]\). By the equality

\[
(aa' abb' bb'(a' abb')^\dagger a^\dagger)^\dagger = (aa' abb' (a' abb')^\dagger a^\dagger)^\dagger = (a')^\dagger a' abb' (a' abb')^\dagger a^\dagger = abb' (a' abb')^\dagger a', \\
(b^t(a' abb')^\dagger a^\dagger aa' abb' b)^\dagger = (b^t(a' abb')^\dagger a' abb' b)^\dagger = b'(a' abb')^\dagger a' abb' (b')^\dagger = b'(a' abb')^\dagger a' ab 
\]

and the assumption \((ab)^\dagger = b'(a' abb')^\dagger a^\dagger\), we observe that \((b^t(a' abb')^\dagger a^\dagger \in (aa' abb'b)[3, 4]\). Hence, \(aa' abb'b \in \mathcal{R}^1\) and \((aa' abb'b)^\dagger = b^t(a' abb')^\dagger a^\dagger. \)
(e9) ⇒ (a2): We can get that $b'((a'abb')^+a') \in (ab)[1,2]$ in the following way

$$abbb'(a'abb')^+a'ab = (a^\dagger)'(a'abb'(a'abb')^+a'abb')(b^\dagger)' = (a^\dagger)'a'abb'(b^\dagger)' = ab,$$

$$b'(a'abb')^+a'abb'(a'abb')^+a' = b'(a'abb')^+a'.$$

From the hypothesis $(aa'abb'b)^+ = b'(a'abb')^+a^\dagger$ we obtain

$$\begin{align*}
(aabb'(a'abb')^+a^\dagger)' &= ((a^\dagger)'a'abb'(a'abb')^+a^\dagger)' = aa'abb'(a'abb')^+a^\dagger \\
&= aa'abb'b^\dagger(a'abb')^+a^\dagger \text{ is self–adjoint}
\end{align*}$$

and

$$\begin{align*}
(b'(a'abb')^+a'ab)' &= (b'(a'abb')^+a'abb'(b^\dagger)')' = b^\dagger(a'abb')^+a'abb'b \\
&= b^\dagger(a'abb')^+a'aa'abb'b \text{ is self–adjoint.}
\end{align*}$$

Thus, $ab \in \mathcal{R}^1$ and $(ab)^+ = b'(a'abb')^+a'$, i.e. the statements (a2) is satisfied.

(f1) ⇒ (f2): First, we will prove that $a^\dagger abb' \in \mathcal{R}^1$. From

$$a^\dagger abb'(b^\dagger)'^+a^\dagger abb' = (a^\dagger abb'(a^\dagger abb')^+a^\dagger abb')bb' = a^\dagger abb'bb' = a^\dagger abb',$$

$$b^\dagger(b^\dagger)'(a^\dagger abb')^+a^\dagger abb'(b^\dagger)'^+a^\dagger abb' = (b^\dagger)'b^\dagger(a^\dagger abb')^+a^\dagger abb'(a'abb')^+ = (b^\dagger)'b^\dagger(a^\dagger abb')^+,$$

we have that $(b^\dagger)'b^\dagger(a'abb')^+ \in (a'abb)[1,2,3]$. Using the assumption $(a^\dagger ab)^+ = b^\dagger(a^\dagger abb')^+$, we get $b^\dagger(a^\dagger abb')^+a'ab$ is self-adjoint and

$$\begin{align*}
(b^\dagger)'b^\dagger(a^\dagger abb')^+a^\dagger abb' &= (bb^\dagger(a^\dagger abb')^+a^\dagger abb')^+ = (a^\dagger abb')^+a^\dagger abb'bb^\dagger \\
&= (a^\dagger abb')^+a^\dagger abb' \text{ is self–adjoint.}
\end{align*}$$

Therefore, $a^\dagger abb' \in \mathcal{R}^1$ and $(a^\dagger abb')^+ = (b^\dagger)'b^\dagger(a^\dagger abb')^+$. By this equality and (f1) we obtain

$$\begin{align*}
(a^\dagger ab)^+ &= b^\dagger(a^\dagger abb')^+ = b^\dagger(b^\dagger)'b^\dagger(a^\dagger abb')^+ = b^\dagger(a^\dagger ab)^+.
\end{align*}$$

