On Slowly Varying Sequences

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Abstract. In this paper we investigate the connection between the class $SV$ of slowly varying sequences (in the sense of Karamata) and the slow equivalence, strong asymptotic equivalence, selection principles and game theory.

1. Introduction and results

Real functions $f, g : [a, +\infty) \rightarrow \mathbb{R}, (a > 0)$, are mutually inversely asymptotic, in denotation $f(x) \sim g(x)$, as $x \rightarrow +\infty$ (see e.g. [1, 5, 7]), if for each $\lambda > 1$, there is an $x_0 = x_0(\lambda) \geq a$ such that the inequality

$$ f\left(\frac{x}{\lambda}\right) \leq g(x) \leq f(\lambda x), \quad (1) $$

is satisfied for each $x \geq x_0$.

In particular, real functions $f, g : [a, +\infty) \rightarrow (0, +\infty), (a > 0)$, are mutually slowly equivalent (see e.g. [8]), in denotation $f(x) \sim g(x)$, as $x \rightarrow +\infty$, if

$$ \lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{g(x)} = 1 \quad (2) $$

and

$$ \lim_{x \rightarrow +\infty} \frac{g(\lambda x)}{f(x)} = 1 \quad (3) $$

hold for each $\lambda > 1$.

Sequences of positive real numbers $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ are mutually slowly equivalent, in denotation $c_n \sim d_n$, as $n \rightarrow +\infty$, if

$$ \lim_{n \rightarrow +\infty} \frac{c_{[\lambda n]}}{d_n} = 1 \quad (4) $$

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Remark 1.5. \( \leq \)  

A measurable real function \( f : [a, +\infty) \mapsto (0, +\infty) \), \( (a > 0) \) is slowly varying in sense of Karamata (see e.g. [9]) if

\[
\lim_{x \to +\infty} \frac{f(\lambda x)}{f(x)} = 1,
\]

holds for each \( \lambda > 0 \). The set of all these functions is denoted by \( SV_f \). The class \( SV_f \) is very important in asymptotic analysis (see [12]).

A sequence of positive real numbers \( c = (c_n)_{n \in \mathbb{N}} \) is slowly varying in sense of Karamata (see e.g. [1]) if

\[
\lim_{n \to +\infty} \frac{c_{[\lambda n]}}{c_n} = 1,
\]

holds for each \( \lambda > 0 \). The set of all these sequences important in asymptotic analysis is denoted by \( SV_s \) (see [1]).

In this paper the set of all positive real sequences will be denoted with \( S \) (see e.g. [2]).

**Proposition 1.1.** Let sequences \( c = (c_n)_{n \in \mathbb{N}} \) and \( d = (d_n)_{n \in \mathbb{N}} \) be elements from \( S \). If \( c_n \leq d_n \) as \( n \to +\infty \), then \( c \in SV_s \) and \( d \in SV_s \).

**Proposition 1.2.** Relation \( \leq \) is a relation of equivalence in \( SV_s \).

The next definition is well-known definition of \( \alpha_i \)-selection principles (see e.g. [11]).

**Definition 1.3.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be nonempty subfamilies of the set \( S \). The symbol \( \alpha_i(\mathcal{A}, \mathcal{B}), i \in \{2, 3, 4\} \), denotes the following selection hypotheses: for each sequence \( (A_n)_{n \in \mathbb{N}} \) of elements from \( \mathcal{A} \) there is an element \( B \in \mathcal{B} \) such that:

1. \( \alpha_2(\mathcal{A}, \mathcal{B}) \): the set \( \text{Im}(A_n) \cap \text{Im}(B) \) is infinite for each \( n \in \mathbb{N} \);
2. \( \alpha_3(\mathcal{A}, \mathcal{B}) \): the set \( \text{Im}(A_n) \cap \text{Im}(B) \) is infinite for infinitely many \( n \in \mathbb{N} \);
3. \( \alpha_4(\mathcal{A}, \mathcal{B}) \): the set \( \text{Im}(A_n) \cap \text{Im}(B) \) is nonempty for infinitely many \( n \in \mathbb{N} \),

where \( \text{Im} \) denotes the image of the corresponding set.

We need also the definition of an interesting game related to the \( \alpha_2 \) selection principle (see e.g. [11]; see also [4]).

