Coordinate Finite Type Rotational Surfaces in Euclidean Spaces

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Abstract. Submanifolds of coordinate finite-type were introduced in [10]. A submanifold of a Euclidean space is called a coordinate finite-type submanifold if its coordinate functions are eigenfunctions of $\Delta$. In the present study we consider coordinate finite-type surfaces in $\mathbb{E}^4$. We give necessary and sufficient conditions for generalized rotation surfaces in $\mathbb{E}^4$ to become coordinate finite-type. We also give some special examples.

1. Introduction

Let $M$ be a connected $n$-dimensional submanifold of a Euclidean space $\mathbb{E}^m$ equipped with the induced metric. Denote $\Delta$ by the Laplacian of $M$ acting on smooth functions on $M$. This Laplacian can be extended in a natural way to $\mathbb{E}^m$-valued smooth functions on $M$. Whenever the position vector $x$ of $M$ in $\mathbb{E}^m$ can be decomposed as a finite sum of $\mathbb{E}^m$-valued non-constant functions of $\Delta$, one can say that $M$ is of finite type. More precisely the position vector $x$ of $M$ can be expressed in the form $x = x_0 + \sum_{i=1}^{k} x_i$, where $x_0$ is a constant map $x_1, x_2, ..., x_k$ non-constant maps such that $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If $\lambda_1, \lambda_2, ..., \lambda_k$ are different, then $M$ is said to be of $k$-type. Similarly, a smooth map $\phi$ of an $n$-dimensional Riemannian manifold $M$ of $\mathbb{E}^m$ is said to be of finite type if $\phi$ is a finite sum of $\mathbb{E}^m$-valued eigenfunctions of $\Delta$ ([2], [3]). For the position vector field $\vec{H}$ of $M$ it is well known (see eg. [3]) that $\Delta x = -n\vec{H}$, which shows in particular that $M$ is a minimal submanifold in $\mathbb{E}^m$ if and only if its coordinate functions are harmonic. In [13] Takahashi proved that an $n$-dimensional submanifold of $\mathbb{E}^m$ is of 1-type (i.e., $\Delta x = \lambda x$) if and only if it is either a minimal submanifold of $\mathbb{E}^m$ or a minimal submanifold of some hypersphere of $\mathbb{E}^m$. As a generalization of T. Takahashi’s condition, O. Garay considered in [8], submanifolds of Euclidean space whose position vector field $x$ satisfies the differential equation $\Delta x = Ax$, for some $m \times m$ diagonal matrix $A$ with constant entries. Garay called such submanifolds coordinate finite type submanifolds. Actually coordinate finite type submanifolds are finite type submanifolds whose type number $s$ are at most $m$. Each coordinate function of a coordinate finite type submanifold $m$ is of 1-type, since it is an eigenfunction of the Laplacian [10].

In [7] by G. Ganchev and V. Milousheva considered the surface $M$ generated by a W-curve $γ$ in $\mathbb{E}^3$. They have shown that these generated surfaces are a special type of rotation surfaces which are introduced first by C. Moore in 1919 (see [12]). Vranceanu surfaces in $\mathbb{E}^4$ are the special type of these surfaces [14].
This paper is organized as follows: Section 2 gives some basic concepts of the surfaces in \( \mathbb{E}^4 \). Section 3 tells about the generalised surfaces in \( \mathbb{E}^3 \). Further this section provides some basic properties of surfaces in \( \mathbb{E}^3 \) and the structure of their curvatures. In the final section we consider coordinate finite type surfaces in euclidean spaces. We give necessary and sufficient conditions for generalised rotation surfaces in \( \mathbb{E}^4 \) to become coordinate finite type.

