Some Remarks on Incomplete Gamma Type Function $\gamma^*(\alpha, x_-)$

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Abstract. The incomplete gamma type function $\gamma^*(\alpha, x_-)$ is defined as locally summable function on the real line for $\alpha > 0$ by

$$
\gamma^*(\alpha, x_-) = \begin{cases} 
\int_{0}^{x} u^{\alpha-1} e^{-u} \, du, & x \leq 0, \\
0, & x > 0
\end{cases}
$$

the integral diverging for $\alpha \leq 0$ and by using the recurrence relation

$$
\gamma^*(\alpha + 1, x_-) = -\alpha \gamma^*(\alpha, x_-) - x^\alpha e^{-x}
$$

the definition of $\gamma^*(\alpha, x_-)$ can be extended to the negative non-integer values of $\alpha$.

Recently the authors [8] defined $\gamma^*(\alpha, x_-)$ for $m = 0, 1, 2, \ldots$. In this paper we define the derivatives of the incomplete gamma type function $\gamma^*(\alpha, x_-)$ as a distribution for all $\alpha < 0$.

1. Introduction

The incomplete gamma function $\gamma(\alpha, x)$ is defined for $\alpha > 0$ and $x \geq 0$ by

$$
\gamma(\alpha, x) = \int_{0}^{\infty} t^{\alpha-1} e^{-t} \, dt
$$

see [7], the integral diverging for $\alpha \leq 0$. The incomplete gamma function can be defined for $\alpha < 0$ and $\alpha \neq -1, -2, -3, \ldots$ by using the recurrence formula

$$
\gamma(\alpha + 1, x) = \alpha \gamma(\alpha, x) - x^\alpha e^{-x}.
$$

By regularization we have

$$
\gamma(\alpha, x) = \int_{0}^{\infty} t^{\alpha-1} \left[ e^{-t} - \sum_{i=0}^{m-1} \frac{(-t)^i}{i!} \right] \, dt + \sum_{i=0}^{m-1} \frac{(-1)^i x^{\alpha+i}}{(\alpha+i)!}.
$$

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for $-m < \alpha < -m + 1$ and $x > 0$. It follows from the definition of gamma function that

$$\lim_{x \to \infty} \gamma(\alpha, x) = \Gamma(\alpha)$$

for $\alpha \neq 0, -1, -2, \ldots$, see [4, 6, 9].

In the following we let $N$ be the neutrix $[1, 4, 8, 9]$ having domain $N' = \{ \varepsilon : 0 < \varepsilon < \infty \}$ and range $N''$ the real numbers, with negligible functions finite linear sums of the functions

$$\varepsilon^\lambda \ln^{-1} \varepsilon, \quad \ln^r \varepsilon \quad (\lambda < 0, \ r \in \mathbb{Z}^+)$$

and all functions of $\varepsilon$ which converge to zero in the normal sense as $\varepsilon$ tends to zero.

If $f(\varepsilon)$ is a real (or complex) valued function defined on $N'$ and if it is possible to find a constant $\beta$ such that $f(\varepsilon) - \beta$ is in $N$, then $\beta$ is called the neutrix limit of $f(\varepsilon)$ as $\varepsilon \to 0$ and we write $N \lim_{\varepsilon \to 0} f(\varepsilon) = \beta$.

Note that if a function $f(\varepsilon)$ tends to $\beta$ in the normal sense as $\varepsilon$ tends to zero, it converges to $\beta$ in the neutrix sense.

On using equation (2), the incomplete gamma function $\gamma(\alpha, x)$ was also defined by

$$\gamma(\alpha, x) = N \lim_{\varepsilon \to 0} \int_x^\varepsilon u^{\alpha-1} e^{-u} \, du$$

for all $\alpha \in \mathbb{R}$ and $x > 0$, and it was shown that $\lim_{\varepsilon \to 0} \gamma(-m, x) = \Gamma(-m)$ for $m \in \mathbb{N}$, see [4, 9].