**Definition 1.4.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be nonempty subfamilies of the set \( S \). The symbol \( \mathcal{G}_{\alpha_2}(\mathcal{A}, \mathcal{B}) \) denotes the following infinitely long game for two players who play a round for each natural number \( n \). In the first round the first player plays an arbitrary element \( A_1 = (A_{1, j})_{j \in \mathbb{N}} \) from \( \mathcal{A} \), and the second one chooses an elements from the subsequence \( y_{r_1} = (A_{1, r_1(j)})_{j \in \mathbb{N}} \) of the sequence \( A_1 \). At the \( k \)th round, \( k \geq 2 \), the first player plays an arbitrary element \( A_k = (A_{k, j})_{j \in \mathbb{N}} \) from \( \mathcal{A} \) and the second one chooses an elements from the subsequence \( y_{r_k} = (A_{k, r_k(j)})_{j \in \mathbb{N}} \) of the sequence \( A_k \), such that \( \text{Im}(r_1(j)) \cap \text{Im}(r_k(j)) = \emptyset \) is satisfied, for each \( p \leq k-1 \). We will say that the second player wins a play \( A_1, y_{r_1}; \ldots; A_k, y_{r_k}; \ldots \) if and only if all elements from the \( Y = \bigcup_{k \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} A_{k, r_k(j)} \), with respect to second index, form a subsequence of the sequence \( y = (y_m)_{m \in \mathbb{N}} \in \mathcal{B} \).

**Remark 1.5.** Let sequence \( c = (c_n)_{n \in \mathbb{N}} \in SV_s \). We will introduce the next set

\[
[c]_\leq = \{ d = (d_n)_{n \in \mathbb{N}} \in SV_s \mid c_n \leq d_n, \text{as } n \to +\infty \}.
\]
Proposition 1.6. The second player has a winning strategy in the game $G_{α_2}(\lfloor c \rfloor, \lfloor c \rfloor)$ for each fixed sequence $c \in SV_\lambda$.

Corollary 1.7. The selection principle $α_2(\lfloor c \rfloor, \lfloor c \rfloor)$ is satisfied, where the sequence $c \in SV_\lambda$ is given and fixed.

Remark 1.8. (1) From Corollary 1.7 and [2] it follows that the selection principles $α_i(\lfloor c \rfloor, \lfloor c \rfloor)$ are satisfied for $i \in \{3, 4\}$, where the sequence $c \in SV_\lambda$ is arbitrary pre-selected and fixed.

(2) From the proof of Proposition 1.6 we have that $c_n \sim d_{n+1}$, as $n \to +\infty$, whenever sequences $c = (c_n)_{n \in \mathbb{N}}$ and $d = (d_n)_{n \in \mathbb{N}}$ belong to the class $SV_\lambda$. (The symbol $\sim$ denotes strong asymptotic equivalence (see e.g. [1]).)

(3) The assertion of Corollary 1.7 has already been given in [3], but in a different form. Actually, in [3] only the sketch of the proof of this corollary is given.

The following is the definition of one of classical selection principles (see e.g. [10]).

Definition 1.9. Let $\mathcal{A}$ and $\mathcal{B}$ be a nonempty subfamilies of the set $\mathbb{N}$. The symbol $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the next selection hypothesis: for each sequence $(A_n)_{n \in \mathbb{N}}$ from $\mathcal{A}$ there is a sequence $B \in \mathcal{B}$ which consists of some numbers from the double sequence $(A_n)_{n \in \mathbb{N}}$ such that sequences $B$ and $(A_n)_{n \in \mathbb{N}}$ have finitely many common elements for each $n \in \mathbb{N}$.

In the following definition we define a new interesting two-person game.

Definition 1.10. Let $\mathcal{A}$ and $\mathcal{B}$ be a nonempty subfamilies of the set $\mathbb{N}$. By $G^*_{fin}(\mathcal{A}, \mathcal{B})$ we denote the following infinitely long game for two players: In the first round the first player plays elements $A_1 \in \mathcal{A}$, and the second player chooses $k_1$ ($k_1 \in \mathbb{N}$) elements from the sequence $A_1$, i.e. elements $b_{11}, b_{12}, \ldots, b_{1k_1}$. At $s^{th}$ round, $s \geq 2$, the first player chooses an element $A_s \in \mathcal{A}$, and the second player responses by choosing $k_s$ ($k_s \in \mathbb{N} \cup \{0\}$) elements from the sequence $A_{s-1}$, i.e. $b_{s-1k_1+1}, b_{s-1k_1+2}, \ldots, b_{s-1k_1+k_s-1}$, and $k_s^{th}$ element from the sequence $A_s$, say $b_{sk_s}$. If we form the sequence $(b_i)_{i \in \mathbb{N}}$ from such chosen elements