2. Basic Concepts

Let \( M \) be a smooth surface in \( \mathbb{E}^n \) given with the patch \( X(u, v) : (u, v) \in D \subseteq \mathbb{E}^2 \). The tangent space to \( M \) at an arbitrary point \( p = X(u,v) \) of \( M \) span \( \{X_u, X_v\} \). In the chart \((u,v)\) the coefficients of the first fundamental form of \( M \) are given by

\[
E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle, \quad \tag{1}\]

where \( \langle \cdot, \cdot \rangle \) is the Euclidean inner product. We assume that \( W^2 = EG - F^2 \neq 0 \), i.e. the surface patch \( X(u, v) \) is regular. For each \( p \in M \), consider the decomposition \( T_p \mathbb{E}^n = T_p M \oplus T_p^\perp M \) where \( T_p^\perp M \) is the orthogonal component of \( T_p M \) in \( \mathbb{E}^n \). Let \( \nabla \) be the Riemannian connection of \( \mathbb{E}^4 \). Given orthonormal local vector fields \( X_1, X_2 \) tangent to \( M \).

Let \( \chi(M) \) and \( \chi^+(M) \) be the space of the smooth vector fields tangent to \( M \) and the space of the smooth vector fields normal to \( M \), respectively. Consider the second fundamental map: \( h : \chi(M) \times \chi(M) \to \chi^+(M) \);

\[
h(X_i, X_j) = \nabla_{X_i} X_j - \nabla_{X_j} X_i \quad 1 \leq i, j \leq 2, \quad \tag{2}\]

where \( \nabla \) is the induced. This map is well-defined, symmetric and bilinear.

For any arbitrary orthonormal normal frame field \( \{N_1, N_2, ..., N_{n-2}\} \) of \( M \), recall the shape operator \( A : \chi^+(M) \times \chi(M) \to \chi(M) \);

\[
A Ni X_j = - (\nabla N_i N_j)^T, \quad X_j \in \chi(M), \quad 1 \leq k \leq n-2, \quad \tag{3}\]

This operator is bilinear, self-adjoint and satisfies the following equation:

\[
\langle A Ni X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = h_{ij}^{k}, \quad 1 \leq i, j \leq 2. \tag{4}\]

The equation (2) is called Gaussian formula, and

\[
h(X_i, X_j) = \sum_{k=1}^{n-2} h_{ij}^{k} N_k, \quad 1 \leq i, j \leq 2 \tag{5}\]

where \( h_{ij}^{k} \) are the coefficients of the second fundamental form.

Further, the Gaussian and mean curvature vector of a regular patch \( X(u, v) \) are given by

\[
K = \sum_{k=1}^{n-2} (h_{11}^{k} h_{22}^{k} - (h_{12}^{k})^2), \quad \tag{6}\]

and

\[
H = \frac{1}{2} \sum_{k=1}^{n-2} (h_{11}^{k} + h_{22}^{k}) N_k, \quad \tag{7}\]

respectively, where \( h \) is the second fundamental form of \( M \). Recall that a surface \( M \) is said to be minimal if its mean curvature vector vanishes identically [2]. For any real function \( f \) on \( M \) the Laplacian of \( f \) is defined by

\[
\Delta f = - \sum_i (\nabla_{e_i} \nabla_{e_i} f - \nabla_{e_i} f) = - \sum_i (\nabla_e \nabla_e f - \nabla_e f). \tag{8}\]
3. Generalised Rotation Surfaces in $\mathbb{E}^4$

Let $\gamma = \gamma(s) : I \rightarrow \mathbb{E}^4$ be a W-curve in Euclidean 4-space $\mathbb{E}^4$ parametrized as follows:

$$\gamma(v) = (a \cos cv, a \sin cv, b \cos dv, b \sin dv), \quad 0 \leq v \leq 2\pi,$$

where $a, b, c, d$ are constants ($c > 0, d > 0$). In [7] G. Ganchev and V. Milousheva considered the surface $M$ generated by the curve $\gamma$ with the following surface patch:

$$X(u, v) = (f(u) \cos cv, f(u) \sin cv, g(u) \cos dv, g(u) \sin dv), \quad (9)$$

where $u \in [0, 2\pi]$ and $f(u)$ and $g(u)$ are arbitrary smooth functions satisfying