The $r$-th derivative of $\gamma(\alpha, x)$ was similarly defined by

$$\gamma^{(r)}(\alpha, x) = N \lim_{\varepsilon \to 0} \int_x^\varepsilon u^{\alpha-1} \ln^r u e^{-u} \, du$$

for all $\alpha$ and $r = 0, 1, 2, \ldots$, provided that the neutrix limit exists, see [9].

The incomplete gamma function with negative arguments are difficult to compute, see [5]. In [10] Thompson gave the algorithm for accurately computing the incomplete gamma function $\gamma(\alpha, x)$ in the cases where $\alpha = n + 1/2, n \in \mathbb{Z}$ and $x < 0$.

However, it was pointed out in [3] that equation (1) could be replaced by the equation

$$\gamma(\alpha, x) = \int_0^x |u|^{\alpha-1} e^{-u} \, du$$

and this equation was used to define $\gamma(\alpha, x)$ for all $x$ and $\alpha > 0$, the integral again diverging for $\alpha \leq 0$.

### 2. The Locally Summable Function $\gamma_*(\alpha, x)$

The locally summable function $\gamma_*(\alpha, x)$ is defined on the real line for $\alpha > 0$ by

$$\gamma_*(\alpha, x) = \begin{cases} \int_0^x |u|^{\alpha-1} e^{-u} \, du, & x \leq 0, \\ 0, & x > 0 \end{cases}$$

$$= \int_0^{-x} |u|^{\alpha-1} e^{-u} \, du$$

see [3, 8] and can be defined as a distribution for $\alpha < 0$ and $\alpha \neq -1, -2, \ldots$ by recurrence formula

$$\gamma_*(\alpha + 1, x) = -\alpha \gamma_*(\alpha, x) - x^\alpha e^{-x}.$$  \hspace{1cm} (6)

If $-m < \alpha < -m + 1, m \in \mathbb{N}$ it is defined by

$$\gamma_*(\alpha, x) = \int_0^{-x} |u|^{\alpha-1} \left[ e^{-u} - \sum_{i=0}^{m-1} \frac{(-u)^i}{i!} \right] \, du - \sum_{i=0}^{m-1} \frac{x^{\alpha+i}}{(\alpha+i)!}.$$  \hspace{1cm} (7)
It was noted in [8] that the function \( \gamma_{\alpha}(x, \alpha) \) can be defined by

\[
\gamma_{\alpha}(x, \alpha) = N \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} |u|^{\alpha - 1} e^{-u} \, du
\]

and this suggested that the incomplete gamma type function \( \gamma_{\alpha}(-m, x) \) be defined by

\[
\gamma_{\alpha}(-m, x) = N \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} |u|^{-m-1} e^{-u} \, du
\]

for \( x < 0 \) and \( m \in \mathbb{N} \). Using equation (7) and taking the neutrix limit, it was shown that

\[
\gamma_{\alpha}(-m, x) = N \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} |u|^{-m-1} e^{-u} \, du
\]

\[
= \int_{0}^{\infty} |u|^{-m-1} [e^{-u} - \sum_{i=0}^{m} \frac{(-u)^i}{i!}] \, du - \sum_{i=0}^{m-1} \frac{x^{i-m}}{(m-i)!} - \frac{1}{m!} \ln x
\]

and also written in the form

\[
\gamma_{\alpha}(-m, x) = \int_{0}^{-1} |u|^{-m-1} [e^{-u} - \sum_{i=0}^{m} \frac{(-u)^i}{i!}] \, du + \int_{-1}^{\infty} |u|^{-m-1} e^{-u} \, du + \sum_{i=0}^{m-1} \frac{1}{(m-i)!}.
\]

If \( m = 0 \), then

\[
\gamma_{\alpha}(0, x) = N \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} |u|^{-1} e^{-u} \, du
\]

\[
= \int_{0}^{\infty} |u|^{-1} (e^{-u} - 1) \, du - \ln x.
\]

Taking the derivative of \( \gamma_{\alpha}(x, \alpha) \), we have

\[
\gamma_{\alpha}^{(r)}(x, \alpha) = N \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} |u|^{\alpha-r-1} \ln^r |u| e^{-u} \, du
\]

for all \( \alpha < 0, r = 0, 1, 2, \ldots \) and \( x < 0 \).