$$b_{11}, b_{12}, \ldots, b_{1k_1}, \ldots, b_{s-1k_1+1}, b_{s-1k_1+2}, \ldots, b_{s-1k_1+k_s-1}, b_{sk_s}, \ldots$$

then we say that the second player wins a play

$$A_1, b_{11}, b_{12}, \ldots, b_{1k_1}, \ldots, A_s, b_{s-1k_1+1}, b_{s-1k_1+2}, \ldots, b_{s-1k_1+k_s-1}, b_{sk_s}, \ldots$$

if the sequence $(b_i)_{i \in \mathbb{N}}$ belongs to $\mathcal{B}$.

Proposition 1.11. The second player has a winning strategy in the game $G^*_{fin}(\lfloor c \rfloor, \lfloor c \rfloor)$ for each fixed $c \in SV_\lambda$.

An important game, denoted by $G^*_{fin}(\mathcal{A}, \mathcal{B})$, was considered in [6]. The game $G^*_{fin}(\mathcal{A}, \mathcal{B})$ introduced in the previous definition is a special case of the game $G^*_{fin}(\mathcal{A}, \mathcal{B})$.

Corollary 1.12. The second player has a winning strategy in the game $G^*_{fin}(\lfloor c \rfloor, \lfloor c \rfloor)$ for each fixed $c \in SV_\lambda$.

Remark 1.13. From the previously mentioned we have that the selection principle $S_{fin}(\lfloor c \rfloor, \lfloor c \rfloor)$ is satisfied for each fixed sequence $c \in SV_\lambda$.

2. Proofs of the results

Proof of Proposition 1.1. Firstly, we have

$$\lim_{n \to +\infty} \frac{c_{\lfloor \lambda \rfloor}}{c_n} = \lim_{n \to +\infty} \frac{c_{\lfloor \lambda \rfloor}}{d_{\lfloor \lambda \rfloor - 1} n} = \lim_{n \to +\infty} \frac{c_{\lfloor \lambda \rfloor} - 1}{d_{\lfloor \lambda \rfloor - 1} n} = 1 \cdot 1 = 1,$$

since $[\lambda] - 1 > 1$ and $\frac{1}{\lfloor \lambda \rfloor} > 1$, for $\lambda \geq 3$. 

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Now, let us observe the function \( c_{[\lambda]} \), for \( x \geq 1 \), where \( x \) is real number. Let \( \varepsilon > 1 \). We will prove that there exists an interval \([A, B] \subseteq (3, 4)\), \((A < B)\), depending on \( \varepsilon \), such that the inequality \( \frac{1}{\varepsilon} < \frac{c_{[\lambda]}}{c_n} < \varepsilon \) holds uniformly for \( \lambda \in [A, B] \), for sufficiently large \( n \in \mathbb{N} \). Hence, we will define \( n_{\lambda} \), \((n_{\lambda} \in \mathbb{N})\) as follows

\[
N_{\lambda} = \begin{cases} 
1, & \text{if } \frac{1}{\varepsilon} < \frac{c_{[\lambda]}}{c_n} < \varepsilon, \text{ for each } n \in \mathbb{N}; \\
1 + \max \left\{ n \in \mathbb{N} \mid \frac{c_{[\lambda]}}{c_n} \geq \varepsilon \text{ or } \frac{c_{[\lambda]}}{c_n} \leq \frac{1}{\varepsilon} \right\}, & \text{otherwise},
\end{cases}
\]

for each \( \lambda \in (3, 4) \). Note that \( 1 \leq n_{\lambda} < +\infty \).

Also, we will define a sequence \( (A_k)_{k \in \mathbb{N}} \) of sets \( A_k = \{ \lambda \in (3, 4) \mid n_{\lambda} > k \} \), \( k \in \mathbb{N} \). This is a non-increasing sequence which satisfies that \( \bigcap_{k=1}^{+\infty} A_k = \emptyset \). Not all sets from this sequence are dense in \((3, 4)\), i.e. there exists a set \( A_k \) for some \( k \in \mathbb{N} \) which is not dense in \((3, 4)\). To prove the previously mentioned we must, firstly, emphasize that at least one of the two following inequalities is true: \( \frac{1}{\varepsilon} \geq \frac{c_{[\lambda]}}{c_{n_{\lambda} - 1}} \geq \varepsilon \),

for each \( \lambda \in A_k \) and for fixed \( k \in \mathbb{N} \). Also, there exists \( \delta_{\lambda} > 0 \) for which at least one of the following inequalities is true: \( \frac{1}{\varepsilon} \geq \frac{c_{[\lambda]}}{c_{n_{\lambda} - 1}} \geq \varepsilon \), or \( \frac{c_{[\lambda]}}{c_{n_{\lambda} - 1}} \geq \varepsilon \), for each \( t \in [\lambda, \lambda + \delta_{\lambda}] \).