$$c^2 f'^2 + d^2 g'^2 > 0 \quad \text{and} \quad (f')^2 + (g')^2 > 0.$$

These surfaces are first introduced by C. Moore in [12], called general rotation surfaces. Note that $X_u$ and $X_v$ are always orthogonal and therefore we choose an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that $e_1, e_2$ are tangent to $M$ and $e_3, e_4$ normal to $M$ in the following (see, [7]):

$$e_1 = \frac{X_u}{\|X_u\|}, \quad e_2 = \frac{X_v}{\|X_v\|},$$

$$e_3 = \frac{1}{\sqrt{(f')^2 + (g')^2}}(g' \cos cv, g' \sin cv, -f' \cos dv, -f' \sin dv),$$

$$e_4 = \frac{1}{\sqrt{c^2 f'^2 + d^2 g'^2}}(-dg \sin cv, dg \cos cv, cf \sin dv, -cf \cos dv).$$

Hence the coefficients of the first fundamental form of the surface are

$$E = \langle X_u, X_u \rangle = (f')^2 + (g')^2$$

$$F = \langle X_u, X_v \rangle = 0$$

$$G = \langle X_v, X_v \rangle = c^2 f'^2 + d^2 g'^2$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{E}^4$. Since

$$EG - F^2 = (f')^2 + (g')^2 \left(c^2 f'^2 + d^2 g'^2\right)$$

does not vanish, the surface patch $X(u, v)$ is regular. Then with respect to the frame field $\{e_1, e_2, e_3, e_4\}$, the Gaussian and Weingarten formulas (2)-(3) of $M$ look like (see, [6]):

$$\nabla_{e_1} e_1 = -A(u)e_2 + h_{11}^1 e_3,$$

$$\nabla_{e_1} e_2 = A(u)e_1 + h_{12}^1 e_4,$$

$$\nabla_{e_1} e_3 = h_{11}^2 e_3,$$

$$\nabla_{e_1} e_4 = h_{12}^2 e_4,$$

and

$$\nabla_{e_2} e_1 = -h_{11}^1 e_1 + B(u)e_4,$$

$$\nabla_{e_2} e_2 = -h_{12}^2 e_2 - B(u)e_3,$$

$$\nabla_{e_2} e_3 = -h_{12}^1 e_2,$$

$$\nabla_{e_2} e_4 = -h_{12}^2 e_1,$$

(11)

(12)

(13)
where
\[
A(u) = \frac{c^2 f f' + d^2 g g'}{\sqrt{(f')^2 + (g')^2}(c^2 f'^2 + d^2 g'^2)},
\]
\[
B(u) = \frac{c d(f f' + g g')}{\sqrt{(f')^2 + (g')^2}(c^2 f'^2 + d^2 g'^2)},
\]
\[
h_{11}^1 = \frac{d^2 f f' - c^2 f g'}{\sqrt{(f')^2 + (g')^2}(c^2 f'^2 + d^2 g'^2)},
\]
\[
h_{12}^1 = \frac{g f f'' - f' g''}{((f')^2 + (g')^2)^2},
\]
\[
h_{12}^2 = \frac{c d(f g' - g f')}{\sqrt{(f')^2 + (g')^2}(c^2 f'^2 + d^2 g'^2)},
\]
\[
h_{11}^2 = h_{22}^1 = h_{12}^1 = 0.
\]
are the differentiable functions. Using (6)-(7) with (14) one can get the following results;

**Proposition 3.1.** [1] Let \( M \) be a generalised rotation surface given by the parametrization (9), then the Gaussian curvature of \( M \) is
\[
K = \frac{(c^2 f'^2 + d^2 g'^2)(g' f'' - f' g'')(d^2 g f' - c^2 f g') - c^2 d^2 (g f' - f g')^2((f')^2 + (g')^2)}{((f')^2 + (g')^2)^2(c^2 f'^2 + d^2 g'^2)}.
\]

An easy consequence of Proposition 3.1 is the following.

**Corollary 3.2.** [1] The generalised rotation surface given by the parametrization (9) has vanishing Gaussian curvature if and only if the following equation
\[
(c^2 f'^2 + d^2 g'^2)(g' f'' - f' g'')(d^2 g f' - c^2 f g') - c^2 d^2 (g f' - f g')^2((f')^2 + (g')^2) = 0,
\]
holds.