The distribution \( x_{(-m)} \) is defined by

\[
x_{(-m)} = \frac{1}{(m-1)!} (\ln x)^{(m)}.
\]

The definition of \( x_{(-m)} \) here is not the same as Gelfand and Shilov’s definition of \( x_{(-m)} \) which we will denote by \( F(x, -m) \) and it is shown that

\[
x_{(-m)} = F(x, -m) + \frac{\phi(m-1)}{(m-1)!} \delta^{(m-1)}(x)
\]

for \( m = 1, 2, \ldots \), see [2], where

\[
\phi(m) = \left\{ \begin{array}{ll}
0, & m = 0, \\
\sum_{i=1}^{m} i^{-1}, & m > 0.
\end{array} \right.
\]

The following two equations are easily satisfied;
and thus
\[ \lim_{\varepsilon \to 0} \int_{-\infty}^{-\varepsilon} |x|^\alpha \varphi(x) \, dx = \langle x^\alpha, \varphi(x) \rangle \] (14)
if \(-m - 1 < \alpha < -m\) for \(m = 1, 2, \ldots\) and
\[ \lim_{\varepsilon \to 0} \int_{-\infty}^{-\varepsilon} x^{-m} \varphi(x) \, dx = (-1)^m \langle F(x, -m), \varphi(x) \rangle \] (15)
for arbitrary \(\varphi \in \mathcal{D}\) and \(m = 1, 2, \ldots\).

In fact, we have
\[
\int_{-\infty}^{\varepsilon} |x|^\alpha \varphi(x) \, dx = \int_{-\infty}^{\varepsilon} |x|^\alpha \left[ \varphi(x) - \sum_{i=0}^{m-1} \frac{\varphi^{(i)}(0)}{i!} x^i \right] \, dx + \sum_{i=0}^{m-1} \frac{\varphi^{(i)}(0)}{i!} \int_{-\infty}^{\varepsilon} (-x)^{i+1} \, dx
\]
and thus
\[ \lim_{\varepsilon \to 0} \int_{-\infty}^{\varepsilon} |x|^\alpha \varphi(x) \, dx = \int_{-\infty}^{0} |x|^\alpha \left[ \varphi(x) - \sum_{i=0}^{m-1} \frac{\varphi^{(i)}(0)}{i!} x^i \right] \, dx = \langle |x|^\alpha, \varphi(x) \rangle \]
proving equation (14).

Similarly
\[
\int_{-\infty}^{\varepsilon} x^{-m} \varphi(x) \, dx = \int_{-\infty}^{\varepsilon} x^{-m} \left[ \varphi(x) - \sum_{i=0}^{m-2} \frac{\varphi^{(i)}(0)}{i!} x^i - H(x+1) \frac{\varphi^{(m-1)}(0) x^{m-1}}{m-1!} \right] \, dx + \sum_{i=0}^{m-2} \frac{\varphi^{(i)}(0)}{i!} \int_{-\infty}^{\varepsilon} x^{-m+i} \, dx + \frac{\varphi^{(m-1)}(0)}{(m-1)!} \int_{-\infty}^{\varepsilon} x^{-1} \, dx
\]
and it follows that
\[ \lim_{\varepsilon \to 0} \int_{-\infty}^{\varepsilon} x^{-m} \varphi(x) \, dx = \int_{-\infty}^{0} x^{-m} \left[ \varphi(x) - \sum_{i=0}^{m-2} \frac{\varphi^{(i)}(0)}{i!} x^i - H(x+1) \frac{\varphi^{(m-1)}(0) x^{m-1}}{m-1!} \right] \, dx = (-1)^m \langle F(x, -m), \varphi(x) \rangle \]
proving equation (15).