Since, from inequality \( n_t \geq (n_{\lambda} - 1) + 1 > k \) we obtain that \( t \in A_k \), for this \( k \). Moreover, from \( \lambda \in A_k \) we have that \((\lambda, \lambda + \delta_{\lambda}) \subseteq A_k \). Therefore, if the set \( A_k \) is dense in the interval \((3, 4)\), then the set \( \text{Int}A_k \) is also dense in the interval \((3, 4)\). If we assume that each set \( A_k \) is dense in \((3, 4)\) we obtain that \( (\text{Int}A_k)_{k \in \mathbb{N}} \) is a sequence of dense and open sets in \((3, 4)\), also, and all of these sets are of the second category in \((3, 4)\).

Consequently, \( \bigcap_{k=1}^{+\infty} \text{Int}A_k \) is a dense set in \((3, 4)\), so it is nonempty. That is a contradiction. Hence, there is a set \( A_{n_0} \) for some \( n_0 \in \mathbb{N} \) which is not dense in \((3, 4)\) and there is an interval \([A, B] \subseteq (3, 4)\) such that \([A, B] \subseteq (3, 4) \setminus A_{n_0} = \{ \lambda \in (3, 4) \mid n_{\lambda} \leq n_0 \} \). Now, we have that \( n_{\lambda} < n_0 \), for each \( \lambda \in [A, B] \), and from that it follows \( \frac{1}{\varepsilon} < \frac{c_{[\lambda]}}{c_{n_{\lambda}}} < \varepsilon \), for each \( n \geq n_0 \geq n_{\lambda} \) and \( \lambda \in [A, B] \).

Finally, we obtain that inequalities \( \lim_{x \to +\infty} \frac{c_{[\lambda]}}{c_{[\lambda]}} \geq \frac{1}{\varepsilon} \cdot 1 = \frac{1}{\varepsilon} \) and \( \lim_{x \to +\infty} \frac{c_{[\lambda]}}{c_{[\lambda]}} \leq \varepsilon \cdot 1 = \varepsilon \) are true, for each \( \lambda \geq 12 \), where \( \varepsilon > 1 \) is arbitrary and pre-selected. Therefore, we have that \( \lim_{x \to +\infty} \frac{c_{[\lambda]}}{c_{[\lambda]}} = 1 \) is satisfied, for each \( \lambda \geq 12 \), and the function \( c_{[\lambda]}, x \geq 1 \) is the element of the class \( SV_f \) (see e.g. [1]). The sequence \((c_n)_{n \in \mathbb{N}}\) is the restriction of this function to \( \mathbb{N} \), so it is an element of the class \( SV_f \).

Proof of Proposition 1.2.

1. (Reflexivity) The asymptotic relation \( \lim_{n \to +\infty} \frac{c_{[\lambda]}}{c_n} = 1 \) is satisfied, for each sequence \( c = (c_n)_{n \in \mathbb{N}} \in SV_f \) and \( \lambda > 1 \). Hence, \( c_{[\lambda]} \overset{\sim}{\sim} c_{[\lambda]} \), as \( n \to +\infty \).

2. (Symmetry) Relation \( \overset{\sim}{\sim} \) is symmetric in \( S \), therefore it is symmetric in \( SV_f \subseteq S \), also.