The following results are well-known;

**Proposition 3.3.** [1] Let \( M \) be a generalised rotation surface given by the parametrization (9), then the mean curvature vector of \( M \) is
\[
\overline{H} = \frac{1}{2}(h_{11}^1 + h_{12}^2)e_3
= \left(\frac{(c^2 f'^2 + d^2 g'^2)(g' f'' - f' g'')(d^2 g f' - c^2 f g')((f')^2 + (g')^2)}{2((f')^2 + (g')^2)^{3/2}(c^2 f'^2 + d^2 g'^2)}\right)e_3.
\]

An easy consequence of Proposition 3.3 is the following.

**Corollary 3.4.** [1] The generalised rotation surface given by the parametrization (9) is minimal surface in \( E^4 \) if and only if the equation
\[
(c^2 f'^2 + d^2 g'^2)(g' f'' - f' g'') + (d^2 g f' - c^2 f g')((f')^2 + (g')^2) = 0,
\]
holds.

**Definition 3.5.** The generalised rotation surface given by the parametrization
\[
f(u) = r(u) \cos u, \quad g(u) = r(u) \sin u, \quad c = 1, d = 1.
\]
is called Vranceanu rotation surface in Euclidean 4-space \( E^4 \) [14].
Remark 3.6. Substituting (15) into the equation given in Corollary 3.2 we obtain the condition for Vranceanu rotation surface which has vanishing Gaussian curvature;

\[ r(u)''(u) - (r'(u))^2 = 0. \tag{16} \]

Further, and easy calculation shows that \( r(u) = \lambda e^{\mu u}, (\lambda, \mu \in \mathbb{R}) \) is the solution is this second degree equation. So, we get the following result.

Corollary 3.7. [15] Let \( M \) is a Vranceanu rotation surface in Euclidean 4-space. If \( M \) has vanishing Gaussian curvature, then \( r(u) = \lambda e^{\mu u} \), where \( \lambda \) and \( \mu \) are real constants. For the case, \( \lambda = 1, \mu = 0 \), \( r(u) = 1 \), the surface \( M \) is a Clifford torus, that is it is the product of two plane circles with same radius.

Corollary 3.8. [1] Let \( M \) is a Vranceanu rotation surface in Euclidean 4-space. If \( M \) is minimal then

\[ r(u)r''(u) - 3(r'(u))^2 - 2r(u)^2 = 0. \]

holds.

Corollary 3.9. [1] Let \( M \) is a Vranceanu rotation surface in Euclidean 4-space. If \( M \) is minimal then

\[ r(u) = \pm \frac{1}{\sqrt{a \sin 2u - b \cos 2u}}, \tag{17} \]

where, \( a \) and \( b \) are real constants.

Definition 3.10. The surface patch \( X(u, v) \) is called pseudo-umbilical if the shape operator with respect to \( H \) is proportional to the identity (see, [2]). An equivalent condition is the following:

\[ \langle h(X_i, X_j), H \rangle = \lambda^2 \langle X_i, X_j \rangle, \tag{18} \]

where, \( \lambda = ||H|| \). It is easy to see that each minimal surface is pseudo-umbilical.

The following results are well-known;

Theorem 3.11. [1] Let \( M \) be a generalised rotation surface given by the parametrization (9) is pseudo-umbilical then

\[ (c^2 f^2 + d^2 g^2)(g' f'' - f' g'') - (d^2 g f' - c^2 f g')(f')^2 + (g')^2 = 0. \tag{19} \]

The converse statement of Theorem 3.11 is also valid.

Corollary 3.12. [1] Let \( M \) be a Vranceanu rotation surface in Euclidean 4-space. If \( M \) pseudo-umbilical then \( r(u) = \lambda e^{\mu u} \), where \( \lambda \) and \( \mu \) are real constants.