The following theorem was given in [3].

**Theorem 2.1.**

\[ \lim_{\varepsilon \to 0} \int_{-\infty}^{\varepsilon} \varphi(x) \int_{-\varepsilon}^{\varepsilon} |u|^\alpha \, du \, dx = \frac{\langle x^{\alpha+1}, \varphi(x) \rangle}{\alpha + 1} \] (16)
if \(-m - 1 < \alpha < -m\), \(m = 1, 2, \ldots\) and
\[ \lim_{\varepsilon \to 0} \int_{-\infty}^{\varepsilon} \varphi(x) \int_{-\varepsilon}^{\varepsilon} u^{-1} \, du \, dx = \langle \ln x, \varphi(x) \rangle, \] (17)
\[
\begin{align*}
N_{-}\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{-\varepsilon} \varphi(x) \int_{-\varepsilon}^{-\varepsilon} u^{-m} du \, dx & = \frac{\langle F(x_-, -m + 1), \varphi(x) \rangle}{m - 1} \\
& + \frac{(-1)^m \delta^{(m-2)}(x), \varphi(x))}{(m - 1)(m - 1)!} \\
& = \frac{\langle x^{-m+1}, \varphi(x) \rangle}{m - 1} \\
& + \frac{(-1)^m \phi(m - 1)(\delta^{(m-2)}(x), \varphi(x))}{(m - 1)(m - 1)!}
\end{align*}
\] (18)

for \( m = 2, 3, \ldots \) and arbitrary \( \varphi \in \mathcal{D} \).

Equations (7) and (16) suggest that the distribution \( \gamma_\alpha(x, x_-) \) can be defined by

\[
\langle \gamma_\alpha(x, x_-), \varphi(x) \rangle = N_{-}\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{-\varepsilon} \varphi(x) \int_{-\varepsilon}^{-\varepsilon} |u|^{a-1} e^{-u} du \, dx
\] (19)

if \( -m - 1 < \alpha < -m \) for \( m = 1, 2, \ldots \) and \( \varphi \in \mathcal{D} \).

As consequence of equation (19), we define \( \gamma_{\alpha, -m, x_-} \) as follows.

**Definition 2.2.** The distribution \( \gamma_{\alpha, -m, x_-} \) is defined by

\[
\langle \gamma_{\alpha, -m, x_-}, \varphi(x) \rangle = N_{-}\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{-\varepsilon} \varphi(x) \int_{-\varepsilon}^{-\varepsilon} |u|^{-m+1} e^{-u} du \, dx
\]

for \( m = 1, 2, \ldots \) and \( \varphi \in \mathcal{D} \).

**Theorem 2.3.**

\[
\langle \gamma_{\alpha, 0}(x, x_-), \varphi(x) \rangle = (-1)^m N_{-}\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{-\varepsilon} \varphi(x) \int_{-\varepsilon}^{-\varepsilon} |u|^{a-1} e^{-u} du \, dx
\]

\[
= (-1)^m \langle \gamma_{\alpha}, (x, x_-), \varphi(0) \rangle
\]

if \( -m - 1 < \alpha < -m \) for \( m = 1, 2, \ldots \) and \( \varphi \in \mathcal{D} \).

**Proof.**

\[
(-1)^m N_{-}\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{-\varepsilon} \varphi(0)(x) \int_{-\varepsilon}^{-\varepsilon} |u|^{a-1} e^{-u} du \, dx
\]

\[
= (-1)^m N_{-}\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{-\varepsilon} \varphi(0)(x) \int_{-\varepsilon}^{-\varepsilon} |u|^{a-1} \left[ e^{-u} - \sum_{i=0}^{m-1} \frac{(-u)^i}{i!} \right] du \, dx
\]

\[
+ (-1)^m N_{-}\lim_{\varepsilon \to 0} \sum_{i=0}^{m-1} \int_{-\varepsilon}^{-\varepsilon} \frac{e^{a+i} - x^{a+i}}{(a + i)!} \varphi(0)(x) \, dx
\]