In the same way from the equalities

$$\begin{align*}
a^\dagger abb'(a^\dagger abb')^+a^\dagger y'a^\dagger abb' &= a'(a'abb' a^\dagger abb')^+a^\dagger abb' = a'aa'a^\dagger abb' = a'aabb', \\
(a^\dagger abb')^+a^\dagger y'a^\dagger abb'(a^\dagger abb')^+a^\dagger y'a^\dagger abb' &= (a^\dagger abb')^+a^\dagger abb'(a^\dagger abb')^+a^\dagger y'a^\dagger abb' = (a^\dagger abb')^+a^\dagger y'a^\dagger abb', \\
(a^\dagger abb')^+a^\dagger y'a^\dagger abb' &= (a^\dagger abb')^+a^\dagger y'a^\dagger abb',
\end{align*}$$

we deduce $(a^\dagger abb')^+a^\dagger y'a^\dagger \in (a'abb')[1,2,4]$. The hypothesis $(abb')^+ = (a^\dagger abb')^+a^\dagger$ implies that $a^\dagger abb'(a^\dagger abb')^+a^\dagger$ is self-adjoint and then

$$\begin{align*}
a^\dagger abb'(a^\dagger abb')^+a^\dagger y'a^\dagger &= (a^\dagger abb'(a^\dagger abb')^+a^\dagger a^\dagger)^+ = a^\dagger aa'a^\dagger abb'(a^\dagger abb')^+ \\
&= a^\dagger abb'(a^\dagger abb')^+ \text{ is self–adjoint.}
\end{align*}$$
Thus, we get that $a'abb^* \in \mathcal{R}$, $(a'abb^*)^\dagger = (a''abb^*) a^\dagger (a')^\dagger$ and, by (f1),

$$(abb^*)^\dagger = (a''abb^*) a^\dagger = (a''abb^*) a^\dagger (a')^\dagger a' = (a''abb^*) a'.$$

So, the condition (f2) is satisfied.

$f2 \Rightarrow f1$: Since

$$a'abb^*bb' (a'abb')^\dagger a'abb^* = (a''abb')(a''abb')(b')b^\dagger = a''abb'(b')b^\dagger = a''abb^*,$$

$$bb'(a'abb')^\dagger a'abb^*bb'(a'abb')^\dagger = bb'(a'abb')^\dagger a'abb' (a'abb')^\dagger = bb'(a'abb')^\dagger,$$

we conclude that $bb'(a'abb')^\dagger \in (a''abb^*)_{1, 2, 3}$. By the equality $(a'abb)^\dagger = b'(a'abb')^\dagger$, we have that $b'(a'abb')^\dagger a'abb$ is self-adjoint and then

$$bb'(a'abb')^\dagger a'abb^* = ((b')b'(a''abb^*)^\dagger a'abb')^\dagger = (a''abb')^\dagger a'abb^* b'b^\dagger$$

$$= (a''abb')^\dagger a'abb^*$$

Hence, $a'abb^* \in \mathcal{R}$ and $(a'abb^*)^\dagger = bb'(a'abb')^\dagger$. Now, by (f2) and the last equality,

$$(a''abb)^\dagger = b'(a'abb')^\dagger = b'b' (a'abb')^\dagger = b'(a'abb')^\dagger.$$

Similarly, from the equalities

$$a'abb^* (a'abb^*)^\dagger a'aa^\dagger abb^* = a^\dagger (a')^\dagger (a''abb^*) a'abb^* = a^\dagger (a')^\dagger a'abb^* = a''abb^*,$$

$$(a'abb^*)^\dagger a'aa^\dagger abb^* (a'abb^*)^\dagger a' a = (a''abb^*) a''abb^* (a'abb^*)^\dagger a' a = (a''abb^*)^\dagger a' a$$

$$(a''abb^*)^\dagger a'aa^\dagger abb^* = (a''abb^*)^\dagger a'abb^*$$

we obtain that $(a''abb^*)^\dagger a' a \in (a''abb^*)_{1, 2, 4}$. Using the condition $(abb^*)^\dagger = (a''abb^*) a'$, the element $abb^*(a''abb^*) a'$ is self-adjoint and now

$$a'abb^* (a'abb^*)^\dagger a' a$$

$$= (a''abb^*) (a''abb^*)^\dagger a' (a')^\dagger = a''a' a' a''abb^* (a''abb^*)^\dagger$$

$$= a''abb^* (a''abb^*)^\dagger$$

is self-adjoint.