3. (Transitivity) Let us assume that \( c_{[\lambda]} \overset{\sim}{\sim} d_n \), as \( n \to +\infty \), and \( d_n \overset{\sim}{\sim} c_n \), as \( n \to +\infty \) are satisfied, for given sequences \( c = (c_n)_{n \in \mathbb{N}} \), \( d = (d_n)_{n \in \mathbb{N}} \) and \( c = (c_n)_{n \in \mathbb{N}} \) from the class \( SV_f \). Therefore, we obtain that \( \lim_{n \to +\infty} \frac{c_{[\lambda]}}{c_{n_{\lambda}}} = \lim_{n \to +\infty} \frac{c_{[\lambda]}}{c_{n_{\lambda}}} \cdot \lim_{n \to +\infty} \frac{d_{[\gamma_{\lambda}]}(n_{\lambda})}{d_{[\gamma_{\lambda}]}(n_{\lambda})} \cdot \lim_{n \to +\infty} \frac{d_{[\gamma_{\lambda}]}(n_{\lambda})}{d_{[\gamma_{\lambda}]}(n_{\lambda})} \cdot \lim_{n \to +\infty} \frac{c_{[\lambda]}}{c_{n_{\lambda}}} = 1 \cdot 1 \cdot 1 = 1 \), for each \( \lambda > 1 \), since \( d_{[\gamma_{\lambda}]} \sim c_{[\gamma_{\lambda}]} \), as \( n \to +\infty \), and \( \lim_{n \to +\infty} \frac{c_{[\lambda]}}{c_{n_{\lambda}}} = 1 \) is uniform limit, for each \( t \in [a, b] \subseteq (0, +\infty) \), \((a < b)\), (see e.g. [1]) and consequently for each \( t \in \left[ \frac{\sqrt{\lambda} - 1}{2}, \sqrt{\lambda} \right] \), and for some \( \lambda > 1 \), which is arbitrary pre-selected and fixed. In an analogous way it can be proved that \( \lim_{n \to +\infty} \frac{e_{[\lambda]}}{e_n} = 1 \), for each \( \lambda > 1 \). Hence, we obtain
that $c_n \sim c_n$, as $n \to +\infty$. Finally, we will prove that $d_{\sqrt{\eta n}} \sim c_{\sqrt{\eta n}}$ is satisfied, as $n \to +\infty$, for $\lambda > 1$.

Namely, it holds that $\lim_{n \to +\infty} \frac{d_{\sqrt{\eta n}}}{c_{\sqrt{\eta n}}} = \lim_{n \to +\infty} \frac{d_n}{c_n} = 1$, for $\lambda > 1$. This completes the proof. □

**Proof of Proposition 1.11.** Let $c = (c_n)_{n \in \mathbb{N}}$ be an arbitrary and fixed sequence from $SV_\sigma$ and let $[c] = \{d = (d_n)_{n \in \mathbb{N}} \in SV_\sigma : d_n \sim c_n, as n \to +\infty\}$.

**(1st step)** Let $c = (c_n)_{n \in \mathbb{N}} \in SV_\sigma$ and $d = (d_n)_{n \in \mathbb{N}} \in SV_\sigma$, and $c_n \sim d_n$, as $n \to +\infty$. Hence, we obtain

$\lim_{n \to +\infty} \frac{c_n}{d_n} = \lim_{n \to +\infty} \frac{c_n}{d_n} \cdot \frac{1}{\frac{1}{c_n} - \frac{1}{d_n}} = 1 \cdot 1 = 1$ for each $\lambda > 1$ i.e. $c_n \sim d_n$, as $n \to +\infty$. Inversely, let $c = (c_n)_{n \in \mathbb{N}} \in SV_\sigma$ and $d = (d_n)_{n \in \mathbb{N}}$ and $c_n \sim d_n$, as $n \to +\infty$. We have that $\lim_{n \to +\infty} \frac{c_n}{d_n} = \lim_{n \to +\infty} \frac{c_n}{d_n} \cdot \lim_{n \to +\infty} \frac{d_n}{d_n} = 1 \cdot 1 = 1$ is satisfied, for $\lambda > 1$. In the similar way, we can prove that $\lim_{n \to +\infty} \frac{d_n}{c_n} = 1$ holds, for $\lambda > 1$, so we obtain $c_n \sim d_n$ as $n \to +\infty$.

**(2nd step)** Let sequence $c = (c_n)_{n \in \mathbb{N}} \in SV_\sigma$ and the class $[c]$, be given. Also, let $\sigma$ be the strategy of the second player. First player chooses the sequence $x_1 = (x_{1,i})_{i \in \mathbb{N}}$ in $[c]$, arbitrary. Then the second player chooses the subsequence $\sigma(x_1) = (x_{1,j(i)})_{i \in \mathbb{N}}$ of the sequence $x_1$ where $Im(k_i)$ is the set of natural numbers which are divisible with 2 and not divisible with $2^i$. The first player chooses the sequence $x_i = (x_{i,j(i)})_{i \in \mathbb{N}} \in [c]$, arbitrary. Then the second player chooses the subsequence $\sigma(x_i) = (x_{i,j(i)})_{i \in \mathbb{N}}$ of the sequence $x_i$, so that $Im(k_i)$ is the set of natural numbers greater or equal to $j_i$, so that they are divisible with $2^i$, and not divisible with $2^{i+1}$, and $j_i$ exists in $\mathbb{N}$ (because of the 1st step of this proof) and it is given by: $1 - \frac{1}{2^i} \leq \frac{x_{i,j}}{x_i,j} \leq 1 + \frac{1}{2^i}$, for each $i \geq j_i$. Now, we will observe the set $Y = \bigcup_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} X_{i,j}(i)$ of positive real numbers indexed by two indexes. Elements of the set $Y$ we can consider as the subsequence of the sequence $y = (y_{n,m})_{n \in \mathbb{N}}$ given by:

$y_m = \begin{cases} 1 & \text{if } m = k_i(j) \text{ for some } i, j \in \mathbb{N}; \\ x_{1,m} & \text{otherwise.} \end{cases}$

By the construction of the sequence $y$ we have that $y \in \mathcal{S}$. Also, the intersection between $y$ and $x_{i,j} (i \in \mathbb{N})$ is an infinite set of common elements. Let us prove that $y_m \sim x_{1,m}$ as $m \to +\infty$.

Let $\varepsilon \in (0, 1)$. Let us choose the smallest natural number $i$ satisfying $\frac{1}{2^i} \varepsilon < \varepsilon$. For each $k \in \{1, 2, \ldots, i-1\}$ there is $j_k \in \mathbb{N}$ so that inequality $1 - \varepsilon \leq \frac{x_{i,j}}{x_i,j} \leq 1 + \varepsilon$ is satisfied, for each $j \geq j_k$. Let $j^* = \max\{j_1, j_2, \ldots, j_{i-1}\}$. Therefore, the inequality $1 - \varepsilon \leq \frac{x_{1,m}}{y_m} \leq 1 + \varepsilon$ is satisfied, for each $m \geq j^*$. Then, from $x_{1,m} \sim y_m$, as $m \to +\infty$ we obtain $y_m \sim c_m$, as $m \to +\infty$, since $\varepsilon \in (0, 1)$ is arbitrary. From the 1st step of this proof we obtain that $y_m \sim c_m$, as $m \to +\infty$, i.e. $y \in [c]$. This completes the proof. □

**Proof of Proposition 1.11.** Let $\sigma$ be the strategy of the second player.

**(1st round)** Let the first player choose an arbitrary sequence $x_1 = (x_{1,i})_{i \in \mathbb{N}}$ from the class $[c]$. Then the second player plays $\sigma(x_1) = x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}$, where $1 - \frac{1}{2} \leq \frac{x_{1,k}}{x_{1,k}} \leq 1 + \frac{1}{2}$ holds, for each $k \geq k_1$. This is possible according to the 1st step of the proof of Proposition 1.6.

**(2nd round**, $i \geq 2$) Let the first player choose an arbitrary sequence $x_i = (x_{i,j})_{j \in \mathbb{N}}$ from the class $[c]$. Then the second player plays $\sigma(x_i) = x_{i-1,k_i+1}, x_{i-1,k_i+2}, \ldots, x_{i-1,k_i+\epsilon_i}, x_{i,k_i}$, where $1 - \frac{1}{2} \leq \frac{x_{i,k}}{x_{i,k}} \leq 1 + \frac{1}{2}$ holds,
for $k \geq k_i$ and $k_i = 1 + k_{i-1} + k^*_{i-1}$. Thus, the second player forms the sequence $y = (y_m)_{m \in \mathbb{N}}$ given by $x_{1,1}, \ldots, x_{1,k_1}, \ldots, x_{2,k_2}, \ldots, x_{i,k_i}, \ldots$ which belongs to $S$ and has a finite number of elements in common with each of the sequences $x_i, i \in \mathbb{N}$. Let $\varepsilon \in (0,1)$. Then $\frac{1}{2\varepsilon} < \varepsilon$ holds, for some $i \in \mathbb{N}$. Therefore, the inequality $1 - \varepsilon \leq \frac{c_m}{y_m} \leq 1 + \varepsilon$ holds, for each $m \geq 1 + k_1 + k^*_1 + \cdots + k^*_{i-1}$, and we have that $c_m \sim y_m$, as $m \to +\infty$ is true. From the 1st step of the proof of Proposition 1.6, we obtain $y \in [c]_\varepsilon$. This means that the second player wins using the strategy $\sigma$. This completes the proof. □

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