3.1. Coordinate Finite Type Surfaces in Euclidean Spaces

In the present section we consider coordinate finite type surfaces in Euclidean spaces \( \mathbb{E}^{m+2} \). A surface \( M \) in Euclidean \( m \)-space is called coordinate finite type if the position vector field \( X \) satisfies the differential equation

\[ \Delta X = AX, \tag{20} \]

for some \( m \times m \) diagonal matrix \( A \) with constant entries. Using the Beltrami formula’s \( \Delta X = -2H \) with (7) one can get

\[ \Delta X = - \sum_{k=1}^{n} (h^k_1 + h^k_2)N_k. \tag{21} \]
So, using (20) with (21) the coordinate finite type condition reduces to

\[ AX = - \sum_{k=1}^{n} (h_{11}^k + h_{22}^k) N_k \]  

(22)

For a non-compact surface in \( \mathbb{E}^4 \) O.J. Garay obtained the following:

**Theorem 3.13.** [9] The only coordinate finite type surfaces in Euclidean 4-space \( \mathbb{E}^4 \) with constant mean curvature are the open parts of the following surfaces:

i) a minimal surface in \( \mathbb{E}^4 \),

ii) a minimal surface in some hypersphere \( S^3(r) \),

iii) a helical cylinder,

iv) a flat torus \( S^1(a) \times S^1(b) \) in some hypersphere \( S^3(r) \).

### 3.2. Surface of Revolution of Coordinate Finite Type

A surface in \( \mathbb{E}^3 \) is called a surface of revolution if it is generated by a curve \( C \) on a plane \( \Pi \) when \( \Pi \) is rotated around a straight line \( L \) in \( \Pi \). By choosing \( \Pi \) to be the \( xz \)-plane and line \( L \) to be the \( x \) axis the surface of revolution can be parameterized by

\[ X(u, v) = (f(u), g(u) \cos v, g(u) \sin v), \]  

(23)

where \( f(u) \) and \( g(u) \) are arbitrary smooth functions. We choose an orthonormal frame \( \{e_1, e_2, e_3\} \) such that \( e_1, e_2 \) are tangent to \( M \) and \( e_3 \) normal to \( M \) in the following:

\[ e_1 = \frac{X_u}{\|X_u\|}, \quad e_2 = \frac{X_v}{\|X_v\|}, \quad e_3 = \frac{1}{\sqrt{(f')^2 + (g')^2}} (g', -f' \cos v, -f' \sin v), \]  

(24)

By covariant differentiation with respect to \( e_1, e_2 \) a straightforward calculation gives

\[ \nabla_{e_1} e_1 = h_{11}^1 e_3, \]

\[ \nabla_{e_2} e_2 = -A(u)e_1 + h_{22}^2 e_3, \]

\[ \nabla_{e_1} e_2 = A(u)e_2, \]

\[ \nabla_{e_1} e_2 = 0, \]  

(25)

where

\[ A(u) = \frac{g'}{\sqrt{(f')^2 + (g')^2}}, \]

\[ h_{11}^1 = \frac{g'' f' - f' g''}{((f')^2 + (g')^2)^{\frac{3}{2}}}, \]

\[ h_{22}^1 = \frac{g'}{\sqrt{(f')^2 + (g')^2}}, \]

\[ h_{12}^1 = 0. \]  

(26)

are the differentiable functions. Using (6)-(7) with (26) one can get

\[ \vec{H} = \frac{1}{2} (h_{11}^1 + h_{22}^1) e_3 \]  

(27)

where \( h_{11}^1 \) and \( h_{22}^1 \) are the coefficients of the second fundamental form given in (26).

A surface of revolution defined by (23) is said to be of polynomial kind if \( f(u) \) and \( g(u) \) are polynomial functions in \( u \) and it is said to be of rational kind if \( f \) is a rational function in \( g \), i.e., \( f \) is the quotient of two polynomial functions in \( g \) [4].

For finite type surfaces of revolution B.Y. Chen and S. Ishikawa obtained in [5] the following results;
Theorem 3.14. [5] Let $M$ be a surface of revolution of polynomial kind. Then $M$ is a surface of finite type if and only if either it is an open portion of a plane or it is an open portion of a circular cylinder.