Therefore, we show that $(a'abb^*)^\dagger = (a'abb^*) a' a$ and then we get, by (f2),

$$(abb^*)^\dagger = (a''abb^*) a' = (a''abb^*) a' aa^\dagger = (a''abb^*) a'. $$

Thus, the condition (f1) holds.
3. Reverse Order Law in $C^*$-algebras

Now, we consider some additional equivalent conditions for the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ for elements of $C^*$-algebras. First, we have the following result.

Lemma 3.1. Let $\mathcal{A}$ be a unital $C^*$-algebra and let $a, b \in \mathcal{A}^-$. Then the following statements are equivalent:

1. $ab \in \mathcal{A}^-$;
2. $a^\dagger abb^\dagger \in \mathcal{A}^-$;
3. $(1 - bb^\dagger)(1 - a^\dagger a) \in \mathcal{A}^-$;
4. $(a^\dagger)^\dagger b \in \mathcal{A}^-$;
5. $a(b^\dagger)^\dagger \in \mathcal{A}^-$;
6. $b^\dagger a^\dagger \in \mathcal{A}^-$;
7. $(1 - a^\dagger a)(1 - bb^\dagger) \in \mathcal{A}^-$;
8. $a^\dagger ab \in \mathcal{A}^-$;
9. $abb^\dagger \in \mathcal{A}^-$.

Proof. Using Theorem 1.1, Theorem 1.2 and Lemma 1.3, we can easily get these equivalences. Notice that the condition $a^\dagger abb^\dagger \in \mathcal{A}^-$ implies $bb^\dagger a^\dagger a = (a^\dagger abb^\dagger)^\dagger \in \mathcal{A}^\dagger$. Since $a^\dagger a, bb^\dagger \in \mathcal{P}(\mathcal{A})$, then, by Lemma 1.3, the condition $a^\dagger abb^\dagger \in \mathcal{A}^\dagger$ is equivalent to $(1 - bb^\dagger)(1 - a^\dagger a) \in \mathcal{A}^\dagger$, that is $a^\dagger abb^\dagger \in \mathcal{A}^- \iff (1 - bb^\dagger)(1 - a^\dagger a) \in \mathcal{A}^-$. 

Theorem 3.2. Let $\mathcal{A}$ be a unital $C^*$-algebra and let $a, b, ab \in \mathcal{A}^-$. Then the following statements are equivalent:

(a1) $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$;

(a3) $(ab)^\dagger = b^\dagger a^\dagger - b^\dagger[(1 - bb^\dagger)(1 - a^\dagger a)]^\dagger a^\dagger$;

(b3) $[(a^\dagger)^\dagger b]^\dagger = b^\dagger a^\dagger - b^\dagger[(1 - bb^\dagger)(1 - a^\dagger a)]^\dagger a^\dagger$;

(c3) $[a(b^\dagger)^\dagger]^\dagger = b^\dagger a^\dagger - b^\dagger[(1 - bb^\dagger)(1 - a^\dagger a)]^\dagger a^\dagger$;

(d3) $(b^\dagger a^\dagger)^\dagger = ab - a[(1 - a^\dagger a)(1 - bb^\dagger)]^\dagger b$;

(f3) $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a - b^\dagger[(1 - bb^\dagger)(1 - a^\dagger a)]^\dagger a^\dagger a$ and $(abb^\dagger)^\dagger = bb^\dagger a^\dagger - bb^\dagger[(1 - bb^\dagger)(1 - a^\dagger a)]^\dagger a^\dagger$. 

Proof. By Lemma 2.1, the hypothesis \( ab \in \mathcal{A}^* \) implies regularity of suitable elements. Let (b1), (c1), (d1), (f1) be conditions from Theorem 2.1. The equivalences (a1) \( \iff \) (b1) \( \iff \) (c1) \( \iff \) (d1) \( \iff \) (f1) follow from Theorem 2.1.