On using Taylor’s theorem we have

\[
N_{-}\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{-\varepsilon} e^{a+i} \varphi(0)(x) \, dx
\]

\[
= \lim_{\varepsilon \to 0} e^{a+i} \left[ (\psi(-\varepsilon) - \psi(-\infty)) \right]
\]

\[
= \lim_{\varepsilon \to 0} e^{a+i} \sum_{j=0}^{m-2} \frac{(-\varepsilon)^j \psi^{(j)}(0)}{j!} + (-1)^m \lim_{\varepsilon \to 0} e^{m+i} \varphi^{(m)}(x) \frac{(-\varepsilon)}{(m - 1)!}
\]

\[= 0\]
where \( \psi(x) \) is the primitive of \( q^{(0)}(x) \). Thus

\[
(-1)^r N \lim_{t \to 0} \int_{-\infty}^{-t} q^{(r)}(x) \int_{-x}^{x} |u|^{m-1} e^{-u} du \, dx = \\
= (-1)^r N \lim_{t \to 0} \int_{-\infty}^{-t} q^{(r)}(x) \int_{-x}^{x} |u|^{m-1} e^{-u} \left( -\sum_{i=0}^{m-1} \frac{(-u)^i}{i!} \right) du \, dx \\
= (-1)^r \sum_{i=0}^{m-1} \frac{1}{(\alpha + i)!} (\chi^{\alpha+i}, q^{(r)}(x)) \\
= (-1)^r \langle \gamma_r(\alpha, x), q^{(r)}(x) \rangle.
\]

\[\Box\]

Theorem 2.3 suggests the following definition.

**Definition 2.4.** The distribution \( \gamma_r^{(m)}(-m, x) \) is defined by

\[
\langle \gamma_r^{(m)}(-m, x), \varphi(x) \rangle = (-1)^r N \lim_{t \to 0} \int_{-\infty}^{-t} q^{(r)}(x) \int_{-x}^{x} |u|^{m-1} e^{-u} du \, dx \\
= (-1)^r \langle \gamma_r(-m, x), q^{(r)}(x) \rangle
\]

for arbitrary \( \varphi \in \mathcal{D} \) and \( r, m = 1, 2, \ldots \).

**Theorem 2.5.** The following equations

\[
\langle \gamma_r^{(0)}(0, x), \varphi(x) \rangle = (-1)^r \int_{-\infty}^{0} q^{(r)}(x) \int_{0}^{x} |u|^{m-1} (e^{-u} - 1) du \\
- (-1)^r \langle \ln x, q^{(r)}(x) \rangle \\
= (-1)^r \langle \gamma_r(0, x), q^{(r)}(x) \rangle
\]

and

\[
\langle \gamma_r^{(m)}(-m, x), \varphi(x) \rangle = (-1)^r \langle \gamma_r(-m, x), q^{(r)}(x) \rangle \\
= (-1)^r \int_{-\infty}^{-x} q^{(r)}(x) \int_{0}^{x} |u|^{m-1} \left[ e^{-u} - \sum_{i=0}^{m} \frac{(-u)^i}{i!} \right] du \, dx \\
- (-1)^r \sum_{i=0}^{m-1} \frac{1}{(m-i)!} \langle F(x, -m+i), q^{(r)}(x) \rangle \\
- (-1)^r \sum_{i=0}^{m-1} \frac{\phi(m-i-1)}{(m-i)!} \langle \delta^{(m-i-1)}(x), q^{(r)}(x) \rangle \\
- (-1)^r \frac{1}{m!} \langle \ln x, q^{(r)}(x) \rangle
\]

hold for arbitrary \( \varphi \in \mathcal{D} \) and \( m = 1, 2, \ldots \) and \( r = 0, 1, 2, \ldots \).

**Proof.** Equation (20) follows from equation (11) and Definition 2. Similarly Equation (21) follows from equations (9) and (13) and Definition 2.4. \( \Box \)
References