Theorem 3.15. [5] Let $M$ be a surface of revolution of rational kind. Then $M$ is a surface of finite type if and only if $M$ is an open portion of a plane.

T. Hasanis and T. Vlachos proved the following.

Theorem 3.16. [10] Let $M$ be a surface of revolution. If $M$ has constant mean curvature and is of finite type then $M$ is an open portion of a plane, of a sphere or of a circular cylinder.

We proved the following result;

Lemma 3.17. Let $M$ be a surface of revolution given with the parametrization (23). Then $M$ is a surface of coordinate finite type if and only if diagonal matrix $A$ is of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

(28)

where

$$a_{11} = \frac{-g'(g(gf'' - f'g'') + f'((f')^2 + (g')^2))}{fg((f')^2 + (g')^2)^2}$$

(29)

$$a_{22} = a_{33} = \frac{f'(g(gf'' - f'g'') + f'((f')^2 + (g')^2))}{g^2((f')^2 + (g')^2)^2}$$

are constant functions.

Proof. Assume that the surface of revolution $M$ given with the parametrization (23). Then, from the equality (21)

$$\Delta X = -(h_{11}^1 + h_{22}^1)\mathbf{e}_3.$$  

(30)

Further, substituting (26) into (30) and using (24) we get the

$$\Delta X = \psi \begin{bmatrix} -g' \\ -f' \cos v \\ -f' \sin v \end{bmatrix}$$

(31)

where

$$\psi = \frac{-g(gf'' - f'g'') + f'((f')^2 + (g')^2)}{g((f')^2 + (g')^2)^2}$$

is differentiable function. Similarly, using (23) we get

$$AX = \begin{bmatrix} a_{11}g & a_{22}g \cos v & a_{11}g \sin v \\ a_{22}g \cos v & a_{22}g & a_{33}g \sin v \end{bmatrix}. $$

(32)

Since, $M$ is coordinate finite type then from the definition it satisfies the equality $AX = \Delta X$. Hence, using (31) and (32) we get the result. $\square$
Remark 3.18. If the diagonal matrix \( A \) is equivalent to a zero matrix then \( M \) becomes minimal. So the surface of revolution \( M \) is either an open portion of a plane or an open portion of a catenoid.

Minimal rotational surfaces are of coordinate finite type.

For the non-minimal case we obtain the following result;

Theorem 3.19. Let \( M \) be a non-minimal surface of revolution given with the parametrization (23). If \( M \) is coordinate finite type surface then

\[
ff' + \lambda gg' = 0 \tag{33}
\]

holds, where \( \lambda \) is a nonzero constant.

Proof. Since the entries \( a_{11}, a_{22} \) and \( a_{33} \) of the diagonal matrix \( A \) are real constants then from the equality (29) one can get the following differential equations

\[
-\frac{g'(g'f'' - f'g'') + f'((f')^2 + (g')^2)}{fg((f')^2 + (g')^2)^2} = c_1
\]

\[
\frac{f'(g'f'' - f'g'') + f'((f')^2 + (g')^2)}{g^2((f')^2 + (g')^2)^2} = c_2.
\]

where \( c_1, c_2 \) are nonzero real constants. Further, substituting one into another we obtain the result. \( \square \)

Example 3.20. The round sphere given with the parametrization \( f(u) = r \cos u, g(u) = r \sin u \) satisfies the equality (33). So it is a coordinate finite type surface.

Example 3.21. The cone \( f(u) = g(u) \) satisfies the equality (33). So it is a coordinate finite type surface.

3.3. Generalised Rotation Surfaces of Coordinate Finite Type

In the present section we consider generalised rotation surfaces of coordinate finite type surfaces in Euclidean 4-spaces \( \mathbb{E}^4 \).