(a1) \( \iff \) (a3): Since \( a^*a, bb^+ \in \mathcal{P}(\mathcal{A}) \), then, by Theorem 1.4, we obtain the formula
\[
(a^*abb^+)^+ = bb^+a^+a - bb^+[(1 - bb^+)(1 - a^+)a]^+a^+a,
\]
which gives the equality
\[
b^+(a^*abb^+)a^+ = b^+(bb^+a^+a - bb^+[(1 - bb^+)(1 - a^+)a]^+a^+)a^+ \\
= b^+a^+ - b^+[(1 - bb^+)(1 - a^+)a]^+a^+.
\]
(23)

Now, we deduce that \( (ab)^+ = b^+(a^*abb^+)a^+ \) if and only if \( (ab)^+ = b^+a^+ - b^+[(1 - bb^+)(1 - a^+)a]^+a^+ \). Therefore, the statement (a1) is equivalent to (a3).

(b1) \( \iff \) (b3): Multiplying the equality (24) by \( aa^* \) from the right side, we get
\[
b^+(a^*abb^+)a^+ = b^+a^+ - b^+[(1 - bb^+)(1 - a^+)a]^+a^*.
\]
So, \( [(a^*b)]^+ = b^+(a^*abb^+)a^* \) and \( [(a^*b)]^+ = b^+a^+ - b^+[(1 - bb^+)(1 - a^+)a]^+a^* \) are equivalent, that is (b1) \( \iff \) (b3).

(c1) \( \iff \) (c3): Multiplying the equality (24) by \( bb^+ \) from the left side, we have
\[
b^+(a^*abb^+)a^+ = b^+a^+ - b^+[(1 - bb^+)(1 - a^+)a]^+a^+
\]
which yields this equivalence.

(d1) \( \iff \) (d3): Using Theorem 1.4, we observe that
\[
(bb^+a^+a)^+a^* = a^*abb^+ - a^*a[(1 - a^+)a(1 - bb^+)]^+bb^+.
\]
Multiplying this equality by \( a \) from the left side and by \( b \) from the right side we get
\[
a(bb^+a^+a)^+b = ab - a[(1 - a^+)a(1 - bb^+)]^+b.
\]
The equivalence (d1) \( \iff \) (d3) easy follows.

(f1) \( \iff \) (f3): Multiplying the equality (23) first by \( b^+ \) from the left side, we have
\[
b^+(a^*abb^+) = b^+a^+a - b^+[(1 - bb^+)(1 - a^+)a]^+a^+a,
\]
and then by \( a^* \) from the right side, we obtain
\[
(a^*abb^+)^+a^* = bb^+a^* - bb^+[(1 - bb^+)(1 - a^+)a]^+a^*.
\]
Now, this part of proof easy follows.

As a consequence of Theorem 1.5 and Theorem 2.1 we get the following result.
Corollary 3.3. Let $R$ be a ring with involution and let $a, b \in R^t$. Then the following statements are equivalent:

(a1) $ab, a^tbb^t \in R^t$ and $(ab)^t = b^t(a^tbb^t)a^t$;

(e1) $ab, a^tbb^t \in R^t$ and $(ab)^t = (a^tbb^t)a^t = b^t(a^tbb^t)$;

(e2) $(a^t)b, a^tbb^t \in R^t$ and $[(a^t)b]^t = (a^t)(bb^t)^t$;

(e3) $a(b^t), a^tbb^t \in R^t$ and $[a(b^t)]^t = [a^t](bb^t)^t$;

(e4) $b^t, bb^t \in R^t$ and $(bb^t)^t = (bb^t)b = a(b^t)a$;

(e5) $ab, a^tbb^t \in R^t$ and $(ab)^t = (a^tbb^t)a^t = b^t(ab^t)$;

(e6) $(a^t)b, (a^t)b, (a^t)b \in R^t$ and $[(a^t)b]^t = [(a^t)b]^t$;

(e7) $a(b^t), a^tbb^t \in R^t$ and $[a(b^t)]^t = [a^t](bb^t)^t$;

(e8) $(b^t, a^t(bb^t), (a^t)(bb^t) \in R^t$ and $[(b^t, a^t(bb^t)])^t = (a^t)(bb^t)^t$.

References