We proved the following result;

Lemma 3.22. Let \( M \) be a generalised rotation surface given with the parametrization (9). Then \( M \) is a surface of coordinate finite type if and only if diagonal matrix \( A \) is of the form

\[
A = \begin{bmatrix}
  a_{11} & 0 & 0 & 0 \\
  0 & a_{22} & 0 & 0 \\
  0 & 0 & a_{33} & 0 \\
  0 & 0 & 0 & a_{44}
\end{bmatrix} \tag{34}
\]

where

\[
a_{11} = a_{22} = \frac{-f'((f')^2 + (g')^2)((f')^2 + (g')^2) + (g'f'' - f'g'')(c^2f^2 + d^2g^2))}{f((f')^2 + (g')^2)^2(c^2f^2 + d^2g^2)},
\]

\[
a_{33} = a_{44} = \frac{f'((f')^2 + (g')^2)((f')^2 + (g')^2) + (g'f'' - f'g'')(c^2f^2 + d^2g^2))}{g((f')^2 + (g')^2)^2(c^2f^2 + d^2g^2)},
\]

are constant functions.
Proof. Assume that the generalised rotation surface given with the parametrization (9). Then, from the equality (21)
\[
\Delta X = -\left( h^1_{11} + h^1_{22} \right) e_3 - \left( h^2_{11} + h^2_{22} \right) e_4.
\]
(36)
Further, substituting (14) into (36) and using (10) we get the
\[
\Delta X = \varphi\begin{bmatrix}
g' \cos cv \\
g' \sin cv \\
-f' \cos dv \\
-f' \sin dv
\end{bmatrix}
\]
(37)
where
\[
\varphi = \frac{\left( d^2 f' g - c^2 f g' \right) \left( (f')^2 + (g')^2 \right) + \left( \dot{g}' f'' - \ddot{g}' f' \right) \left( c^2 f^2 + d^2 g^2 \right)}{(f')^2 + (g')^2 \left( c^2 f^2 + d^2 g^2 \right)}
\]
is differentiable function. Also using (9) we get
\[
AX = \begin{bmatrix}
a_{11} f \cos cv \\
a_{22} f \sin cv \\
a_{33} g \cos dv \\
a_{44} g \sin dv
\end{bmatrix}
\]
(38)
Since, \( M \) is coordinate finite type then from the definition it satisfies the equality \( AX = \Delta X \). Hence, using (37) and (38) we get the result. \( \Box \)

If the matrix \( A \) is a zero matrix then \( M \) becomes minimal. So minimal rotational surfaces are of coordinate finite type.

For the non-minimal case we obtain the following result;

**Theorem 3.23.** Let \( M \) be a generalised rotation surface given by the parametrization (9). If \( M \) is a coordinate finite type then
\[
ff' = \mu gg'
\]
holds, where, \( \mu \) is a real constant.

Proof. Since the entries \( a_{11}, a_{22}, a_{33} \) and \( a_{44} \) of the diagonal matrix \( A \) are real constants then from the equality (29) one can get the following differential equations
\[
\frac{-f' \left( (f')^2 + (g')^2 \right) \left( f(f' - f') + g(g' - g') \right) \left( c^2 f^2 + d^2 g^2 \right)}{g((f')^2 + (g')^2) \left( c^2 f^2 + d^2 g^2 \right)} = d_1,
\]
\[
\frac{f' \left( (f')^2 + (g')^2 \right) \left( f(f' - f') + g(g' - g') \right) \left( c^2 f^2 + d^2 g^2 \right)}{g((f')^2 + (g')^2) \left( c^2 f^2 + d^2 g^2 \right)} = d_2
\]
where \( d_1, d_2 \) are nonzero real constants. Further, substituting one into another we obtain the result. \( \Box \)

An easy consequence of Theorem 3.23 is the following,

**Corollary 3.24.** Let \( M \) be a Vranceanu rotation surface in Euclidean 4-space. If \( M \) is a coordinate finite type, then
\[
rr' \left( \cos^2 u + c \sin^2 u \right) = r^2 \cos u \sin u (1 + c)
\]
holds, where, \( c \) is a real constant.

In [11] C. S. Houh investigated Vranceanu rotation surfaces of finite type and proved the following

**Theorem 3.25.** [11] A flat Vranceanu rotation surface in \( \mathbb{E}^4 \) is of finite type if and only if it is the product of two circles with the same radius, i.e. it is a Clifford torus.
References


